TWO SEPARATION CRITERIA FOR SECOND ORDER ORDINARY OR PARTIAL DIFFERENTIAL OPERATORS

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We generalize a well-known separation condition of Everitt and Giertz to a class of weighted symmetric partial differential operators defined on domains in $\mathbb{R}^n$. Also, for symmetric second-order ordinary differential operators we show that $\limsup\limits_{t\to c}(pq')'/q^2 = \theta < 2$ where c is a singular point guarantees separation of $-(py')' + qy$ on its minimal domain and extend this criterion to the partial differential setting. As a particular example it is shown that $-\Delta y + qy$ is separated on its minimal domain if q is superharmonic. For $n = 1$ the criterion is used to give examples of a separation inequality holding on the domain of the minimal operator in the limit-circle case.

Keywords: separation, ordinary or partial differential operator, limit-point, essentially self-adjoint

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1. Introduction

In this paper we investigate separation properties of unbounded operators determined by the ordinary or partial differential expressions

$$M_w[y] := w^{-1}[-(py')' + qy], \tag{1.1}$$

$$M_{w,n}[y] := w^{-1}[-\operatorname{div}(P\nabla y) + qy]. \tag{1.2}$$

For (1.1) we assume that p, q, and w satisfy the so-called *minimal conditions* of Naimark [24]; that is, they are real valued functions defined on an interval $I = (a,b)$, $-\infty \leqslant a < b \leqslant \infty$ such that $w > 0$ a.e. and p^{-1}, q, and $w > 0$ are locally integrable functions. In (1.2) ∇y denotes the gradient of y where the differentiation is understood in the sense of distributions. w, q are real-valued functions defined on a domain (open set) $\Omega \subseteq \mathbb{R}^n$; w remains positive, but w, q are $C^2(\Omega)$ and P is a $n \times n$ real matrix valued function such that P is positive semi-definite (and hence symmetric) in the sense that $[P(x)v, v]_n \geqslant 0$ for $x \in \Omega$ where $[\cdot,\cdot]_n$ denotes the euclidean inner product on C^n and the components $\{p_{ij}\}$ are $C^2(\Omega)$.

Suppose $\mathcal{D}_0$ and $\mathcal{D}$ denote the domains of the minimal and maximal operators L_0 and L determined by (1.1) or (1.2) on I or Ω. (Precise definitions of these concepts will be given below.) Then M_w or $M_{w,n}$ is said to be separated on $\mathcal{D}_0$ or $\mathcal{D}$ if for $J = I$ or Ω

$$y \in \mathcal{D}_0 \text{ or } \mathcal{D} \Longrightarrow w^{-1}qy \in L^2(w; J), \tag{1.3}$$

where $L^2(w; J)$ signifies the usual Hilbert space of equivalence classes of all complex Lebesgue square integrable functions f with norm $\|f\|_{w,J}$ and inner product $[f, g]_{w,J}$ given by

$$\|f\|_{w,J} = \Big(\int_J w|f|^2 \,\mathrm{d}x\Big)^{1/2},$$
$$[f, g]_{w,J} = \int_J wfg \,\mathrm{d}x.$$

A property equivalent to separation is the following.

Definition 1. L or L_0 satisfies a separation inequality on $\mathcal{D}$ or $\mathcal{D}_0$ if whenever $y \in \mathcal{D}$ or $y \in \mathcal{D}_0$ then there are constants $A, C, K > 0$, $B \geqslant 0$, and a constant L, all independent of y, such that

$$\begin{aligned} A\|w^{-1}(py')'\|^2_{w,I} + B\|w^{-1}\sqrt{pq}y'\|^2_{w,I} + C\|w^{-1}qy\|^2_{w,I} \\ \leqslant K\|M_w[y]\|^2_{w,I} + L\|y\|^2_{w,I} \end{aligned} \tag{1.4}$$

or

$$\begin{aligned} A\|w^{-1}\operatorname{div}(P\nabla y)\|^2_{w,\Omega} + B\|w^{-1}(q[P\nabla y, \nabla y]_n)^{1/2}\|^2_{w,\Omega} + C\|w^{-1}qy\|^2_{w,\Omega} \\ \leqslant K\|M_{w,n}[y]\|^2_{w,\Omega} + L\|y\|^2_{w,\Omega} \end{aligned} \tag{1.5}$$

hold.

Clearly (1.4), or (1.5) implies (1.3). But if (1.3) holds then a closed graph theorem argument shows that L_0 or L satisfies either (1.4) or (1.5) with $A = C = 1$, $B = 0$, and $K = L$. See [3, Proposition 1] for a proof in the ordinary case. The proof in $\mathbb{R}^n$, $n > 1$, is similar.

If $w = 1$ several criteria for separation in the ordinary case have been given by Everitt and Giertz in a series of pioneering papers [12–16], also see Everitt, Giertz, and Weidmann [17], and Atkinson [1]. More recent results (that include weighted cases) may be found in Brown and Hinton [3]. Some extensions of these criteria to the partial differential case may be found in Everitt and Giertz [16] and Evans and Zettl [9]

One of the principal results of this paper for the ordinary case is that under various conditions on p, q, and w, then the condition

$$(\mathrm{S}_1) \qquad -\infty \leqslant \limsup_{t\to c} w(p(w^{-1}q)')'/q^2 = \theta < 2,$$

where c is a singular endpoint of I implies separation at least on $\mathcal{D}_0$. We will show that the same is true for the partial differential expression (1.2) under the basic conditions assumed above on w, q and P if (S_1) is replaced by

$$(\mathrm{S}_n) \qquad \sup_{t\in\Omega} w\,\mathrm{div}(P\nabla(w^{-1}q))/q^2 = \theta < 2.$$

One easy consequence of (S_1) and standard theory is that M_w will be separated even on $\mathcal{D}$ if $w = p = 1$ and q is bounded below, increasing, and concave downward. Similarly we can prove that $M_{w,n}$ is separated at least on $\mathcal{D}_0$ (and if essentially self-adjoint on $\mathcal{D}$ also) if $w^{-1}q$ is superharmonic on Ω.

A second sufficient condition for separation on $\mathcal{D}_0$ for $n > 1$ involves the condition

$$(|\mathrm{S}_\mathrm{n}^*|) \qquad [P(x)\nabla(w^{-1}q), \nabla(w^{-1}q)]_n^{1/2} \leqslant \theta w^{-1}|q(x)|^{3/2}, \quad 0 < \theta < 2.$$

This result generalizes a separation result in [3] as well as theorems given by Everitt and Giertz in the unweighted case when $P = I$. It is also closely related in form to a result of Evans and Zettl [9] but our proof appears to be simpler and applies to a larger class of potentials q.

The precise statement of these and other results will be given in Sections 3 and 4. The background needed to state and prove them is given immediately below.

2. Preliminaries

Since our results are more comprehensive when $n = 1$ we choose to treat this theory separately from the multidimensional case, even though (1.1) is formally a special case of (1.2). Under the minimal conditions[1] stated above M_w naturally

[1] Naimark only considers the case $w = 1$; however the extension to general weights is routine.

determines minimal and maximal operators L_0 and L in the following way. L_0 is the closure of the "preminimal operator" L_0' which is the restriction of M_w to the compact support functions $\mathcal{D}_0' \subset \mathcal{D}$ where

$$\mathcal{D} := \{y \in L^2(w;I) \cap AC_{\mathrm{loc}}(I) \colon\ py' \in AC_{\mathrm{loc}}(I); M_w[y] \in L^2(w,I)\}.$$

Here $AC_{\mathrm{loc}}(I)$ denotes the locally[2] absolutely continuous functions on I.

The maximal operator L is then given by M_w acting on $\mathcal{D}$. With these definitions it can be shown that:

(i) $L_0 \subset L$,
(ii) $L_0'^* = L_0^* = L$,
(iii) $L^* = L_0 = \overline{L_0'}$.

Thus L_0', L_0, and L are densely defined; L_0', L_0 are symmetric, and L_0, L are respectively the "smallest" and "largest" closed operators in $L^2(w;I)$ naturally generated by M_w. The density of the domains $\mathcal{D}_0', \mathcal{D}_0$, and $\mathcal{D}$ is easy to verify if the coefficients q, p are smooth enough that $C_0^\infty \subseteq \mathcal{D}_0'$; otherwise this is not obvious and is a consequence of the adjoint relationships (ii) and (iii).

If p^{-1}, q are locally integrable on $[a,c)$ or $(c,b]$ for $a < c < \infty$ we say that a or b are *regular*; otherwise they are *singular*. In our setting a or b may be either regular or singular and we signal the regular case at either or both end-points by writing I as a semi-closed or closed interval $[a,b)$, $(a,b]$, or $[a,b]$. We regard an infinite end-point as singular.

M_w is said to be *limit-point* or LP at the singular end-point a or b if there is at most one solution of $M_w[y] = 0$ which is in $L^2(a,c)$ or $L^2(c,b)$ for $a < c < b$. M_w is *limit-circle* or LC at an end-point if both solutions are in $L^2(w;J)$ for a neighborhood J containing the point. If one end-point is regular and the other singular the LP case can be shown equivalent to the property that $\mathcal{D}$ is exactly a two dimensional extension of $\mathcal{D}_0$; while if M_w is limit-circle, then $\mathcal{D}$ is a four dimensional extension of $\mathcal{D}_0$. Still another characterization of the LP property at a singular point (say b) which is sometimes taken as the definition is the vanishing of the Lagrange bilinear form $\{y,z\}$ at the point. We define this form by the identity (proven by two integration by parts)

$$\int_s^t w M_w[y]\bar{z} - \int_t^s wy\overline{M_w[z]} = \{y,z\}(t) - \{y,z\}(s),$$

where $t, s \in I$ and $\{y,z\}(t) := (yp\bar{z}' - py'\bar{z})(t)$. That M_w is limit-point at b is equivalent to the property

$$\lim_{t\to b}\{y,z\}(t) := 0$$

[2] Any local property will be labeled with the subscript "loc"; thus $L^2_{\mathrm{loc}}(\Omega)$ will denote the the locally square integrable functions on Ω.

for all $y, z \in \mathcal{D}$. A more restrictive condition at b which implies LP is the "strong limit-point" (SLP) property which means that

$$\lim_{t \to b} (yp\bar{z}')(t) = 0$$

for all $y, z \in \mathcal{D}$. That in our setting M_w must be either limit-point or limit-circle is called the Weyl alternative after the inventor of these concepts.[3] The SLP property has been extensively studied by Everitt; see e.g. [10–11] and [17]. For LP criteria see Read [26] and Kauffman, Read, and Zettl [22].

If M_w is limit-point at the singular end-points one can show that separation on $\mathcal{D}_0$ implies separation on $\mathcal{D}$. Further if L is separated then M_w is SLP at the singular endpoints. Proofs of these statements may be found in [3, Proposition 2].

A version of minimal conditions that applies to the expression $-\operatorname{div}(P\nabla y)+qy$ has been given by E. B. Davies using quadratic form methods in the book [5]. But most results of interest to us have been proven using some variant of the basic conditions give above. In particular appropriate smoothness[4] is required for P and it is assumed that $q \in L^2_{\text{loc}}(\Omega)$. Under such hypotheses $\mathcal{D}_0' \supseteq C_0^\infty(\Omega)$, $L_0'^* = L$, and $L^* = L_0 = \overline{L_0'}$, where L as in the ordinary case is defined by $M_{w,n}$ on

$$\mathcal{D} := \{u \in L^2(w;\Omega) \colon\ M_{w,n}[y] \in L^2(w;\Omega)\},$$

where the differentiation in $M_{w,n}$ is interpreted in the distributional sense. For the details of this development see [5] or [7]. We remark however that for consistency in the discussion of operators determined by M_w and $M_{w,n}$ we shall call L_0 the "minimal operator", while most other writers use this term to denote L_0' in the partial case. When $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+ := \mathbb{R}^n \backslash \{0\}$, $n \geqslant 2$, the idea which replaces the LP condition is the concept that L_0' is "essentially self-adjoint". This means that $L_0 \equiv \overline{L_0'} = L$. Thus since $L^* = L_0$, L is self-adjoint. Equivalently L_0 has a unique self-adjoint extension; for if T is any self-adjoint extension of L_0, then

$$T = T^* \subseteq L_0^* = L = L_0 \subseteq T.$$

Many sufficient conditions have been given for essential self-adjointness. For instance, Simon [27] showed that the basic Schrödinger operator $-\Delta y + qy$ is essentially self-adjoint if $q = q_1 + q_2$, where $0 \leqslant q_1 \in L^2(\mathbb{R}^n)$ and $q_2 \in L^\infty$. Successively more

[3] Likewise the nomenclature "limit-point" or "limit-circle" is due to Weyl and results from his technique which associates these cases with nested families of circles in the complex plane which converge respectively either to a point or a circle. See e.g. Coddington and Levinson [4, Chapter 9] for an account of Weyl's method.

[4] One can usually get by with $P \in C^{1+\alpha}(\Omega)$ for some $\alpha > 0$ rather than our assumption that $P \in C^2(\Omega)$.

powerful extensions of this result were given by Kato [21], Eastham, Evans, and McLeod [7], and Evans [8]. Since these results are rather complicated and are peripheral to our main interest we will not state them here. Some of these papers allow considerable oscillation of q at ∞, but not potentials which are strongly singular at 0. This gap was covered by Kalf [19] and Kalf et al. [20] who showed that $-\Delta y + qy$ is essentially self-adjoint on $\mathbb{R}^n_+$ if q satisfies a local Stummel condition and

$$q \geqslant (1 - [(n-2)/2]^2)|x|^{-2} - \gamma|x|^2,$$

with $\gamma \geqslant 0$. Essential self-adjointness criteria for L_0' on a subdomain $\Omega \subset \mathbb{R}^n$ can be found in Jörgens [18].

Our purpose in this paper is to improve the following two separation results obtained in [3] in the ordinary setting.

Theorem A. *Suppose $p^{-1} \in L_{\text{loc}}(I)$, w is a positive function in $L_{\text{loc}}(I)$, $pq \geqslant 0$, and $q \in AC_{\text{loc}}(I)$, where $I = [a, b)$, $-\infty < a < b \leqslant \infty$. Then the separation inequality* (1.4) *holds for all $y \in \mathcal{D}_0$ with certain constants $A, C < 1$, $B < 2$, $K = 1$ and $L = 0$ under the condition*

$$(|\mathrm{S}_1^*|) \qquad \limsup_{t \to b} \left| w p^{1/2} (w^{-1} q)' / q^{3/2} \right| = \theta < 2.$$

Theorem B. *Suppose p and w satisfy the minimal conditions stated above on $I = [a, \infty)$ and additionally that $pq \geqslant 0$, and q, p are differentiable on I, Then the separation inequality* (1.4) *holds on $\mathcal{D}_0$ with certain constants $A, C < 1$, $B < 2$, $K = 1$, and $L = 0$ if*

$$(|\mathrm{S}_1|) \qquad \limsup_{t \to \infty} \left| w (p (w^{-1} q)')' / q^2 \right| = \theta.$$

for some $0 \leqslant \theta < 2$.

Our proof of Theorem A closely followed an argument due to Everitt and Giertz who considered the case $w = p = 1$. Theorem B on the other hand appears to be new. It was motivated by a claim of Dunford and Schwartz who in [6, Chapter XIII, 9.B5, p. 1541] state without giving a proof or reference that M_w is separated on $\mathcal{D}$ when $I = [0, \infty)$ if

$$\limsup_{t \to \infty} |(pq')'|q^2 < 1.$$

As noted by Everitt and Giertz in 1974 [14] this condition may be a misprint since $p(x) = 1$ and $q(x) = -x$ for $x \in [0, \infty)$ satisfies the condition and yet as is shown by them in [12] separation does not occur. Our version is in a weighted setting and proves (but on $\mathcal{D}_0$ only) a result that may have been intended.

Our extensions of the above theorems are given in Sections 3 and 4. In Theorem 1 of Section 3 we prove a version of Theorem B in the ordinary case which replaces ($|\mathrm{S}_1|$) by the condition (S_1) which differs from the previous condition in omitting the absolute value sign. This allows more freedom in the choice of p, q and w. Such a result parallels a version of Theorem A proven by Atkinson in [1] which allows some negativity in $|\mathrm{S}_1^*|$. Here it was shown that if $w = p = 1$ then separation occurs on $\mathcal{D}$ if

$$-4/\sqrt{15} < q'/q^{3/2} < 4/\sqrt{15}.$$

Further we allow a and/or b to be singular or finite and (with some additional tightening of the assumptions on p, q and w) pq to be nonpositive. Examples of Theorem 1 will include limit-circle cases satisfying a separation inequality on $\mathcal{D}_0$ but not on $\mathcal{D}$ and which additionally do not satisfy the Everitt and Giertz-type criterion of Theorem A. In Section 4 we turn to the multidimensional case and prove separation theorems for weighted Schrödinger-type operators. The first result (Theorem 2) extends Theorem A to this setting. The argument is similar to that given by Everitt and Giertz [16], but the class of operators we consider is wider. Our separation criterion is also of the same general type as that given by Evans and Zettl [9] but because we work on $\mathcal{D}_0$ we do not require essential self-adjointness at the outset and so our assumptions are less complicated and we permit strongly singular potentials such as those considered in [19–20]. Theorem 3 is an $\mathbb{R}^n$ extension of the the simplest part of Theorem 1. A Corollary will imply that the minimal operator corresponding to $-\Delta y + qy$ is separated if $\Delta q \leqslant 0$, in other words if q is superharmonic (i.e., $-\Delta q \geqslant 0$, where Δ signifies the Laplacean). The paper ends with an example showing that in Theorems 1–3 the conditions $\theta \leqslant 2$ or $\theta < 2$ are necessary for separation on $\mathcal{D}$ in all dimensions.

3. A separation result for second order symmetric ordinary differential operators

Let λ denote a real parameter. We call λ *admissible* if $\lambda \geqslant 1$ and for some $\delta \in (-\infty, 1)$, $2\delta - \delta^2/\lambda > \theta$, where θ is defined by (S_1). Also set $Q_\lambda := 2\lambda pqw - p(p'w^{-1})'$, and define

$$\{Q_\lambda\}_-(x) = \begin{cases} |Q_\lambda(x)|, & \text{if } Q_\lambda(x) < 0 \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

We consider the following conditions on p, q and w which may hold for an admissible λ on $I_s = [s, b)$ or $I_s = (a, s]$ for s sufficiently close to a singular point $c = a$ or b.

(C0) $pq \geqslant 0$.

(C1) $Q_\lambda \geqslant 0$.

(C2) $\sup\limits_{t\in I_s}\left(\int_t^s \{Q_\lambda\}_-\,\mathrm{d}x\right)\left(\int_a^t wp^{-2}\,\mathrm{d}x\right) \leqslant \frac14$ or

$\sup\limits_{t\in I_s}\left(\int_s^t \{Q_\lambda\}_-\,\mathrm{d}x\right)\left(\int_t^b wp^{-2}\,\mathrm{d}x\right) \leqslant \frac14$.

(C3) $\sup\limits_{t\in I_s}\left(\int_a^t \{Q_\lambda\}_-\,\mathrm{d}x\right)\left(\int_t^s wp^{-2}\,\mathrm{d}x\right) \leqslant \frac14$ or

$\sup\limits_{t\in I_s}\left(\int_t^b \{Q_\lambda\}_-\,\mathrm{d}x\right)\left(\int_s^t wp^{-2}\,\mathrm{d}x\right) \leqslant \frac14$.

(C4) There exists a positive continuous function f such that for $\varepsilon > 0$

$$\sup_{t\in I_s} f(t)^2\left([\varepsilon f(t)]^{-1}\int_t^{t+\varepsilon f(t)}\{Q_\lambda\}_-\,\mathrm{d}x\right)\left([\varepsilon f(t)]^{-1}\int_t^{t+\varepsilon f(t)} wp^{-2}\,\mathrm{d}x\right) < \infty,$$

$$\limsup_{t\to c} f(t)^{-2}\left([\varepsilon f(t)]^{-1}\int_t^{t+\varepsilon f(t)}\{Q_\lambda\}_-\,\mathrm{d}x\right)\left([\varepsilon f(t)]^{-1}\int_t^{t+\varepsilon f(t)} wq^{-2}\,\mathrm{d}x\right) = 0.$$

(C5) $q \geqslant 0$ and $-Q_\lambda \leqslant E(\lambda)p < \infty$, where $E(\lambda)$ is a positive constant depending on λ.

Given these conditions we can state:

Theorem 1. *Suppose p, q and w are twice differentiable on I. Then $M_w[y]$ on $\mathcal{D}_0$ is separated and satisfies an inequality of the form* (1.4) *with $A = C > 0$, and $B = 0$ under one of* (C0)–(C5) *provided also that* (S_1) *holds.*

P r o o f. We begin by choosing s large enough as needed so that the conditions (C0)–(C5) hold, and so that in (S_1)

$$\frac{w(p(w^{-1}q)')'(t)}{q(t)^2} \leqslant \frac{\lambda^2-(\lambda-\delta)^2}{\lambda} \leqslant 2\delta - \frac{\delta^2}{\lambda} < 2 - \frac{\delta^2}{\lambda} \tag{3.2}$$

for a convenient admissible λ.

Let $M_{w,\lambda}[y]$ be given by the expression $w^{-1}[-(py')'+\lambda qy]$. We define the maximal and minimal operators L and L_0 corresponding to $M_{w,\lambda}$ as above, but on I_s. Let $C_0^\infty(I_s)$ denote the infinitely differentiable functions with compact support on I_s. Then $C_0^\infty(I_s) \subset \mathcal{D}_0'$ relative to I_s. Suppose $y \in C_0^\infty(I_s)$ and and $\lambda > 1$. Repeated integrations by parts and evaluation of $M^2_{w,\lambda}$ show that

$$\begin{aligned}(3.3)\qquad \|M_{w,\lambda}[y]\|^2_{w,I_s} &= \int_{I_s} wM^2_{w,\lambda}[y]y\,\mathrm{d}x\\ &= \|w^{-1}(py')'\|^2_{w,I_s} + \int_{I_s}\Big[2\lambda pqw^{-1}|y'|^2\\ &\quad + (\lambda q)^2 w^{-1}\Big(1-\frac{w(p(w^{-1}q')')'}{\lambda q^2}\Big)|y|^2\Big]\,\mathrm{d}x.\end{aligned}$$

Alternatively,

$$\begin{aligned}(3.4)\qquad \|M_{w,\lambda}[y]\|^2_{w,I_s} &= \int_{I_s}\Big\{(w^{-1}p^2y'')'' - (2\lambda pqw^{-1} - p(p'w^{-1})')y'\\ &\quad + ((\lambda q)^2w^{-1} - \lambda(p(w^{-1}q)')')y\Big\}\bar{y}\,\mathrm{d}x\\ &= \int_{I_s}\Big\{w^{-1}p^2|y''|^2 + (2\lambda pqw^{-1} - p(p'w^{-1})')|y'|^2\\ &\quad + ((\lambda q)^2w^{-1} - (\lambda p(w^{-1}q)')')|y|^2\Big\}\,\mathrm{d}x\\ &\geqslant \int_{I_s}\Big\{(2\lambda pqw^{-1} - p(p'w^{-1})')|y'|^2\\ &\quad + (\lambda q)^2w^{-1}\Big(1-\frac{w(p(w^{-1}q')')'}{\lambda q^2}\Big)|y|^2\Big\}\,\mathrm{d}x.\end{aligned}$$

It then follows from (3.2) together with (3.3) and (C0) or (3.1), (3.4), and (C1) that

$$(3.5)\qquad \|M_{w,\lambda}[y]\|^2_{w,I_s} \geqslant (\lambda-\delta)^2\|w^{-1}qy\|^2_{w,I_s}.$$

However, it is also true that

$$(3.6)\qquad \|M_{w,\lambda}[y]\|_{w,I_s} \leqslant \|M_w[y\|_{w,I} + (\lambda-1)\|w^{-1}qy\|_{w,I_s}.$$

And therefore

$$\|M_w[y]\|_{w,I_s} \geqslant (1-\delta)\|w^{-1}qy\|_{w,I_s}.$$

If the conditions (C2) or (C3) are satisfied instead of (C1), it follows from [25, Theorems 1.14 and 6.2] that there is the Hardy-type inequality

$$\int_{I_s}\{Q_\lambda\}_-|y'|^2\,\mathrm{d}x \leqslant C\int_{I_s} w^{-1}p^2|y''|^2\,\mathrm{d}x,$$

where $C<1$. This together with (3.4) yields that

$$\|M_{w,\lambda}[y]\|^2_{w,I_s} \geqslant (1-C)\int_{I_s}\{w^{-1}p^2|y''|^2 + [(\lambda^2)w^{-1}q^2 - (\lambda p(w^{-1}q)')']|y|^2\}\,\mathrm{d}x$$

and the proof is completed as before. □

If (C4) is satisfied, it follows from [2, Theorem 2.1] that there is a sum inequality of the form

$$\left\| \sqrt{\{Q_\lambda\}_-}\, y' \right\|_{I_s}^2 \leqslant \varepsilon \left\{ \|w^{-1}qy\|_{w,I_s}^2 + \|w^{-1}py''\|_{w,I_s}^2 \right\}.$$

Again, using (3.4) gives the inequality

$$\|M_{w,\lambda}[y]\|_{w,I_s}^2 \geqslant (1-\varepsilon)\int_{I_s} \left\{ w^{-1}p^2|y''|^2 + [(\lambda^2-\varepsilon)w^{-1}q^2 - (\lambda p(w^{-1}q)')']|y|^2 \right\} \mathrm{d}x.$$

With large enough λ and small enough ε we obtain that

$$\begin{aligned} \|M_{w,\lambda}[y]\|_{w,I} &\geqslant \left[\sqrt{(\lambda-\delta)^2-\varepsilon}\,\right] \|w^{-1}qy\|_{w,I_s} \\ &> \left[(\lambda-\delta)-\sqrt{\varepsilon}\,\right] \|w^{-1}qy\|_{w,I_s}, \end{aligned}$$

which combined with (3.6) gives that

$$\|M_w[y]\|_{w,I_s} \geqslant \left[(1-\delta)-\sqrt{\varepsilon}\,\right] \|w^{-1}qy\|_{w,I_s}$$

with $[(1-\delta)-\sqrt{\varepsilon}\,] > 0$.

Finally, under (C5) we rearrange (3.4) so that

$$\|M_{w,\lambda}[y]\|_{w,I_s}^2 + E(\lambda)\int_{I_s} p|y'|^2\,\mathrm{d}x \geqslant \int_{I_s} (\lambda q)^2 w^{-1}\Big(1 - \frac{w(p(w^{-1}q')'}{\lambda q^2}\Big)|y|^2\,\mathrm{d}x.$$

Combining this with the inequalities

$$\int_{I_s} p|y'|^2\,\mathrm{d}x \leqslant [M_{w,\lambda}[y],y]_{w,I_s} \leqslant (\tfrac{1}{2}\varepsilon)\|M_{w,\lambda}[y]\|_{I_s}^2 + (\tfrac{1}{2\varepsilon})\|y\|_{w,I_s}^2$$

(the last of which being a consequence of Cauchy-Schwartz and the arithmetic-geometric mean inequality) gives that

$$\begin{aligned} (1+\tfrac{1}{2}E(\lambda)\varepsilon)\|M_{w,\lambda}[y]\|_{w,I_s}^2 &+ \tfrac{E(\lambda)}{2\varepsilon}\|y\|_{w,I_s} \\ &\geqslant \int_{I_s} (\lambda q)^2 w^{-1}\Big(1 - \frac{w(p(w^{-1}q')'}{\lambda q^2}\Big)|y|^2\,\mathrm{d}x \end{aligned}$$

and the proof is repeated as before.

Thus under any of these assumptions we have obtained a separation inequality for C_0^∞ functions on I_s. Now let L_0'' denote the restriction of L_0' to $C_0^\infty(I_s)$. We sketch a standard argument showing that that $\overline{L_0''} = L_0$. It is clear that $L \subseteq L_0''^*$. If we can show that $L_0''^* \subseteq L$, it will follow that $L^* = \overline{L_0''^*} = L_0$. Suppose (α,β) belongs to the graph of $L_0''^*$ so that $[L_0''y, \alpha]_{w,I_s} = [y, \beta]_{w,I_s}$. Making use of the differentiability of p we write $-(py')' = -p'y' - py''$. Integration by parts then gives $[y'', z]_{w,I_s} = 0$, where

$$z = \int_a^t p'\alpha \,\mathrm{d}s + \int_a^t (t-s)(q\alpha - \beta)\,\mathrm{d}s - p\alpha.$$

The Fundamental Lemma of the calculus of variations implies that z is a linear function. Since z' is absolutely continuous, two differentiations show that $\alpha \in \mathcal{D}$ and $\beta = L(\alpha)$. Thus $L_0''^* = L$. Since $L^* = \overline{L_0''^*} = L_0$, we can approximate $y \in \mathcal{D}_0$ and $M_{w,\lambda}[y]$ by sequences $\{y_n\}$, $M_{w,\lambda}[y_n]$, where the $y_n \in C_0^\infty(I_s)$. From this it will follow (cf. [9, p. 313] or [3, Lemma 1]) that the inequality is true on $\mathcal{D}_0$ defined relative to I_s.

Next we want to extend these results to I. To this end, define a pair of smooth compact support functions φ_1, φ_2 on $[s, b)$ or $(a, s]$ such that $\varphi_1(s) = 1$, $\varphi_1'(s) = 0$ and $\varphi_2(s) = 0$, $\varphi_2'(s) = 1$. Then for a given y in $\mathcal{D}_0$ (on I), the function $\tilde{y} = y\chi_{I_s} - \psi$, where $\psi = y(s)\varphi_1 + y'(s)\varphi_2$ is in $\mathcal{D}_0$ on I_s. By the previous reasoning there is an inequality of the form

$$\|w^{-1}q\tilde{y}\|_{w,I_s} \leqslant K\|M_w[\tilde{y}]\|_{w,I_s}.$$

However this together with the triangle inequality implies that

$$\|w^{-1}qy\|_{w,I_s} \leqslant K\{\|M_w[y]\|_{w,I_s} + \|M_w[\psi]\|_{w,I_s}\} + \|w^{-1}q\psi\|_{w,I_s}.$$

Since ψ has compact support the last two norms are finite, so that $\|w^{-1}qy\|_{w,I_s} < \infty$. As we pointed out above this fact and a closed graph argument gives the inequality for $\mathcal{D}_0$ (on I_s)

$$\begin{aligned} \|w^{-1}qy\|_{w,I_s} &\leqslant K\{\|M_w[y]\|_{w,I_s} + \|y\|_{w,I_s}\} \\ &\leqslant K\{\|M_w[y]\|_{w,I} + \|y\|_{w,I}\}. \end{aligned} \tag{3.7}$$

However, since the Green's function $G(t, s)$ of M_w is evidently bounded on $[a, s]\times[a, s]$ if a is regular or on $[s, b] \times [s, b]$ if b is regular we can obtain an inequality of the form

$$\|y\|_{w,[a,s]} \leqslant K_1\|M_w[y]\|_{w,[a,s]} \quad \text{or} \quad \|y\|_{w,[s,b]} \leqslant K_1\|M_w[y]\|_{w,[s,b]}$$

for all $y \in \mathcal{D}$ such that $y(a) = y'(a) = 0$ or $y(b) = y'(b) = 0$. Since q, w^{-1} are also bounded on $[a, s]$ it follows that

$$\|w^{-1}qy\|_{w,[a,s]} \leqslant K_1K_2\|M_w[y]\|_{w,[a,s]} \leqslant K_1K_2\|M_w[y]\|_{w,I}, \tag{3.8}$$

where K_2 is a bound on $w^{-1}q$. (3.7), (3.8) together followed by application of the triangle inequality gives that

$$\|w^{-1}(py')'\|_{w,I} \leqslant (K_1K_2 + K)\|M_w[y]\|_{w,I} + K\|y\|_{w,I}.$$

R e m a r k 1 . The hypotheses (C1)–(C4) of Theorem 1 can viewed as examples of conditions which guarantee either that the spectrum of a certain minimal operator is nonnegative or that a certain quadratic form is nonnegative. Let $\widetilde{M}_{w,\lambda}[y] := w^{-1}[-(Py')' + Q_\lambda y]$, where $P = w^{-1}p^2$. Assume that P and Q_λ satisfy minimal conditions and let $\widetilde{L}_{0,\lambda,s}$ signify the minimal operator determined by $\widetilde{M}$ on I_s. We also define the quadratic form $\Phi_{\lambda,s}$ by

$$\Phi_{\lambda,s}(z) = \int_{I_s} \left[P|z'|^2 + Q_\lambda |z|^2\right] \,\mathrm{d}x.$$

We then consider the conditions

(C6) For sufficiently large λ, s $\widetilde{L}_{0,\lambda,s}$ has nonnegative continuous spectrum.

(C7) If $z = y'$, where $y \in C_0^\infty(I_s)$ then $\Phi_{\lambda,s}(z) \geqslant 0$.

It is well known that (C6) $\Longrightarrow$ (C7).

Corollary 1. *Let p, q, and w satisfy the hypotheses of Theorem* 1. *Then M_w is separated and the inequality of Theorem* 1 *holds under* (C6) *or* (C7) *provided* (S_1) *is satisfied. In* (C6) *P and Q_λ need not satisfy minimal conditions.*

P r o o f. We repeat the proof of Theorem 1 noting that (C6) and (C7) can replace (C1)–(C4) in that they guarantee that

$$\int_{I_s} \left[w^{-1}p^2|y''|^2 + \left(2\lambda pqw^{-1} - p(p'w^{-1})'\right)|y'|^2\right] \mathrm{d}x \geqslant 0,$$

if $y' \in C_0^\infty(I_s)$. □

Corollary 2. *If $I = [a, \infty)$, $w = 1$, and $pq \geqslant 0$ then M is separated on $\mathcal{D}_0$ if $(pq')' \leqslant 0$. If $p > 0$ and q is bounded below then M is also separated on $\mathcal{D}$.*

P r o o f. That M is separated on $\mathcal{D}_0$ is immediate from Theorem 1 using (C0). That M is limit-point if $p > 0$ and q is bounded below is well known (see e.g. [6, XIII.6.14, p. 1405]; consequently M is separated on $\mathcal{D}$. □

E x a m p l e s . In all the cases that follow $w^{-1}q$ is unbounded since otherwise separation holds trivially.

1. Let $p(t) = t^\alpha$, $w(t) = t^\delta$, $q(t) = Ct^\beta$, and $I = [a, \infty)$, $a > 0$, where C is a positive constant. Then (C0) is satisfied for all $\lambda > 0$ and (S_1) holds if $(\alpha-\delta+\beta-1)(\beta-\delta) \leqslant 0$, $\beta > \alpha - 2$, or $\beta = \alpha - 2$ and $(2\alpha - \delta - 3)(\alpha - 2 - \delta) < 2C$. Thus if $p(t) = t^\alpha$ and $\alpha \leqslant 2$ we can let $q(t) = t^\beta$ for $\beta > 0$. In both cases the operator is limit-point at ∞ so that separation will also hold on $\mathcal{D}$.

2. Let I, $p(t)$, w, and C be as above, but take $q(t) = -Ct^{\beta}$. (C1) holds if $\alpha(\alpha - \delta - 1) < 0$ and $\beta < \alpha - 2$. (S_1) holds if $(\alpha - \delta + \beta - 1)(\beta - \delta) \geqslant 0$. We note that in the unweighted case we cannot obtain from (C1) any nontrivial example of separation. For $\delta = 0$ implies that $\alpha \in (0, 1)$ and therefore $\beta < -1$ so that q is bounded.

3. Let $I = [0, \infty)$, $p(t) = \mathrm{e}^{\alpha t}$, $w(t) = \mathrm{e}^{\delta t}$, and $q(t) = C\mathrm{e}^{\beta t}$, where $C > 0$. (C0) of Theorem 1 holds and (S_1) is satisfied if $(\beta - \delta)(\beta + \alpha - \delta) > 0$ and $\beta > \alpha$, or $(\beta - \delta)(\beta + \alpha - \delta) \leqslant 0$, or $0 < (\alpha - \delta)(2\alpha - \delta) < 2$ if $\beta = \alpha$.

4. Let everything be as in Example 3 but take $q(t) = -C\mathrm{e}^{\beta t}$. For (C1) to be satisfied we need that $0 < \alpha < \delta$ and $\beta < \alpha$. (2.1) implies that $(\beta - \delta)(\beta + \alpha - \delta) < 0$ and $\beta > \alpha$, or $(\beta - \delta)(\beta + \alpha - \delta) \geqslant 0$, or $0 > (\alpha - \delta)(2\alpha - \delta) > -2$ if $\beta = \alpha$.

5. If $w = 1$, $p = (q')^{-1}$, $q', q \geqslant 0$, and $I = [a, \infty)$ separation on $\mathcal{D}_0$ is a consequence of Theorem A. Under the same assumptions on w and q, if $p = (q')^{-r}$ for $r > 1$, and $q'' > 0$ then (C0) and (S_1) hold so there is separation at least on $\mathcal{D}_0$.

6. If $w = p = 1$, $q = -t^{-2}/8$, and $I = (0, \infty)$ we find that

$$\frac{q''}{q^2} = -48.$$

Consequently $\lambda = 1$ is admissible if $\delta > -6$. A calculation shows that the second condition of (C3) applies with $s = 0$. Equivalently, the classical Hardy inequality yields that

$$2\int_I \{q\}_- |y'|^2 \,\mathrm{d}x \leqslant \int_I |y''|^2 \,\mathrm{d}x$$

so that (C7) holds. We conclude that separation occurs on $\mathcal{D}_0$ and by (3.5)–(3.6) there is the inequality

$$\int_I t^{-2} |y|^2 \,\mathrm{d}x \leqslant \frac{64}{49} \int_I \left| y'' + (\tfrac{1}{8}t^{-2})y \right|^2 \,\mathrm{d}x.$$

The solutions of $M[y] = 0$ are of the form $y = t^{\alpha}$, where $\alpha = 1/2 \pm \sqrt{2}/4$. Both solutions are square integrable near 0 so that M is limit-circle at 0. Therefore we have an example of separation holding on $\mathcal{D}_0$ but not on $\mathcal{D}$. Note also that since

$$\left| \frac{q'}{q^{3/2}} \right| = 4\sqrt{2},$$

Theorem A does not apply.

7. Let $I = (0, 1]$, $p = -ct^{1/2}$, $w = 1$, $q = \frac{1}{8}ct^{-3/2} - \frac{1}{2}$, where $c > 0$ is a constant. A calculation with $\lambda = 1$ shows that (C5) is satisfied and that (S_1) holds because

$(pq')' = -\frac{3}{8}c^2t^{-3} < 0$. This example does not satisfy a version of $|S_1^*|$ formulated for the singular point 0 since θ is found to be $8^{3/2}(\frac{3}{16})^{2/3} \approx 7.413$. Moreover M is limit-circle at 0 since it is a perturbation of an Euler operator with two L^2 integrable solutions at 0.

4. Partial differential operators

We write

$$T(y) := \sum_{i,j=1}^{n} D_i(p_{ij}(x)D_j y) \equiv \operatorname{div}(P\nabla y)$$

so that $M_{w,n}[y] = w^{-1}[-T(y)+qy]$. Our goal will be to prove separation inequalities on $\mathcal{D}_0' \equiv C_0^\infty(\Omega)$ of the form (1.5) by generalizing Theorem A and Theorem 1. Since $L^* = L_0 \equiv \overline{L_0'}$ a closure argument like that given in [16, Lemma 2] will show that the inequality holds on $\mathcal{D}_0$. Finally, if L_0' is essentially self-adjoint (so that $L_0 = L = L^*$) the inequality will hold on $\mathcal{D}$. We note, however, that separation is a stronger property than essential self-adjointness. Let $T_{w,0}$ and T_w respectively denote the minimal and maximal operators on a domain Ω determined by $w^{-1}T$.

Lemma 1. *Suppose $T_{w,0}'$ is essentially self-adjoint and that L is separated. Then L_0 is essentially self-adjoint.*

Proof. We need show only that L is self-adjoint. Let $(u, v) \in \operatorname{Graph}(L^*) = \operatorname{Graph}(L_0)$. Then $[Ly, u]_{w,\Omega} = [y, v]_{w,\Omega}$. Since L is separated, the Cauchy-Schwartz inequality implies that $[w^{-1}T(y), u]_{w,\Omega}$ and $[w^{-1}qy, u]_{w,\Omega}$ are finite. Hence by the essential self-adjointness of $T_{w,0}'$ and self-adjointness of multiplication operators

$$[w^{-1}T(y), u]_{w,\Omega} = [y, w^{-1}T(u)]_{w,\Omega} \quad \text{and} \quad [w^{-1}qy, u]_{w,\Omega} = [y, w^{-1}qu]_{w,\Omega}.$$

It follows that

$$[Ly, u]_{w,\Omega} = [y, Lu]_{w,\Omega} = [y, v]_{w,\Omega},$$

and so since $\mathcal{D}$ is dense $v = Lu$. □

Theorem 2. *Under condition ($|S_n^*|$) $M_{w,n}$ satisfies the separation inequality (1.5) on $\mathcal{D}_0$ with certain coefficients $A > 1, C < 1$, $B < 2$, and $L = 0$.*

Proof. Without loss of generality we can as in [16] and by the remarks at the beginning of this section give the proof only for real functions in $C_0^\infty(\Omega)$. Let

$y \in C_0^\infty(\Omega)$. We begin with the identity

$$\int_\Omega \left\{wM_{n,w}^2[y] + \gamma(wM_{n,w}[y])(w^{-1}T[y])\right\} \,\mathrm{d}x = \int_\Omega \left\{w^{-1}(1-\gamma)T[y]^2 + w^{-1}(\gamma-2)T[y]qy + w^{-1}q^2y^2\right\} \,\mathrm{d}x, \tag{4.1}$$

where $\gamma \in (0,1)$. Application of the arithmetic-geometric mean inequality to the the term $\gamma(wM_{n,w})(w^{-1}T[y])$ in (4.1) gives for $\delta > 0$ the estimate

$$\left|\int_\Omega (wM_{n,w}[y])(w^{-1}T[y])\,\mathrm{d}x\right| \leqslant \tfrac{1}{2}\left\{\delta\|M_{n,w}[y]\|_{w,\Omega}^2 + \delta^{-1}\|w^{-1}T[y]\|_{w,\Omega}^2\right\}. \tag{4.2}$$

Next integration by parts, the condition ($|\mathrm{S}_n^*|$), and the arithmetic-geometric mean inequality applied to $w^{-1}T[y]qy$ yields successively that

$$\begin{aligned}
\int_\Omega w^{-1}T[y]qy\,\mathrm{d}x &= \int_\Omega \sum_{i,j=1}^n D_i(p_{ij}(x)D_jy)(w^{-1}q)y\,\mathrm{d}x \\
&= -\int_\Omega [P(x)\nabla y, \nabla(w^{-1}q)]_n y\,\mathrm{d}x - \int_\Omega w^{-1}[P(x)\nabla y, \nabla y]_n q\,\mathrm{d}x \\
&\leqslant \int_\Omega \left|[P(x)\nabla y, \nabla(w^{-1}q)]_n\right| |y|\,\mathrm{d}x - \int_\Omega w^{-1}\,|[P(x)\nabla y, \nabla y]_n q| \,\mathrm{d}x \\
&\leqslant \int_\Omega \|P(x)^{1/2}\nabla y\|_n \|P(x)^{1/2}\nabla(w^{-1}q)\|_n\,|y|\,\mathrm{d}x \\
&\qquad - \int_\Omega w^{-1}\,|[P(x)\nabla y, \nabla y]_n q| \,\mathrm{d}x \\
&\leqslant \theta\int_\Omega \|P(x)^{1/2}\nabla y\|_n w(x)^{-1}q(x)^{3/2}|y|\,\mathrm{d}x \\
&\qquad - \int_\Omega w^{-1}\,|[P(x)\nabla y, \nabla y]_n q| \,\mathrm{d}x \qquad (\text{by } (|\mathrm{S}_n^*|)) \\
&\leqslant \theta\left(\int_\Omega \|P(x)^{1/2}\nabla y\|_n w(x)^{-1}q(x)\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega w^{-1}q(x)^2y^2\,\mathrm{d}x\right)^{1/2} \\
&\qquad - \int_\Omega w^{-1}\,|[P(x)\nabla y, \nabla y]_n q| \,\mathrm{d}x \\
&\leqslant \tfrac{1}{2}\theta\left[\int_\Omega \|P(x)^{1/2}\nabla y\|_n w(x)^{-1}q(x)\,\mathrm{d}x + \int_\Omega w^{-1}q(x)^2y^2\,\mathrm{d}x\right] \\
&\qquad - \int_\Omega w^{-1}\,|[P(x)\nabla y, \nabla y]_n q| \,\mathrm{d}x.
\end{aligned} \tag{4.3}$$

We now substitute (4.2) and (4.3) into (4.1) to obtain

$$\begin{aligned}(1+\gamma\delta/2)\|M_{n,w}[y]\|^2_{w,\Omega} &\geqslant (1-\gamma-\tfrac{\gamma}{2\delta})\|w^{-1}T[y]\|^2_{w,\Omega}\\ &\quad + (2-\gamma)\bigl\{\|w^{-1}([P\nabla y,\nabla y]_n q)^{1/2}\|^2_{w,\Omega}\\ &\quad - \theta\|w^{-1}([P\nabla y,\nabla y]_n q)^{1/2}\|_{w,\Omega}\,\|w^{-1}qy\|_{w,\Omega}\bigr\}\\ &\quad + \|w^{-1}qy\|^2_{w,\Omega}\\ &\geqslant (1-\gamma-\tfrac{\gamma}{2\delta})\|w^{-1}T[y]\|^2_{w,\Omega}\\ &\quad + (2-\gamma)(1-\tfrac12\theta)\|w^{-1}[P\nabla y,\nabla y]_n^{1/2}q\|^2_{w,\Omega}\\ &\quad + [(1-(2-\gamma)(\tfrac12\theta)]\|w^{-1}qy\|^2_{w,\Omega}.\end{aligned}$$

This is the inequality (1.5) if we choose $\gamma<1$ such that

$$(2-\gamma)(\tfrac12\theta)<1 \Leftrightarrow \gamma>2-\tfrac{2}{\theta}$$

and δ large enough that $(1-\gamma-\frac{\gamma}{2\delta})>0$. □

Theorem 3. *Under condition* (S_n) *and if* $q\geqslant 0$, *then* $M_{w,n}$ *satisfies the separation inequality* (1.5) *on* $\mathcal{D}_0$ *with* $A=C=K=1$ *and* $B,L=0$.

Proof. Let $y\in C_0^\infty(\Omega)$ and set $M_{w,n,\lambda}:=w^{-1}[-T(y)+\lambda qy]$. By a direct computation

$$\begin{aligned}[M^2_{w,n,\lambda}[y],y]_{w,\Omega} &= \int_\Omega \{-T(w^{-1}[-T(y)+\lambda qy])+\lambda q w^{-1}[-T(y)+\lambda qy]\}\bar y\,\mathrm{d}x\\ &= \|w^{-1}T(y)\|^2_{w,\Omega} - \int_\Omega \{T(w^{-1}\lambda qy)\bar y + w^{-1}\lambda qT(y)\bar y\}\,\mathrm{d}x\\ &\quad + \int_\Omega w^{-1}(\lambda q)^2|y|^2\,\mathrm{d}x\\ &\geqslant -2\mathrm{Re}\Bigl(\int_\Omega \mathrm{div}(P\nabla y w^{-1}\lambda q)\bar y\,\mathrm{d}x\Bigr) + \int_\Omega w^{-1}(\lambda q)^2|y|^2\,\mathrm{d}x\\ &= 2\mathrm{Re}\Bigl(\int_\Omega P\nabla y\cdot\nabla(w^{-1}\lambda q\bar y)\,\mathrm{d}x\Bigr) + \int_\Omega w^{-1}(\lambda q)^2|y|^2\,\mathrm{d}x\\ &= 2\mathrm{Re}\Bigl(\int_\Omega \{[P\nabla y,\nabla y]_n w^{-1}\lambda q + [P\nabla y,\nabla(w^{-1}\lambda q)]_n\bar y\}\,\mathrm{d}x\Bigr)\\ &\quad + \int_\Omega w^{-1}(\lambda q)^2|y|^2\,\mathrm{d}x\\ &= 2\mathrm{Re}\Bigl(\int_\Omega [P\nabla y,\nabla y]_n w^{-1}\lambda q\,\mathrm{d}x\Bigr)\\ &\quad + 2\mathrm{Re}\Bigl(\int_\Omega [P\nabla y,\nabla(w^{-1}\lambda q)]_n\bar y\,\mathrm{d}x\Bigr) + \int_\Omega w^{-1}(\lambda q)^2|y|^2\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= 2\lambda \int_\Omega \{[P\nabla y, \nabla y]_n w^{-1} q \,\mathrm{d}x + \lambda \int_\Omega P\nabla(w^{-1}q)\cdot\nabla(|y|^2)\,\mathrm{d}x \\
&\quad + \int_\Omega w^{-1}(\lambda q)^2 |y|^2 \,\mathrm{d}x \\
&\geqslant \int_\Omega [w^{-1}(\lambda q)^2 - \lambda \operatorname{div}(P\nabla(w^{-1}q))]|y|^2\,\mathrm{d}x.
\end{aligned}
$$

The proof is then completed as in the (C0) case of Theorem 1. (Note that the basic assumptions on the matrix P and the nonnegativity of q guarantee that $\int_\Omega [P\nabla y, \nabla y]_n w^{-1} q\,\mathrm{d}x \geqslant 0$. □

The next result parallels Corollary 2 for $n > 1$.

Corollary 3. *If $w = 1$ and $P = I_n$ then there is a separation inequality of form (1.5) if $\Delta q \leqslant 0$.*

R e m a r k 2. We can show that $\theta \leqslant 2$ in Theorem 2 and $\theta < 2$ in Theorems 1 and 3 is a necessary condition for separation on $\mathcal{D}$ for all dimensions n. To see this, let Ω be $\mathbb{R}^n \setminus \overline{B(0,1)}$ ($B(0,1)$ is the unit ball centered at the origin), and set

$$
\begin{aligned}
y &= |x|^\mu, \quad w = |x|^\delta, \\
q &= K_0|x|^\beta, \quad P = |x|^\alpha I_n,
\end{aligned}
$$

where I_n is the identity matrix. Then a calculation shows that

$$
y \in L^2(w;\Omega) \Leftrightarrow \int_\Omega |r|^{\delta+2\mu} r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma < \infty \Leftrightarrow 2\mu + \delta + n - 1 < -1, \tag{4.4}
$$

where σ represents the angular measure in polar coordinates. Also

$$
\int_\Omega w|w^{-1}qy|^2\,\mathrm{d}x = \infty \Leftrightarrow 2\mu \geqslant \delta - 2\beta - n. \tag{4.5}
$$

In Theorem 2 the condition ($|\mathrm{S}_n^*|$) gives

$$
\sup_{x\in\Omega} |K_0|^{-1/2}|\beta - \delta||x|^{(\alpha-\beta)/2-1} = \theta, \tag{4.6}
$$

Suppose in (4.6) that $\theta = 2 + \varepsilon$. We will show that we can choose α, β, δ, and μ such that (4.4) and (4.5) are satisfied. First we suppose that $Ly = 0$. This implies that $K_0 = \mu(\alpha + \mu - 2 + n)$. Next take $\alpha = 2 - n$ so that $K_0 = \mu^2$. Now (4.4) $\Leftrightarrow -2\mu > \delta + n$ and (4.5) $\Leftrightarrow 2\mu \geqslant \delta + n$. In other words, assuming that $\delta < -n$, $y \in \mathcal{D}$ and $\|w^{-1}qy\|_{w,\Omega} = \infty$ if and only if

$$
\tfrac{1}{2}(\delta + n) \leqslant \mu < -\tfrac{1}{2}(\delta + n).
$$

Next if $\beta = \alpha - 2 = -n$, then (4.6) is equivalent to

$$\frac{|-n-\delta|}{|\mu|} \equiv \frac{|n+\delta|}{|\mu|} = \theta \equiv 2 + \varepsilon.$$

This will hold if

$$\tfrac{1}{2}(\delta + n) < (\delta + n)(2+\varepsilon)^{-1} < \mu = -(\delta + n)(2+\varepsilon)^{-1} < -\tfrac{1}{2}(\delta + n).$$

For $n = 1$ (Theorem A) our example bears on question that is implicit in the paper [15] of Everitt and Giertz. They showed [15, Theorem 3] that $M[y] = -y'' + qy$ was separated on $\mathcal{D}$ if in ($|\mathrm{S}_1^*|$) $\theta < 2$ while separation need not happen on $\mathcal{D}$ if $\theta > 4/\sqrt{3}$. But the situation when $\theta \in [2, 4/\sqrt{3})$ was left open. This problem seems still to be open; however our example shows that if nontrivial p, w are allowed θ cannot exceed 2 in Theorem A if separation is to occur on $\mathcal{D}$.

A slightly modified analysis works for Theorems 1 and 3. Here

$$w \operatorname{div}(P\nabla(w^{-1}q)) = K_0(\beta - \delta)(\beta - \delta + \alpha)|x|^{\beta+\alpha-2},$$

and thus (S_n) becomes

$$\sup_{|x\in\Omega|} K_0^{-1}(\beta - \delta)(\beta - \delta + \alpha)|x|^{\alpha-\beta-2} = \theta, \tag{4.7}$$

Suppose $\theta \geqslant 2$. The choice $\beta = -n$, $\alpha = 2 - n$, and μ such that $Ly = 0$ gives in (4.7)

$$\theta = \mu^{-2}(n + \delta)(2n + \delta - 2).$$

Therefore we can take

$$\mu = -\sqrt{\tfrac{1}{\theta}(n + \delta)(2n + \delta - 2)}.$$

If $\delta < -n$ then (4.4) will hold. Moreover

$$2 \leqslant \theta \Leftrightarrow 2\theta^{-1}(n + \delta) \geqslant (n + \delta)$$

and

$$2\theta^{-1}(n + \delta) < -2\sqrt{\tfrac{1}{\theta}(n + \delta)[(n + \delta) + (n - 2)]} = 2\mu$$

so that $2\mu > n + \delta$ and (4.5) also is satisfied.

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