A DUALITY BETWEEN ALGEBRAS OF BASIC LOGIC AND BOUNDED REPRESENTABLE DRl-MONOIDS

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Abstract. BL-algebras, introduced by P. Hájek, form an algebraic counterpart of the basic fuzzy logic. In the paper it is shown that BL-algebras are the duals of bounded representable DRl-monoids. This duality enables us to describe some structure properties of BL-algebras.

Keywords: BL-algebra, MV-algebra, bounded DRl-monoid, representable DRl-monoid, prime spectrum

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1. Connections between BL-algebras and DRl-monoids

Dually residuated lattice ordered monoids (briefly: DRl-monoids) were introduced and studied by K. L. N. Swamy in [16], [17] and [18] as a common generalization of commutative lattice ordered groups (l-groups) and Brouwerian (and hence also Boolean) algebras.

Definition. An algebra $\mathcal{A} = (A, +, 0, \vee, \wedge, -)$ of signature $\langle 2, 0, 2, 2, 2 \rangle$ is called a *DRl-monoid* if it satisfies the following conditions ($x, y, z \in A$):

(1) $(A, +, 0)$ is an abelian monoid;
(2) $(A, \vee, \wedge)$ is a lattice;
(3) $(A, +, \vee, \wedge, 0)$ is an l-monoid;
(4) if $\leqslant$ denotes the order on A induced by the lattice $(A, \vee, \wedge)$ then $x - y$ is the smallest $z \in A$ such that $y + z \geqslant x$;

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(5) $((x - y) \vee 0) + y \leqslant x \vee y$.

N o t e. As is shown in [16], condition (4) is equivalent to the system of identities

$$
\begin{aligned}
x + (y - x) &\geqslant y; \\
x - y &\leqslant (x \vee z) - y; \\
(x + y) - y &\leqslant x,
\end{aligned}
$$

hence DRl-monoids form a variety of algebras of type $\langle 2, 0, 2, 2, 2 \rangle$.

The notion of a DRl-monoid actually includes also other types of algebras.

It is well-known (by C. C. Chang [2]) that the Łukasiewicz infinite valued propositional logic has as its algebraic counterpart the notion of an MV-algebra. Moreover, there are several other types of algebraic structures equivalent to MV-algebras which in this sense can be associated with Łukasiewicz logic. For example, by D. Mundici [8] and [9], MV-algebras are categorically equivalent to abelian lattice ordered groups with strong order units and to bounded commutative BCK-algebras.

In [12] and [14] it was shown that the class of MV-algebras is polynomially equivalent to a variety of bounded DRl-monoids.

The Łukasiewicz infinite valued logic is an axiomatic extension of the basic fuzzy logic. The latter has as its algebraic counterpart the notion of a BL-algebra. (See [6], [7] or [4].) The *basic fuzzy logic* and BL-algebras were introduced by P. Hájek to formalize a part of the reasoning in fuzzy logic. In this paper we will show that also BL-algebras can be equivalently replaced by a class of dually residuated lattice ordered monoids, and that this equivalence makes it possible to use some results of the theory of such lattice ordered monoids in the theory of BL-algebras.

Definition. A *BL-algebra* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of signature $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that

(i) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1;
(ii) $(A, \odot, 1, \vee, \wedge)$ is a commutative lattice ordered monoid;
(iii) $\mathcal{A}$ satisfies the following conditions:
(1) $z \leqslant x \rightarrow y$ iff $x \odot z \leqslant y$, for all $x, y, z \in A$,
(2) $x \wedge y = x \odot (x \rightarrow y)$,
(3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

R e m a r k. a) The BL-algebras also form a variety of algebras of the type considered.

b) A BL-algebra could be also defined equivalently as an algebra $\mathcal{A} = (A, \odot, \rightarrow, 0)$ of signature $\langle 2, 2, 0 \rangle$ (see [4]). We use the above Hájek's definition because it gives a direct possibility to show a duality between the class of BL-algebras and a class of DRl-monoids.

Now we can recognize BL-algebras as dual cases of some DRl-monoids.

Definition. A DRl-monoid $\mathcal{A} = (A, +, 0, \vee, \wedge, -)$ is called *representable* (see [20]) if it is isomorphic to a subdirect product of linearly ordered DRl-monoids (i.e. DRl-chains).

For instance, commutative l-groups and Boolean algebras are representable DRl-monoids.

One can prove (see [20]) that a DRl-monoid $\mathcal{A}$ is representable if and only if $\mathcal{A}$ satisfies the identity

$$(x - y) \wedge (y - x) \leqslant 0.$$

Remark. Comparing two classes of algebras, it will be simpler to use algebras dual to BL-algebras. Namely, an algebra $\mathcal{A} = (A, \vee, \wedge, \oplus, \ominus, 1, 0)$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ is called a *dual BL-algebra* if

(i)^d $(A, \vee, \wedge, 1, 0)$ is a bounded lattice with the greatest element 1 and the least element 0;

(ii)^d $(A, \oplus, 0, \wedge, \vee)$ is a commutative lattice ordered monoid;

(iii)^d $\mathcal{A}$ satisfies the conditions

(1) $z \geqslant x \ominus y$ iff $x \oplus z \geqslant y$, for all $x, y, z \in A$,

(2) $x \vee y = x \oplus (y \ominus x)$,

(3) $(x \ominus y) \wedge (y \ominus x) = 0$.

Let $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a BL-algebra and let $(A, \wedge_d, \vee_d)$ be the lattice dual to the lattice $(A, \wedge, \vee)$, i.e. $x \wedge_d y = x \vee y$ and $x \vee_d y = x \wedge y$ for any $x, y \in A$. Further, set $x \oplus_d y = x \odot y$ and $x \ominus_d y = y \rightarrow x$ for each $x, y \in A$. Then $(A, \vee_d, \wedge_d, \oplus_d, \ominus_d, 0, 1)$ is a dual BL-algebra. Conversely, using the dual considerations, one can obtain a BL-algebra from a given dual BL-algebra. It is obvious that the above processes are mutually inverse and therefore there is a one-to-one correspondence between the BL-algebras and the dual BL-algebras.

Theorem 1. *Let $\mathcal{A} = (A, +, 0, \vee, \wedge, -)$ be an above bounded DRl-monoid with the greatest element* 1. *Then $(A, \vee, \wedge, +, -, 1, 0)$ is a dual BL-algebra if and only if $\mathcal{A}$ is representable.*

Proof. One can easily prove (see e.g. [10], Theorem 1.2.3) that if a DRl-monoid $\mathcal{A}$ is bounded above then it is bounded below too, and, moreover, 0 is the least element in $\mathcal{A}$. If this is the case, then the conditions (i)^d, (ii)^d and (iii)^d(1) are trivially satisfied, and the condition (iii)^d(2) follows from (5) of the definition of a DRl-monoid. If moreover $\mathcal{A}$ is representable, the condition (iii)^d(3) holds.

Conversely, if $\mathcal{A}$ is a bounded DRl-monoid such that $(A, \vee, \wedge, +, -, 1, 0)$ is a dual BL-algebra, then $\mathcal{A}$ is obviously representable. $\square$

Comparing the definitions of BL-algebras and representable DRl-monoids we get the following theorem.

Theorem 2. *If $\mathcal{A} = (A, \vee, \wedge, \oplus, \ominus, 1, 0)$ is a dual BL-algebra then $(A, \oplus, 0, \vee, \wedge, \ominus)$ is a bounded representable DRl-monoid with the greatest element* 1.

R e m a r k. For the class $\mathcal{DR}l_1$ of bounded DRl-monoids (and especially for the class $\mathcal{RDR}l_1$ of bounded representable DRl-monoids) we will consider the greatest element 1 as a new nullary operation and thus we will enlarge the type of those DRl-monoids to $(+, 0, \vee, \wedge, -, 1)$ of signature $\langle 2, 0, 2, 2, 2, 0 \rangle$. Hence the class $\mathcal{DBL}$ of dual BL-algebras is, from this point of view, a subclass of the class $\mathcal{DR}l_1$ which is, by Theorems 1 and 2, equal to the class $\mathcal{RDR}l_1$ of bounded representable DRl-monoids. This means that BL-algebras are in fact the dual algebras of bounded representable DRl-monoids, and therefore one can obtain some results on BL-algebras as consequences of those on DRl-monoids.

Now, let us recall the notion of an MV-algebra.

Definition. An algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ of signature $\langle 2, 1, 0 \rangle$ is called an *MV-algebra* if $\mathcal{A}$ satisfies the following identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(MV2) $x \oplus y = y \oplus x$;
(MV3) $x \oplus 0 = x$;
(MV4) $\neg\neg x = x$;
(MV5) $x \oplus \neg 0 = \neg 0$;
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

As is known, MV-algebras were introduced by C. C. Chang in [2] and [3] as an algebraic counterpart of Łukasiewicz infinite-valued propositional logic.

If we put in any MV-algebra $\mathcal{A}$

$$1 = \neg 0,\ x \ominus y = \neg(\neg x \oplus y),$$
$$x \vee y = \neg(\neg x \oplus y) \oplus y,\ x \wedge y = \neg(\neg x \vee \neg y)$$

for each $x, y \in A$, then $(A, \vee, \wedge, \oplus, \ominus, 1, 0)$ is a dual BL-algebra and so also a bounded representable DRl-monoid.

Moreover, MV-algebras are by [12] and [14] in a one-to-one correspondence with bounded DRl-monoids (the representability is not explicitly required) which satisfy the identity

$$\text{(i)} \qquad 1 - (1 - x) = x.$$

Therefore we get as a consequence a known characterization of MV-algebras in the class of dual BL-algebras:

Corollary 3. *A dual BL-algebra $\mathcal{A}$ is an MV-algebra if and only if $\mathcal{A}$ satisfies*

$$\text{(i}'\text{)} \qquad 1 \ominus (1 \ominus x) = x.$$

N o t e. This corollary corresponds to [7], Definition 3.2.2, where MV-algebras are defined as BL-algebras satisfying the law of double negation $\neg\neg x = x$.

R e m a r k. DRl-monoids (similarly as MV-algebras) in general lack additive idempotents. However, in Brouwerian algebras which are special cases of DRl-monoids, the operations $+$ and $\vee$ coincide, and hence, among others, all elements are additive idempotents. It is known ([2], Theorem 1.17) that additive idempotents in any MV-algebra form a Boolean algebra. Now we can analogously describe the properties of the set of idempotents in any bounded representable DRl-monoid.

Proposition 4. *The set B of additive idempotents of any representable bounded DRl-monoid $\mathcal{A}$ is a Brouwerian algebra.*

P r o o f. Let $\mathcal{A} = (A, +, 0, \vee, \wedge, -, 1)$ be a bounded representable DRl-monoid and $B = \{x \in A;\ x + x = x\}$. Obviously $0, 1 \in B$. Let $x, y \in B$. Then

$$(x + y) + (x + y) = x + y,$$
$$(x \wedge y) + (x \wedge y) = (x + x) \wedge (x + y) \wedge (y + y) = x \wedge y \wedge (x + y) = x \wedge y,$$

hence $x + y$, $x \wedge y \in B$.

For any $x, y \in B$,

$$x + (x \wedge y) = x \wedge (x + y) = x,$$

thus $(B, +, \wedge)$ satisfies both absorption laws. Therefore $(B, +, \wedge)$ is a lattice which is distributive by the definition of a DRl-monoid.

Let $\mathcal{A}$ be a bounded DRl-chain. The order induced on B by the lattice $(B, +, \wedge)$ is clearly the same as that induced on B by $\mathcal{A}$. Hence $(B, +, \wedge)$ is a chain, and so

$$x + y = \sup(x, y) = \max(x, y) = x \vee_A y$$

for any $x, y \in B$.

Moreover, $(B, +, \wedge) = (B, \vee, \wedge)$ is a Brouwerian algebra because for any $x, y \in B$ we have

$$x - y = 0 \quad \text{if} \quad x \leqslant y,$$
$$x - y = x \quad \text{if} \quad x > y.$$

Let now a DRl-monoid $\mathcal{A}$ be a subdirect product of bounded DRl-chains $\mathcal{A}_i$, $i \in I$. If $a = (a_i;\ i \in I) \in A$, then $a \in B$ if and only if $a_i \in B_i$ for each $i \in I$. (B_i is the set of idempotents of $\mathcal{A}_i$.) Hence, if $a, b \in B$ then

$$a + b = (a_i + b_i;\ i \in I) = (\max(a_i, b_i);\ i \in I) = a \vee b,$$

and if we set $a - b = (a_i - b_i;\ i \in I)$ for any $a, b \in B$, we get that $(B, 0, \vee, \wedge, -, 1)$ is a Brouwerian algebra. □

Corollary 5. *The set of multiplicative idempotents of any BL-algebra is a Heyting algebra.*

2. Structure properties of BL-algebras

Recall the notion of a filter of a BL-algebra introduced in [7], Definition 2.3.13:

If $\mathcal{A}$ is a BL-algebra then $\emptyset \neq F \subseteq A$ is called a *filter* of $\mathcal{A}$ if

(a) $\forall a, b \in F;\ a \odot b \in F$,

(b) $\forall a \in F,\ x \in A;\ a \leqslant x \Rightarrow x \in F$.

Further, recall that $\emptyset \neq F \subseteq A$ is called a *deductive system* of a BL-algebra $\mathcal{A}$ if

(a′) $1 \in F$,

(b′) $\forall x, y \in A;\ x \in F,\ x \to y \in F \Longrightarrow y \in F$.

One can easily prove that $\emptyset \neq F \subseteq A$ is a filter of $\mathcal{A}$ if and only if F is a *deductive system* in $\mathcal{A}$.

Note that deductive systems of BL-algebras were introduced in [21] where, moreover, also special types of deductive systems called implicative and weakly implicative were studied.

Let $\mathcal{B}$ be an arbitrary DRl-monoid. For any $x, y \in B$ set $x * y = (x - y) \vee (y - x)$. Then $\emptyset \neq I \subseteq B$ is called an *ideal* of $\mathcal{B}$ if

(c) $\forall a, b \in I;\ a + b \in I$,

(d) $\forall a \in I,\ x \in B;\ x * 0 \leqslant a * 0 \Rightarrow x \in I$.

It is obvious that $0 \leqslant x$ implies $x * 0 = x$ for any x in a DRl-monoid $\mathcal{B}$. Therefore, if $\mathcal{A}$ is a BL-algebra then the filters of $\mathcal{A}$ and the ideals of the DRl-monoid $\mathcal{A}^d$ dual to $\mathcal{A}$ coincide.

Further, the ideals and congruences in any DRl-monoid are in a one-to-one correspondence (see [18]), therefore this holds also for filters and congruences of BL-algebras (see also [7] or [4]).

In [19] some results concerning the lattices of semiregular normal autometrized lattice ordered algebras are obtained. The DRl-monoids are special cases of these algebras, thus the following assertions are consequences of [19], Theorem 6, of the distributivity of Brouwerian lattices, and of the correspondence between the lattice of subvarieties of any variety of algebras $\mathcal{V}$ and the lattice of fully characteristic congruences of the free algebra with countable rank in $\mathcal{V}$.

Theorem 6. *The filters of any BL-algebra form, under the ordering by set inclusion, a complete algebraic Brouwerian lattice.*

Corollary 7. *The variety $\mathcal{BL}$ of BL-algebras is congruence distributive.*

Theorem 8. *The lattice* **BL** *of all varieties of BL-algebras is a complete dually algebraic dually Brouwerian lattice.*

If $\mathcal{A}$ is a BL-algebra then a filter F of $\mathcal{A}$ is called *prime* if F is a finitely meet irreducible element of the lattice $\mathcal{F}(\mathcal{A})$ of all filters of $\mathcal{A}$, i.e., if

$$\forall K, L \in \mathcal{F}(\mathcal{A});\ K \cap L = F \Longrightarrow K = F \ \text{ or } L = F.$$

According to [11], F is a prime filter of $\mathcal{A}$ if and only if

$$\forall x, y \in A;\ x \vee y \in F \Longrightarrow x \in F \ \text{ or } y \in F,$$

and hence by [7] if and only if the quotient algebra of $\mathcal{A}$ by the congruence corresponding to F is linearly ordered.

In [7], Definition 2.3.13, a filter F of a BL-algebra $\mathcal{A}$ is defined to be prime if for each $x, y \in A$,

$$x \to y \in F \quad \text{or} \quad y \to x \in F.$$

Further, in [7], Lemma 2.3.14, the correspondence between congruences and filters of BL-algebras is described and it is shown that the quotient BL-algebra is linearly ordered if and only if it corresponds to a prime filter. Hence our definition of a prime filter is equivalent to that of [7]. Moreover, in [7], Lemma 2.3.15, it is shown that any BL-algebra $\mathcal{A}$ has "enough" prime filters because for any $1 \neq x \in A$ there is a prime filter of $\mathcal{A}$ not containing x.

Let us denote by $\operatorname{Spec} \mathcal{A}$ the prime spectrum of a BL-algebra $\mathcal{A}$, i.e. the set of all proper prime filters of $\mathcal{A}$. As dual BL-algebras form a subvariety of the variety $\mathcal{DR}l_1$ of bounded DRl-monoids, we get, by [13], Corollary 6, the following theorem.

Theorem 9. *If $\mathcal{A}$ is a BL-algebra, then* $\operatorname{Spec} \mathcal{A}$ *endowed with the spectral (i.e. hull-kernel) topology is a compact topological space.*

Let us consider the sets $m(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ of all minimal and maximal, respectively, proper prime filters of a BL-algebra $\mathcal{A}$. Since $\mathcal{A}^d$ is, moreover, a representable DRl-monoid, Theorems 11 and 14 of [13] imply the following properties of spectral topologies on $m(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ induced by the spectral topology of $\operatorname{Spec} \mathcal{A}$.

Theorem 10. *Let $\mathcal{A}$ be a BL-algebra. Then the spectral topologies of $m(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ are T_2-topologies and the space $\mathcal{M}(\mathcal{A})$ is compact.*

Let us recall the notion of a weak Boolean product of algebras.

Definition. An algebra $\mathcal{A}$ is called a *weak Boolean product* (a *Boolean product*) of an indexed family $(\mathcal{A}_x;\, x \in X)$ of algebras over a Boolean space X if $\mathcal{A}$ is a subdirect product of the family $(\mathcal{A}_x;\, x \in X)$ such that

(BP1) if $a, b \in A$ then $[[a = b]] = \{x \in X;\, a(x) = b(x)\}$ is open (clopen);

(BP2) if $a, b \in A$ and U is a clopen subset of X, then $a|_U \cup b|_{X \setminus U} \in A$, where $(a|_U \cup b|_{X\setminus U})(x) = a(x)$ for $x \in U$ and $(a|_U \cup b|_{X\setminus U})(x) = b(x)$ for $x \in X \setminus U$.

(See [1] or [5].)

In the paper [5], Theorem 2.3, it was proved how the ordered prime spectrum of a weak Boolean product (and hence also of a Boolean product) of MV-algebras is composed by the prime ordered spectra of the components of this product. This result was generalized in [15], Theorem 2, to weak Boolean products of arbitrary bounded DRl-monoids. Hence the next theorem is a consequence of [15].

Theorem 11. *Let a BL-algebra $\mathcal{A}$ be a weak Boolean product over a Boolean space X of a system $(\mathcal{A}_x;\, x \in X)$ of BL-algebras. Then the ordered prime spectrum* $(\operatorname{Spec} \mathcal{A}, \subseteq)$ *is isomorphic to the cardinal sum of the ordered prime spectra* $(\operatorname{Spec} \mathcal{A}_x, \subseteq)$, $x \in X$.

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