ON SOME SIMPLE SUFFICIENT CONDITIONS FOR UNIVALENCE

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Abstract. In this paper some simple conditions on $f'(z)$ and $f''(z)$ which lead to some subclasses of univalent functions will be considered.

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1. Introduction and preliminaries

Let A denote the class of analytic functions $f(z)$ in the unit disc $U=\{z\colon\ |z|<1\}$ and normalized so that $f(0)=f'(0)-1=0$.

A function $f(z)\in A$ is said to be *starlike of order* α, i.e., to belong to $S^*(\alpha)$, $0\leqslant\alpha<1$, if and only if

$$\operatorname{Re}\Big\{\frac{zf'(z)}{f(z)}\Big\}>\alpha$$

for all $z\in U$. Then $S^*=S^*(0)$ is the class of *starlike functions* in the unit disc U. Further, $\widetilde{S}^*(\alpha)$, $0<\alpha\leqslant 1$, is the class of *strongly starlike functions of order* α defined by

$$\widetilde{S}^*(\alpha)=\Big\{f(z)\in A\colon\ \Big|\arg\frac{zf'(z)}{f(z)}\Big|<\frac{\alpha\pi}{2},z\in U\Big\}.$$

Also $K(\alpha)$, $0\leqslant\alpha<1$, is the class of *convex functions of order* α which consists of functions $f(z)\in A$ such that

$$\operatorname{Re}\Big\{1+\frac{zf''(z)}{f'(z)}\Big\}>\alpha$$

for all $z\in U$, and $K=K(0)$ is the class of *convex functions* on the unit disc U.

In addition to these classes we will deal also with the following ones:

$$R(\alpha) = \{f(z) \in A\colon \operatorname{Re}\{f'(z)\} > \alpha,\ z \in U\},\ 0 \leqslant \alpha < 1;$$
$$R_\alpha = \left\{f(z) \in A\colon |\arg f'(z)| < \frac{\alpha\pi}{2},\ z \in U\right\},\ 0 < \alpha \leqslant 1.$$

All of the above mentioned classes are subclasses of univalent functions in U and moreover $K \subset S^*$ (see [1]). Further, S^* does not contain R_1 and R_1 does not contain S^* ([2]).

Let $f(z)$ and $g(z)$ be analytic in the unit disc U. Then we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, if there exists a function $\omega(z)$ analytic in U such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in U$. If $g(z)$ is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$ then $f(z) \prec g(z)$.

The problem of finding $\lambda > 0$ such that the condition $|f''(z)| \leqslant \lambda$, $z \in U$, implies $f(z) \in S^*$ was first considered by Mocanu in his paper [3] for $\lambda = 2/3$. Later, Ponnusamy and Singh found a better constant $\lambda = 2/\sqrt{5}$, and recently Obradović in [4] closed this problem with the constant $\lambda = 1$ by proving that this result is sharp. In this paper, using similar techniques as Obradović did in [4] we will study λ such that the condition $|f''(z)| \leqslant \lambda$, $z \in U$, implies that $f(z)$ belongs to one of the classes defined above.

We will also generalize the result that Mocanu gave in [5]: $|f'(z) - 1| < 2/\sqrt{5}$, $z \in U$, implies $f(z) \in S^*$.

For all of this we will need the following two lemmas.

Lemma 1 ([6]). *Let $G(z)$ be convex and univalent in U, $G(0) = 1$. Let $F(z)$ be analytic in U, $F(0) = 1$ and let $F(z) \prec G(z)$ in U. Then for all $n \in \mathbb{N}_0$,*

$$(n+1)z^{-n-1}\int_0^z t^n F(t)\,\mathrm{d}t \prec (n+1)z^{-n-1}\int_0^z t^n G(t)\,\mathrm{d}t.$$

Lemma 2 ([7]). *Let $F(z)$ and $G(z)$ be analytic functions in the unit disc U and $F(0) = G(0)$. If $H(z) = zG'(z)$ is a starlike function in U and $zF'(z) \prec zG'(z)$ then*

$$F(z) \prec G(z) = G(0) + \int_0^z \frac{H(t)}{t}\,\mathrm{d}t.$$

2. Conditions on $f''(z)$

Theorem 1. *If $f(z) \in A$ and $|f''(z)| \leqslant k$, $z \in U$, $0 < k \leqslant 1$, then*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{k}{2-k}z \tag{1}$$

P r o o f. Noting that the condition of the theorem is equivalent to $zf''(z) \prec kz$, from lemma 1, choosing $F(z) = zf''(z) + 1$, $G(z) = kz + 1$ and $n = 0$, we get

$$f'(z) - \frac{f(z)}{z} \prec \frac{kz}{2},$$

which is equivalent to

$$z\Big(\frac{f(z)}{z}\Big)' \prec z\Big(1 + \frac{kz}{2}\Big)' \tag{2}$$

and to

$$\frac{f(z)}{z}\Big(\frac{zf'(z)}{f(z)} - 1\Big) \prec \frac{k}{2}z \tag{3}$$

for $z \in U$. Now, from (2) and lemma 2, taking $F(z) = f(z)/z$ and $G(z) = 1 + kz/2$ we obtain $f(z)/z \prec 1 + kz/2$, which implies $1 - k/2 < |f(z)/z| < 1 + k/2$, $z \in U$. From this relation and from (3) we can conclude that

$$\Big(1 - \frac{k}{2}\Big)\Big|\frac{zf'(z)}{f(z)} - 1\Big| < \Big|\frac{f(z)}{z}\Big|\Big|\frac{zf'(z)}{f(z)} - 1\Big| < \frac{k}{2}, \quad z \in U,$$

i.e.,

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| < \frac{k}{2-k},$$

$z \in U$, and (1) follows. □

Corollary 1. *If $f(z) \in A$ and $|f''(z)| \leqslant 2(1-\alpha)/(2-\alpha) = k$, $z \in U$, $0 \leqslant \alpha < 1$, then $f(z) \in S^*(\alpha)$. The result is sharp.*

P r o o f. It is obvious that the conditions of Theorem 1 are satisfied, and so from (1) we obtain that $\operatorname{Re}\{zf'(z)/f(z)\} > 1 - k/(2-k) = \alpha$, $z \in U$, i.e., $f(z) \in S^*(\alpha)$. Further, the function $f(z) = z + (k+\varepsilon)z^2/2$, $0 < k \leqslant 1$, $0 < \varepsilon < 1$, proves that the result is sharp, i.e., that k defined in the corollary is the biggest for a given α because $|f''(z)| = k + \varepsilon > k$ and

$$\frac{zf'(z)}{f(z)} = \frac{2(1 + (k+\varepsilon)z)}{2 + (k+\varepsilon)z}$$

is smaller than α when z is real and close to -1. Hence $f(z) \notin S^*(\alpha)$. □

R e m a r k 1. For $\alpha = 0$ ($k = 1$) in Corollary 1 we get Theorem 1 from [4].

Corollary 1.1. *Let $f(z) \in A$. Then*
(i) $|f''(z)| \leqslant 4/5$ *implies* $f(z) \in S^*(1/3)$;
(ii) $|f''(z)| \leqslant 2/3$ *implies* $f(z) \in S^*(1/2)$; *and*
(iii) $|f''(z)| \leqslant 1/2$ *implies* $f(z) \in S^*(2/3)$.

Corollary 2. *If $f(z) \in A$ and $|f''(z)| \leqslant 2\sin(\alpha\pi/2)/(1+\sin(\alpha\pi/2)) = k$, $z \in U$, $0 < \alpha \leqslant 1$, then $f(z) \in \widetilde{S}^*(\alpha)$.*

P r o o f. Because the conditions from Theorem 1 are fulfilled, from the subordination (1) we get that $|\arg\{zf'(z)/f(z)\}| < \arcsin(k/(2-k)) = \alpha\pi/2$, $z \in U$, i.e., $f(z) \in \widetilde{S}^*(\alpha)$. □

R e m a r k 2. The question about the sharpness of the result from Corollary 2 is open. It can be subject to further investigation if for given α, $0 < \alpha < 1$, $k = 2\sin(\alpha\pi/2)/(1+\sin(\alpha\pi/2))$ is the biggest number for which $|f''(z)| \leqslant k$, $z \in U$, implies $f(z) \in \widetilde{S}^*(\alpha)$ (in [4] Obradović showed that for $\alpha = 1$, $k = 1$ is the biggest number with this property). The function $f(z) = z + (k+\varepsilon)z^2/2$, $0 < k < 1$, $\varepsilon > 0$, for which $|f''(z)| = k + \varepsilon > k$ cannot be used for proving sharpness because for each k, $0 < k < 1$, there exists an $\varepsilon > 0$ small enought such that $f(z) \in \widetilde{S}^*(\alpha)$. This follows from the fact that for $z = re^{i\theta}$

$$\arg\frac{zf'(z)}{f(z)} = \arctan\frac{r(k+\varepsilon)\sin\theta}{2 + 3r(k+\varepsilon)\cos\theta + r^2(k+\varepsilon)^2}$$

and

$$\sup_{z\in U}\left|\arg\frac{zf'(z)}{f(z)}\right| = \arcsin\frac{k+\varepsilon}{2-(k+\varepsilon)^2},$$

which is smaller than $\arcsin(k/(2-k)) = \alpha\pi/2$ for $\varepsilon > 0$ small enought.

Corollary 2.1. *Let $f(z) \in A$. Then*
(i) $|f''(z)| \leqslant 2/3$ *implies* $f(z) \in \widetilde{S}^*(1/3)$;
(ii) $|f''(z)| \leqslant 2(\sqrt{2}-1) = 0,8284\ldots$ *implies* $f(z) \in \widetilde{S}^*(1/2)$; *and*
(iii) $|f''(z)| \leqslant 2(2\sqrt{3}-3) = 0,9282\ldots$ *implies* $f(z) \in \widetilde{S}^*(2/3)$.

Using the next theorem we will obtain some results on the classes $K(\alpha)$, $R(\alpha)$ and R_α.

Theorem 2. *If $f(z) \in A$ and $|f''(z)| \leqslant k$, $z \in U$, $0 < k \leqslant 1$, then*

$$f'(z) \prec 1 + kz. \tag{4}$$

P r o o f. The condition $|f''(z)| \leqslant k$, $z \in U$, is equivalent to

$$zf''(z) \prec kz \tag{5}$$

$z \in U$, and again, using Lemma 2 for $F(z) = f'(z)$ and $G(z) = 1 + kz$, we get that the subordination (4) is true. □

Corollary 3. *If $f(z) \in A$ and $|f''(z)| \leqslant (1-\alpha)/(2-\alpha) = k$, $z \in U$, $0 \leqslant \alpha < 1$, then $f(z) \in K(\alpha)$. The result is sharp.*

P r o o f. Because the conditions from Theorem 2 are fulfilled we get that (4) and (5) are true, and from (5) with $p(z) = 1 + zf''(z)/f'(z)$ we conclude

$$(p(z) - 1)f'(z) \prec kz \tag{6}$$

for $z \in U$. Now, let us suppose that there exists $z_0 \in U$ such that $p(z_0) = \alpha + \mathrm{i}x$. So from (4) and (6) it follows that

$$1 - k < |f'(z_0)| < 1 + k \tag{7}$$

and

$$|(p(z_0) - 1)f'(z_0)| < k. \tag{8}$$

Further, using (7) we obtain

$$\begin{aligned} |(p(z_0) - 1)f'(z_0)|^2 &= |\alpha - 1 + \mathrm{i}x|^2 |f'(z_0)|^2 \\ &> [(\alpha - 1)^2 + x^2](1 - k)^2 \\ &= (\alpha - 1)^2(1 - k)^2 + x^2(1 - k)^2 \\ &\geqslant (\alpha - 1)^2(1 - k)^2 = k^2 \end{aligned}$$

for $\alpha = (1-2k)/(1-k)$ ($\Leftrightarrow k = (1-\alpha)/(2-\alpha)$), which contradicts to (8). Therefore we have proved that under the conditions of Corollary 3 $\mathrm{Re}\{1 + zf'(z)/f(z)\} > \alpha$ is true for any $z \in U$, i.e., $f(z) \in K(\alpha)$.
The proof that the result is sharp is again done by the function $f(z) = z + (k+\varepsilon)z^2/2$, $0 < k \leqslant 1/2$ and $\varepsilon > 0$, for which $|f''(z)| = k + \varepsilon > k$ and

$$\mathrm{Re}\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} = \frac{1 + 2z(k + \varepsilon)}{1 + z(k + \varepsilon)}$$

is smaller than α when z is real and close to -1, i.e., $f(z) \notin K(\alpha)$. □

R e m a r k 3. For $\alpha = 0$, i.e., $k = 1/2$, Corollary 3 is equivalent to Theorem 3 from [4].

Corollary 3.1. *Let $f(z) \in A$. Then*
(i) $|f''(z)| \leqslant 2/5$ *implies* $f(z) \in K(1/3)$;
(ii) $|f''(z)| \leqslant 1/3$ *implies* $f(z) \in K(1/2)$; *and*
(iii) $|f''(z)| \leqslant 1/4$ *implies* $f(z) \in K(2/3)$.

Corollary 4. *If $f(z) \in A$ and $|f''(z)| \leqslant 1 - \alpha = k$, $z \in U$, $0 \leqslant \alpha < 1$, then $f(z) \in R(\alpha)$. The result is sharp.*

P r o o f. Subordination (4) is true because the conditions from Theorem 2 are fulfilled and hence we conclude that $\operatorname{Re}\{f'(z)\} > 1 - k = \alpha$ for $z \in U$, $f(z) \in R(\alpha)$. Once again, using the function $f(z) = z + (k + \varepsilon)z^2/2$, $0 < k \leqslant 1$ and $\varepsilon > 0$, for which $|f''(z)| = k + \varepsilon > k$ and $f'(z) = 1 + (k + \varepsilon)z$ is smaller than α when z is real and close to -1, we prove that the result of the corollary is sharp. □

Corollary 4.1. *Let $f(z) \in A$. Then*
(i) $|f''(z)| \leqslant 2/3$ *implies* $f(z) \in R(1/3)$;
(ii) $|f''(z)| \leqslant 1/2$ *implies* $f(z) \in R(1/2)$;
(iii) $|f''(z)| \leqslant 1/3$ *implies* $f(z) \in R(2/3)$.

Corollary 5. *If $f(z) \in A$ and $|f''(z)| \leqslant \sin(\alpha\pi/2) = k$, $z \in U$, $0 < \alpha \leqslant 1$, then $f(z) \in R_\alpha$. The result is sharp.*

P r o o f. From the subordination (4), which is true because the conditions of Theorem 2 are fulfilled, we obtain that $|\arg f'(z)| < \arcsin k = \alpha\pi/2$, $z \in U$, i.e., $f(z) \in R_\alpha$. And in this case the proof that the result is sharp is done by considering the function $f(z) = z + (k+\varepsilon)z^2/2$, $0 < k \leqslant 1$ and $\varepsilon > 0$, for which $|f''(z)| = k + \varepsilon > k$ and $\sup_{z\in U} |\arg f'(z)| = \arcsin(k + \varepsilon) > \arcsin k = \alpha\pi/2$ for $\varepsilon > 0$ small enought. □

Corollary 5.1. *Let $f(z) \in A$. Then*
(i) $|f''(z)| \leqslant 1/2$ *implies* $f(z) \in R_{1/3}$;
(ii) $|f''(z)| \leqslant \sqrt{2}/2 = 0,7071\ldots$ *implies* $f(z) \in R_{1/2}$; *and*
(iii) $|f''(z)| \leqslant \sqrt{3}/2 = 0,8660\ldots$ *implies* $f(z) \in R_{2/3}$.

3. Condition on $f'(z)$

Theorem 3. *Let $f(z) \in A$. If $|f'(z) - 1| < \lambda$ for some $0 < \lambda \leqslant 1$ and for all $z \in U$, then $f(z) \in \widetilde{S}^*(\alpha)$, where*

$$\alpha = \frac{2}{\pi} \arcsin \Big(\lambda \sqrt{1 - \frac{\lambda^2}{4}} + \frac{\lambda}{2}\sqrt{1-\lambda^2}\Big),$$

and $|f(z)| < 1 + \lambda/2$ for $z \in U$.

P r o o f. From the condition $f'(z) \prec 1 + \lambda z$ it follows that

$$|\arg f'(z)| < \arcsin \lambda, \quad z \in U. \tag{9}$$

From the same condition, using lemma 1 for $F(z) = f'(z)$, $G(z) = 1 + \lambda z$ and $n = 0$ we get that

$$\frac{f(z)}{z} \prec 1 + \frac{\lambda}{2} z. \tag{10}$$

Consequently,

$$\Big|\arg \frac{f(z)}{z}\Big| < \arcsin \frac{\lambda}{2} \tag{11}$$

for $z \in U$. Now from (9) and (11) we can conclude that

$$\begin{aligned}\Big|\arg \frac{zf'(z)}{f(z)}\Big| &= \Big|\arg \frac{z}{f(z)} + \arg f'(z)\Big| \leqslant \Big|\arg \frac{z}{f(z)}\Big| + |\arg f'(z)| \\ &< \arcsin \frac{\lambda}{2} + \arcsin \lambda = \arcsin \Big(\lambda \sqrt{1 - \frac{\lambda^2}{4}} + \frac{\lambda}{2}\sqrt{1-\lambda^2}\Big),\end{aligned}$$

i.e., $f(z) \in \widetilde{S}^*(\alpha)$ for

$$\alpha = \frac{2}{\pi} \arcsin \Big(\lambda \sqrt{1 - \frac{\lambda^2}{4}} + \frac{\lambda}{2}\sqrt{1-\lambda^2}\Big). \tag{12}$$

Further, from (10) it is easy to infer that for $z \in U$

$$|f(z)| < \Big|\frac{f(z)}{z}\Big| < 1 + \frac{\lambda}{2}.$$

□

We can rewrite Theorem 3 in the following way.

Theorem 3′. *Let* $f(z) \in A$, $0 < \alpha \leqslant 1$ *and let*

$$|f'(z) - 1| < 2a\sqrt{\frac{5 - 4\sqrt{1-a^2}}{16a^2+9}} = \lambda, \tag{13}$$

where $a = \sin(\alpha\pi/2)$. *Then* $f(z) \in \widetilde{S}^*(\alpha)$ *and* $|f(z)| < 1 + \lambda/2$ *for* $z \in U$.

P r o o f. If we put λ from (13) to the right side of (12) we obtain α. □

Corollary 6. *Let* $f(z) \in A$ *and* $|f'(z) - 1| < \lambda$. *Then*

(i) *if* $\lambda = 2\sqrt{5}/5 = 0,8944\ldots$, *then* $f(z) \in \widetilde{S}^*(1) = S^*$ *and* $|f(z)| < 1 + \sqrt{5}/5 = 1,4472\ldots$, *for* $z \in U$;

(ii) *if* $\lambda = \sqrt{21}/7 = 0,6546\ldots$, *then* $f(z) \in \widetilde{S}^*(2/3)$ *and* $|f(z)| < 1 + \sqrt{21}/14 = 1,3273\ldots$, *for* $z \in U$;

(iii) *if* $\lambda = \sqrt{(10 - 4\sqrt{2})/17} = 0,5054\ldots$, *then* $f(z) \in \widetilde{S}^*(1/2)$ *and* $|f(z)| < 1 + \lambda/2 = 1,2527\ldots$, *for* $z \in U$;

(iv) *if* $\lambda = \sqrt{(5 - 2\sqrt{3})/13} = 0,3437\ldots$, *then* $f(z) \in \widetilde{S}^*(1/3)$ *and* $|f(z)| < 1+\lambda/2 = 1,1718\ldots$, *for* $z \in U$;

R e m a r k 4. The result from Corollary 6 (i) is the same as the result from Theorem 2 from [5], but it is obtained by a different method.

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