ON GENERAL SOLVABILITY PROPERTIES OF p-LAPALACIAN-LIKE EQUATIONS

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Abstract. We discuss how the choice of the functional setting and the definition of the weak solution affect the existence and uniqueness of the solution to the equation

$$-\Delta_p u = f \quad \text{in} \quad \Omega,$$

where Ω is a very general domain in $\mathbb{R}^N$, including the case $\Omega = \mathbb{R}^N$.

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1. Introduction

The object of our study is the second order quasilinear elliptic differential operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real number. Note that we define $\Delta_p u = 0$ for $\nabla u = 0$ and $1 < p < 2$. We concentrate on the following basic question: "How the choice of an appropriate function space affects the existence and uniqueness of the weak solution to the equation

$$-\Delta_p u = f \text{ in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$?" Let us point out that Ω is considered to be a bounded, an (unbounded) exterior domain or, possibly, $\Omega = \mathbb{R}^N$. The choice of an appropriate

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function space and the relation between p and the dimension N then play the essential role in the questions of existence, nonexistence or uniqueness of the weak solution to Eq. (1.1). While for Ω a bounded domain the situation seems to be more or less clear and often treated in literature, for $\Omega = \mathbb{R}^N$ or Ω an exterior domain in $\mathbb{R}^N$ we can observe some phenomena which may seem to be surprising without deeper insight of the problem and a careful definition of the notion of a weak solution (cf. [7]). We start our exposition with very general existence and uniqueness results in abstract Banach spaces. Then we consider the typical situations: Ω a bounded domain, an (unbounded) exterior domain and the whole of $\mathbb{R}^N$, and point out some differences between these cases. Let us remind the reader that problems of this type were treated e.g. in [1], [2], [3] or [5].

2. Some general existence and uniqueness results

Let $\Omega \subset \mathbb{R}^N$ be a domain and let $L^{1,p}(\Omega) := \{u \in L^1_{\text{loc}}(\Omega);\ \nabla u \in [L^p(\Omega)]^N\}$. Here $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$, where $\partial_i u := \partial u/\partial x_i$ $(i = 1, \ldots, N)$ is the weak (distributional) derivative of u.

Let X be a linear function space with the following properties:

(X1) $\quad X \subset L^{1,p}(\Omega)$.

(X2) $\quad$ By $\|u\|_X := \|\nabla u\|_{p;\Omega}$ for $u \in X$ a norm is defined on X so that X equipped with this norm is a reflexive Banach space where $\|\cdot\|_{p;\Omega}$ is the usual L^p-norm of $|\nabla u| := \left(\sum_{i=1}^{N} |\partial_i u|^2\right)^{1/2}$.

Let us denote by X^* the dual space, by $\|.\|_{X^*}$ the norm on X^* and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. We define the operator $J\colon X \to X^*$ by

$$\langle J(u), v\rangle_X = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v$$

for any $u, v \in X$. Then the operator J has the following properties:

(J1) $\quad \langle J(u), u\rangle_X = \|u\|_X^p$ for any $u \in X$;

(J2) $\quad \langle J(u) - J(v), u - v\rangle_X > 0$ for any $u, v \in X, u \neq v$;

(J3) $\quad J$ and J^{-1} are continuous operators.

Indeed, the properties (J1) and (J2) as well as the continuity of J are obvious. It then follows from the theory of monotone operators (see e.g. [4]) that J is surjective.

To prove the continuity of J^{-1} we use the inequality

$$\langle J(u) - J(v), u - v\rangle_X \geqslant (\|u\|_X^{p-1} - \|v\|_X^{p-1})(\|u\|_X - \|v\|_X) \tag{2.1}$$

which is an immediate consequence of the Hölder inequality. Let us suppose that $J^{-1}\colon X^* \to X$ is not continuous. Then there exists a sequence $(f_n) \subset X^*$, $f_n \to f$, i.e. strongly, in X^* and

$$\|J^{-1}(f_n) - J^{-1}(f)\|_X \geqslant \delta$$

for some $\delta > 0$. Denote $u_n = J^{-1}(f_n)$, $u = J^{-1}(f)$. It follows from (J1) that

$$\|f_n\|_{X^*}\|u_n\|_X \geqslant \langle f_n, u_n\rangle_X = \langle J(u_n), u_n\rangle_X = \|u_n\|_X^p,$$

i.e. $(u_n) \subset X$ is a bounded sequence. Due to (X2) we can assume (after passing to a subsequence, if necessary) that there exists $\tilde{u} \in X$ such that $u_n \rightharpoonup \tilde{u}$, i.e. weakly, in X. Hence we have

$$\langle J(u_n) - J(\tilde{u}), u_n - \tilde{u}\rangle_X = \langle J(u_n) - J(u), u_n - \tilde{u}\rangle_X + \langle J(u) - J(\tilde{u}), u_n - \tilde{u}\rangle_X \to 0 \tag{2.2}$$

since $J(u_n) = f_n \to f = J(u)$ in X^*. If we set $u = u_n$ and $v = \tilde{u}$ in (2.1) then (2.2) implies $\|u_n\|_X \to \|\tilde{u}\|_X$. Then (X2) yields $u_n \to \tilde{u}$ in X and so by (J2) we get $u = \tilde{u}$, a contradiction. Actually, we have proved

Theorem 2.1. *The operator J is a homeomorphism between X and X^*. In particular, given $f \in X^*$, the equation $J(u) = f$ has a unique solution $u_f \in X$ and $\|u_f\|_X \leqslant \|f\|_{X^*}^{1/(p-1)}$.*

Note that the equation $J(u) = f$ can be interpreted also as an Euler equation of the functional

$$\Phi_f(u) = \frac{1}{p}\|u\|_X^p - \langle f, u\rangle_X, \quad u \in X,$$

and its solution as a minimizer of Φ_f. Indeed, it is easy to verify that $\Phi_f\colon X \to \mathbb{R}$ is a coercive, strictly convex and weakly lower semicontinuous functional. So for arbitrary $f \in X^*$, there exists a unique minimizer $u_f \in X$ of Φ_f which is also its unique critical point.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Define $X := \overline{C_0^\infty(\Omega)}^{\|\nabla\cdot\|_{p;\Omega}} = W_0^{1,p}(\Omega)$ and let $f \in X^*$. It is well known that the space X equipped with the norm $\|\nabla \cdot \|_{p;\Omega}$ satisfies (X1) and (X2). We then define a *weak solution* of (3.1) as a function $u \in X$ for which the identity

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \langle f, v\rangle_X \tag{3.2}$$

holds for every $v \in X$. It follows from Theorem 2.1 that (3.2) is uniquely solvable for any $f \in X^*$.

In what follows, for $1 < p < N$ we set

$$p^* = \frac{Np}{N-p} \text{ (the critical Sobolev exponent)}, \quad p^{*\prime} = \frac{p^*}{p^*-1} = \frac{Np}{Np-N+p}.$$

In the case $p > N$ we set $p^* = \infty$, $p^{*\prime} = 1$, and finally for $p = N$ we put $p^* = q$, $p^{*\prime} = \frac{q}{q-1}$, where $q \in (1, \infty)$ is an arbitrarily chosen number. It follows from the Sobolev imbedding theorem that any $f \in L^{p^{*\prime}}(\Omega)$ can be identified with an $f \in X^*$ and $\langle f, v\rangle_X = \int_\Omega fv$ for any $v \in X$. The above considerations immediately imply

Theorem 3.1. *Let $f \in L^{p^{*\prime}}(\Omega)$. Then the Dirichlet problem* (3.1) *has a unique weak solution $u_f \in X$, i.e.*

$$\int_\Omega |\nabla u_f|^{p-2} \nabla u_f \cdot \nabla v = \int_\Omega fv$$

for any $v \in X$ (or equivalently for any $v \in C_0^\infty(\Omega)$).

For the Neumann problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ |\nabla u|^{p-2} \partial u/\partial \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

(here $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the exterior normal) the situation is different. A weak solution of (3.3) is usually defined by the same integral identity as (3.2) but now with the test space X replaced by $\widetilde{X} := W^{1,p}(\Omega)$, where $W^{1,p}(\Omega) =$

$\{u \in L^p(\Omega);\ \nabla u \in [L^p(\Omega)]^N\}$. Since $\|\nabla \cdot\|_{p;\Omega}$ is only a seminorm on $\widetilde{X}$, we cannot apply Theorem 2.1 as in the case of the Dirichlet problem. Roughly speaking, we have to rule out the constants from $\widetilde{X}$. One possibility is to restrict ourselves (since Ω is bounded) to the subspace $X := \{u \in \widetilde{X};\ \int_\Omega u = 0\}$.

Now, $\|\nabla \cdot\|_{p;\Omega}$ defines a norm on X but additional information about Ω is needed in order to guarantee that $(X, \|\nabla \cdot\|_{p;\Omega})$ is complete. It is proved in [11] that this is the case if and only if the Poincaré inequality

$$\|u\|_{p;\Omega} \leqslant c\|\nabla u\|_{p;\Omega} \qquad \forall\ u \in X \tag{3.4}$$

holds. One of the sufficient conditions for (3.4) to hold is $\partial\Omega \in C^0$ (i.e. for any $x_0 \in \partial\Omega$ there is a neighbourhood $U(x_0) \subset \mathbb{R}^N$ such that $U(x_0) \cap \partial\Omega$ is a C^0 manifold in $\mathbb{R}^N$—see [11]). So, assuming $\partial\Omega \in C^0$, we verify (X1), (X2), and for any $f \in X^*$ there exists a unique $u_f \in X$ satisfying (3.2) with this choice of X.

In order to apply Sobolev's imbedding theorems for X we need now $\partial\Omega \in C^{0,1}$ (the boundary is locally Lipschitzian—this property is defined analogously as $\partial\Omega \in C^0$). Remark also that the norm $\|\nabla \cdot\|_{p;\Omega}$ on X is equivalent to the usual Sobolev norm $\|\cdot\|_{W^{1,p}(\Omega)}$ in this case. If this is the case, any $f \in L^{p^{*\prime}}(\Omega)$ defines $f \in X^*$ satisfying $\langle f, v\rangle_X = \int_\Omega fv$ for any $v \in X$. But now any constant function on Ω is identified with the zero element of $X(X^*)$ and by the same argument any $u \in X$ ($f \in L^{p^{*\prime}}(\Omega)$) is identified with $\tilde{u} = u - \int_\Omega u$ ($\tilde{f} = f - \int_\Omega f$). Thus we have

Theorem 3.2. *Let $\partial\Omega \in C^{0,1}$, $f \in L^{p^{*\prime}}(\Omega)$. Then the Neumann problem* (3.3) *has a unique family of weak solutions $u_{f,c} = u_f + c$, $c \in \mathbb{R}$, where $\int_\Omega u_f = 0$ (i.e.*

$$\int_\Omega |\nabla u_{f,c}|^{p-2}\,\nabla u_{f,c} \cdot \nabla v = \int_\Omega fv$$

for any $v \in W^{1,p}(\Omega)$) if and only if

$$\int_\Omega f = 0.$$

4. The case $\Omega = \mathbb{R}^N$

In this section we discuss the existence of a weak solution of the equation

$$-\Delta_p u = f \text{ in } \mathbb{R}^N. \tag{4.1}$$

For $1 < p < N$ set $\widehat{H}_0^{1,p}(\mathbb{R}^N) := \{u \in L^{1,p}(\mathbb{R}^N);\ u \in L^{p^*}(\mathbb{R}^N)\}$ where $p^* := \frac{Np}{N-p}$. Let us recall some facts from [3], [9], [11] and [12]. In the sense of a direct decomposition we have

$$\begin{cases} L^{1,p}(\mathbb{R}^N) = \widehat{H}_0^{1,p}(\mathbb{R}^N) \oplus \mathbb{R}, \\ u = (u - c_u) + c_u, \\ \text{where } \ (u - c_u) \in \widehat{H}_0^{1,p}(\mathbb{R}^N) \ \text{ and} \\ c_u = \lim\limits_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} u, \text{ where } B_R := \{x \in \mathbb{R}^N;\ |x| < R\}. \end{cases} \tag{4.2}$$

Here, $|B_R|$ denotes the Lebesgue measure of B_R. Moreover, we have

$$\widehat{H}_0^{1,p}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\nabla\cdot\|_{p;\mathbb{R}^N}} \tag{4.3}$$

by the Sobolev imbedding, and $\|\nabla.\|_{p;\mathbb{R}^N}$ is a norm on $X := \widehat{H}_0^{1,p}(\mathbb{R}^N)$ so that X is complete. Thus (X1) and (X2) are verified and we can apply Theorem 2.1. In particular, we have

Theorem 4.1. *Let $f \in L^{p^{*\prime}}(\mathbb{R}^N)$. Then there is a unique $u_f \in X$ such that the integral identity*

$$\int_{\mathbb{R}^N} |\nabla u_f|^{p-2}\, \nabla u_f \cdot \nabla v = \int_{\mathbb{R}^N} fv \tag{4.4}$$

holds for any $v \in X$ (or equivalently for any $v \in C_0^\infty(\mathbb{R}^N)$).

Let us now consider the case $p \geqslant N \geqslant 2$. As is shown in [7], if $f \in L^1(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} f \neq 0$ then there is no $u \in L^{1,p}(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_{\mathbb{R}^N} fv$$

for arbitrary $v \in C_0^\infty(\mathbb{R}^N)$.

A natural question arises: "Does this result contradict Theorem 2.1?" The answer is NO and in the remaining part of this section we will justify it.

Let us recall again some facts from [3], [9], [11] and [12]. For $\emptyset \neq M \subset\subset \mathbb{R}^N$ (i.e. M is an open nonempty and bounded set) define

$$L_M^{1,p}(\mathbb{R}^N) := \left\{u \in L^{1,p}(\mathbb{R}^N);\ \int_M u = 0\right\}.$$

Then in the sense of a direct decomposition

$$\begin{cases} L^{1,p}(\mathbb{R}^N) = L_M^{1,p}(\mathbb{R}^N) \oplus \mathbb{R}, \\ u = (u - m_u) + m_u, \\ m_u := \frac{1}{|M|}\int_M u. \end{cases} \tag{4.5}$$

Moreover, in the case $N \leqslant p < \infty$ we have

$$L_M^{1,p}(\mathbb{R}^N) = \overline{C_{0,M}^{\infty}(\mathbb{R}^N)}^{\|\nabla\cdot\|_{p;\mathbb{R}^N}}, \tag{4.6}$$

where $C_{0,M}^{\infty}(\mathbb{R}^N) := \left\{u \in C_0^{\infty}(\mathbb{R}^N);\ \int_M u = 0\right\}$.

Set $X := L_M^{1,p}(\mathbb{R}^N)$ and let $R > 0$ be such that $M \subset B_{2R}$ and $f \in L^{p'}(\mathbb{R}^N)$ satisfy

$$\int_{\mathbb{R}^N\setminus B_{2R}} |f(x)|^{p'}|x|^{p'}\,\mathrm{d}x < \infty \ \text{ if } \ p > N, \tag{4.7}$$

$$\int_{\mathbb{R}^N\setminus B_{2R}} |f(x)|^{\frac{N}{N-1}}|x|^{\frac{N}{N-1}}\left(\ln\frac{|x|}{R}\right)^{\frac{N}{N-1}}\mathrm{d}x < \infty \ \text{ if } \ p = N. \tag{4.8}$$

Lemma 4.1. *The assumptions of Theorem* 2.1 *are satisfied with X and f given above.*

Proof. Let $\varphi \in C_0^{\infty}(\mathbb{R}^N)$, $\operatorname{supp}\varphi \subset \mathbb{R}^N \setminus \overline{B_{2R}}$. The following auxiliary estimates were proved in [12], Lemma II.9.2, p. 95:

$$\left(\int_{R^N}\frac{|\varphi(x)|^p}{|x|^p}\,\mathrm{d}x\right)^{\frac{1}{p}} \leqslant \frac{p}{|N-p|}\|\nabla\varphi\|_{p;\mathbb{R}^N} \tag{4.9}$$

if $p > 1$, $p \neq N$ and

$$\left(\int_{R^N}\frac{|\varphi(x)|^N}{|x|^N(\ln\frac{|x|}{R})^N}\,\mathrm{d}x\right)^{\frac{1}{N}} \leqslant \frac{N}{|N-1|}\|\nabla\varphi\|_{N;\mathbb{R}^N}. \tag{4.10}$$

Let us also recall the (extended) Poincaré inequality (see [3], estimate (2.12)):

$$\|u\|_{p,B_{R'}} \leqslant c(R,M)\|\nabla u\|_{p,B_{R'}} \tag{4.11}$$

for all $u \in L^{1,p}_M(\mathbb{R}^N)$, valid even for $1 \leqslant p < \infty$ and all R' such that $M \subset B_{R'}$.

We prove that f defines a continuous linear functional on X. Indeed, let $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \eta(x) \leqslant 1$,

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \leqslant 2R, \\ 0 & \text{for } |x| \geqslant 4R. \end{cases}$$

For $\varphi \in X$ we consider

$$\langle f, \varphi\rangle_X = \langle f, \eta\varphi\rangle_X + \langle f, (1-\eta)\varphi\rangle_X.$$

Set $\varphi_1 := \eta\varphi$, $\varphi_2 := (1-\eta)\varphi$. Then

$$|\langle f, \varphi_1\rangle_X| \leqslant \|f\|_{p';\mathbb{R}^N} \|\varphi_1\|_{p;\mathbb{R}^N}$$

and since $\int_M \varphi_1 = 0$, we have

$$\|\varphi_1\|_{p;\mathbb{R}^N} \leqslant c(R,M)\|\nabla\varphi_1|\|_{p;\mathbb{R}^N} \leqslant c(R,M)(\|\,|\eta\nabla\varphi|\,\|_{p;\mathbb{R}^N} + \|\varphi\nabla\eta\|_{p;\mathbb{R}^N}).$$

On the other hand, since $\operatorname{supp}\eta \subset B_{4R}, |\nabla\eta| \leqslant C_R$, we get by (4.11)

$$\|\varphi\nabla\eta\|_{p;\mathbb{R}^N} \leqslant C_R\|\varphi\|_{p,B_{4R}} \leqslant C_R c(R,M)\|\nabla\varphi\|_{p,B_{4R}}$$

and

$$\|\varphi_1\|_{p;\mathbb{R}^N} \leqslant c(R,M)(1 + C_R c(R,M))\|\nabla\varphi\|_{p,B_{4R}}.$$

For φ_2 we get

$$\begin{aligned} |\langle f, \varphi_2\rangle_X| &\leqslant \int_{\mathbb{R}^N} |f(x)||\varphi_2(x)|\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} (|f(x)||x|)(|x|^{-1}|\varphi_2(x)|)\,\mathrm{d}x \\ &\leqslant \left(\int_{\mathbb{R}^N\setminus B_{2R}} |f(x)|^{p'}|x|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}_N} \frac{|\varphi_2(x)|^p}{|x|^p}\,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_{\mathbb{R}^N\setminus B_{2R}} |f(x)|^{p'}|x|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}} \frac{p}{|N-p|}\|\nabla\varphi_2\|_{p;\mathbb{R}^N} \end{aligned}$$

by (4.9) and (4.7) if $p > N$.

Similarly, we get

$$\begin{aligned} |\langle f, \varphi_2\rangle_X| &\leqslant \int_{\mathbb{R}^N} \Big(|f(x)|\,|x|\Big|\ln\frac{|x|}{R}\Big|\Big)\Big(\frac{|\varphi_2(x)|}{|x|\ln(\frac{|x|}{R})}\Big)\,\mathrm{d}x \\ &\leqslant \left(\int_{\mathbb{R}^N\setminus B_{2R}} |f(x)|^{\frac{N}{N-1}}\Big||x|\ln\frac{|x|}{R}\Big|^{\frac{N}{N-1}}\,\mathrm{d}x\right)^{\frac{N-1}{N}} \left(\int_{\mathbb{R}^N} \frac{|\varphi_2(x)|^N}{|x|^N\big|\ln(\frac{|x|}{R})\big|^N}\,\mathrm{d}x\right)^{\frac{1}{N}} \\ &\leqslant \left(\int_{\mathbb{R}^N\setminus B_{2R}} |f(x)|^{\frac{N}{N-1}}\Big||x|\ln\frac{|x|}{R}\Big|^{\frac{N}{N-1}}\,\mathrm{d}x\right)^{\frac{N-1}{N}} \frac{N}{N-1}\|\nabla\varphi_2\|_{N;\mathbb{R}^N} \end{aligned}$$

by (4.10) and (4.8). Now, by (4.11) again

$$\begin{aligned}\|\nabla\varphi_2\|_{p;\mathbb{R}^N} &\leqslant \|\nabla\varphi\|_{p,\mathbb{R}^N\setminus B_{2R}} + C_R\|\varphi\|_{p,B_{4R}} \\ &\leqslant \|\nabla\varphi\|_{p,\mathbb{R}^N\setminus B_{2R}} + C_R c(R,M)\|\nabla\varphi\|_{p,B_{4R}} \leqslant C_1(R,M)\|\nabla\varphi\|_{p;\mathbb{R}^N}.\end{aligned}$$

Thus we have an estimate

$$|\langle f,\varphi\rangle_X| \leqslant c\|\nabla\varphi\|_{p;\mathbb{R}^N}$$

for any $\varphi \in X$, where the constant depends only on $R > 0$, i.e. $f \in X^*$. Since (X1) and (X2) are satisfied, the proof of the lemma is complete. □

R e m a r k 4.1. It follows from Lemma 4.1 and Theorem 2.1 that for any $f \in L^{p'}(\mathbb{R}^N)$ satisfying (4.7) (if $p > N$) and (4.8) (if $p = N$) there exists a unique $u_f \in X$ such that

$$\int_{\mathbb{R}^N} |\nabla u_f|^{p-2}\nabla u_f \cdot \nabla\varphi = \int_{\mathbb{R}^N} f\varphi$$

holds for any $\varphi \in X$.

Theorem 4.2. *Let X and f be as in Lemma* 4.1. *Then $f \in L^1(\mathbb{R}^N)$ and moreover, there is a unique family $u_{f,c} = u_f + c$, $c \in \mathbb{R}$, $u_{f,c} \in L^{1,p}(\mathbb{R}^N)$ satisfying*

$$\int_{\mathbb{R}^N} |\nabla u_{f,c}|^{p-2}\nabla u_{f,c} \cdot \nabla\varphi = \int_{\mathbb{R}} f\varphi \tag{4.12}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ if and only if

$$\int_{\mathbb{R}^N} f = 0.$$

P r o o f. Let $p > N$. Then it follows from Hölder's inequality that for any $T > 2R$ we have

$$\begin{aligned}\int_{\{2R\leqslant|x|\leqslant T\}} |f(x)|\,\mathrm{d}x &= \int_{\{2R\leqslant|x|\leqslant T\}} |f(x)||x|\ |x|^{-1}\,\mathrm{d}x \\ &\leqslant \Big(\int_{\mathbb{R}^N\setminus B_{2R}} |f(x)|^{p'}|x|^{p'}\,\mathrm{d}x\Big)^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^N\setminus B_{2R}} |x|^{-p}\,\mathrm{d}x\Big)^{\frac{1}{p}},\end{aligned}$$

$$\int_{\{2R\leqslant|x|\leqslant T\}} |x|^{-p}\,\mathrm{d}x = \omega_N\int_{2R}^T r^{N-1-p}\,\mathrm{d}r \leqslant \frac{\omega_N}{p-N}(2R)^{N-p}.$$

(Here ω_N is the measure of the unit sphere in $\mathbb{R}^N$.)

Let $p = N$. Then from Hölder's inequality we have for any $T > 2R$

$$\int_{\{2R\leqslant|x|\leqslant T\}} |f(x)|\,\mathrm{d}x \leqslant \Big(\int_{R^N\setminus B_{2R}} |f(x)|^{\frac{N}{N-1}}|x|^{\frac{N}{N-1}}\Big(\ln\frac{|x|}{R}\Big)^{\frac{N}{N-1}}\,\mathrm{d}x\Big)^{\frac{N-1}{N}}$$
$$\times\Big(\int_{R^N\setminus B_{2R}} |x|^{-N}\Big(\ln\frac{|x|}{R}\Big)^{-N}\,\mathrm{d}x\Big)^{\frac{1}{N}},$$

$$\int_{\{2R\leqslant|x|\leqslant T\}} |x|^{-N}\Big(\ln\frac{|x|}{R}\Big)^{-N}\,\mathrm{d}x$$
$$= \omega_N\int_{2R}^{T} r^{-1}\Big(\ln\frac{|r|}{R}\Big)^{-N}\,\mathrm{d}r \leqslant \frac{\omega_N}{R(N-1)}(\ln 2)^{1-N}.$$

Hence from $f \in L^{p'}(\mathbb{R}^N)$, (4.7) (if $p > N$) and (4.8) (if $p = N$) we get that $f \in L^1(\mathbb{R}^N)$.

Assume now $\int_{\mathbb{R}^N} f = 0$. As mentioned above any $\varphi \in L^{1,p}(\mathbb{R}^N)$ splits as

$$\varphi = (\varphi - m_\varphi) + m_\varphi,$$

where $m_\varphi = \frac{1}{|M|}\int_M \varphi$. Then

$$\int_{\mathbb{R}^N} f m_\varphi = 0 = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\cdot\nabla m_\varphi,$$

which together with the fact that (4.11) holds for any $\varphi \in X$ (cf. Remark 4.1) yields

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = \int_{\mathbb{R}^N} f\varphi \tag{4.13}$$

for any $\varphi \in L^{1,p}(\mathbb{R}^N)$ and, in particular, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$.

If conversely, (4.13) holds for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ then we can choose $\varphi = g_k$, where $g_k \in C_0^\infty(\mathbb{R}^N), 0 \leqslant g_k \leqslant 1, g_k(x) = 1$ for $|x| \leqslant k$ and $\|\nabla g_k\|_{p;\mathbb{R}^N} \to 0$ as $k \to \infty$ (cf. [3]). Then

$$\int_{\mathbb{R}_N} |\nabla u|^{p-2}\nabla u\cdot\nabla g_k \to 0 \tag{4.14}$$

and since $fg_k \to f$ a.e. in $\mathbb{R}^N$, $|fg_k| \leqslant |f|$, by Lebesgue's theorem we conclude

$$\int_{\mathbb{R}^N} fg_k \to \int_{\mathbb{R}^N} f.$$

On the other hand, by (4.13), (4.14) $\int_{\mathbb{R}^N} fg_k \to 0$, i.e. $\int_{\mathbb{R}^N} f = 0$. □

Let us assume that $p > N$. Then due to the Morrey estimate (see [6], Theorem 7.17) the space $L^{1,p}_M(\mathbb{R}^N)$ is isometrically isomorphic to

$$\begin{aligned}(4.15)\qquad \widehat{H}^{1,p}_\bullet(\mathbb{R}^N) := \{u \in L^{1,p}(\mathbb{R}^N)\colon\ &|u(x) - u(y)| \\ &\leqslant C(N,p)\|\nabla u\|_{p;\mathbb{R}^N}|x-y|^{1-\frac{N}{p}} \quad \forall x,y \in \mathbb{R}^N, u(0) = 0\}.\end{aligned}$$

The corresponding isometric isomorphism $J_p\colon L^{1,p}_M(\mathbb{R}^N) \to \widehat{H}^{1,p}_\bullet(\mathbb{R}^N)$ is defined by

$$(J_p\tilde{u})(x) := \tilde{u}(x) - \tilde{u}(0),$$

where $\tilde{u}$ denotes the unique continuous respresentative belonging to the equivalence class $u \in L^{1,p}_M(\mathbb{R}^N)$.

Hence for $p > N$ we can alternatively set $X = \widehat{H}^{1,p}_\bullet(\mathbb{R}^N)$ and $(X, \|\nabla \cdot\|_{p;\mathbb{R}^N})$ satisfies (X1) and (X2).

Let $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N\,;\ |x| > 0\}$ and

$$(4.16)\qquad D_{N,p}(\mathbb{R}^N) := \Big\{f \in L^1_{\text{loc}}(\mathbb{R}^N_+)\,;\ \int_{\mathbb{R}^N} |f(x)|\ |x|^{1-\frac{N}{p}}\,\mathrm{d}x < \infty\Big\}.$$

Then by

$$\|f\|_{D_{N,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |f(x)|\ |x|^{1-\frac{N}{p}}\,\mathrm{d}x$$

a norm is defined and $\big(D_{N,p}(\mathbb{R}^N), \|\cdot\|_{D_{N,p}(\mathbb{R}^N)}\big)$ is a Banach space.

Let $u \in X$ and $f \in D_{N,p}(\mathbb{R}^N)$. It follows from (4.15) and (4.16) that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f(x)u(x)\,\mathrm{d}x\right| &\leqslant C(N,p)\|\nabla u\|_{p;\mathbb{R}^N}\int_{\mathbb{R}^N}|f(x)|\ |x|^{1-\frac{N}{p}}\,\mathrm{d}x \\ &= C(N,p)\|f\|_{D_{N,p}(\mathbb{R}^N)}\|\nabla u\|_{p;\mathbb{R}^N},\end{aligned}$$

i.e. $D_{N,p}(\mathbb{R}^N) \subset X^*$.

Theorem 4.3. *Let $p > N$ and X be as above. Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and assume that for some $q > p$ the inequality*

$$\int_{\mathbb{R}^N\setminus B_1} |f(x)|^{q'}|x|^{q'}\,\mathrm{d}x < \infty$$

holds. Then there exists a unique family $u_{f,c} = u_f + c$, $c \in \mathbb{R}$, $u_f \in X$, $X = \widehat{H}^{1,p}_\bullet(\mathbb{R}^N)$, $u_{f,c} \in L^{1,p}(\mathbb{R}^N)$, satisfying

$$\int_{\mathbb{R}^N} |\nabla u_{f,c}|^{p-2}\nabla u_{f,c}\cdot\nabla\varphi = \int_{\mathbb{R}^N} f\varphi$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ *if and only if*

$$\int_{\mathbb{R}^N} f = 0.$$

P r o o f. We prove that $f \in D_{N,p}(\mathbb{R}^N)$. Indeed, by Hölder's inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |f(x)||x|^{1-\frac{N}{p}}\,\mathrm{d}x &\leqslant \int_{B_1} |f(x)|\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus B_1} |f(x)|\,|x|^{1-\frac{N}{p}}\,\mathrm{d}x \\ &\leqslant \|f\|_{1,B_1} + \Big(\int_{\mathbb{R}^N\setminus B_1} |f(x)|^{q'}|x|^{q'}\,\mathrm{d}x\Big)^{\frac{1}{q'}}\Big(\int_{\mathbb{R}^N\setminus B_1} |x|^{-N\frac{q}{p}}\,\mathrm{d}x\Big)^{\frac{1}{p}}.\end{aligned}$$

The rest of the proof follows the lines of the proof of Theorem 4.2. □

R e m a r k 4.2. Our Theorems 4.2 and 4.3 generalize a necessary condition given in [7]. In particular, we get from here that any constant is a weak solution of

$$-\Delta_p u = 0 \quad \text{in} \quad \mathbb{R}^N.$$

5. The case of an exterior domain

Let $G := \mathbb{R}^N \setminus \overline{K}$, where $\emptyset \neq K \subset\subset \mathbb{R}^N$, $0 \in K$. Let us consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = f \text{ in } G, \\ u = 0 \qquad \text{on } \partial G. \end{cases} \tag{5.1}$$

We want to prove existence and uniqueness of a weak solution of (5.1). Define the space

$$\widehat{H}_0^{1,p}(G) := \overline{C_0^\infty(G)^{\|\nabla\cdot\|_{p;G}}}.$$

Let $1 < p < N$. Then due to the Sobolev imbedding we have $\widehat{H}_0^{1,p}(G) \hookrightarrow L^{p^*}(G)$ and therefore $X := \widehat{H}_0^{1,p}(G)$ verifies (X1) and (X2). We can apply the abstract Theorem 2.1 and, in particular, we have the following result.

Theorem 5.1. *Let* $f \in L^{p^{*\prime}}(G)$ *be given. Then there is a unique* $u_f \in X$ *such that*

$$\int_G |\nabla u_f|^{p-2}\nabla u_f \cdot \nabla\varphi = \int_G f\varphi \tag{5.2}$$

holds for any $\varphi \in X$ *(or equivalently, for any* $\varphi \in C_0^\infty(G)$*).*

Let $p \geqslant N$. Then $\widehat{H}_0^{1,p}(G)$ coincides with the space

$$\widehat{H}_{\bullet}^{1,p}(G) := \{u \in L^{1,p}(G);\ u \in L^p(G_R) \text{ for every } R > 0 \text{ and } \\ \eta u \in W_0^{1,p}(G) \text{ for any } \eta \in C_0^\infty(\mathbb{R}^N)\},$$

where $G_R = G \cap B_R$ (see [12], Theorems I. 2.7, I. 2.16). Now, we can literally follow the approach from Section 4, case $p \geqslant N$, to get the following result.

Theorem 5.2. *Let $f \in L^{p'}(G)$, let f satisfy* (4.7) *for $p > N$ and* (4.8) *for $p = N$. Then there exists a unique $u_f \in X$ such that* (5.2) *holds for any $\varphi \in X$ (or equivalently, for any $\varphi \in C_0^\infty(G)$).*

Remark 5.1. Let us point out that contrary to the case of the whole of $\mathbb{R}^N$ we do not need any additional condition of the type "$\int f = 0$" because the constants are ruled out due to the homogeneous Dirichlet boundary conditions.

Let us consider the Neumann problem

$$\begin{cases} -\Delta_p u = f \text{ in } G, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial G. \end{cases}$$

Choose M such that $\emptyset \neq M \subset\subset G$. Then a subspace of $L^{1,p}(G)$ is given by

$$L_M^{1,p}(G) := \left\{u_0 \in L^{1,p}(G);\ \int_M u_0 = 0\right\} \tag{5.3}$$

and in the sense of a direct sum

$$\begin{gathered} L^{1,p}(G) = L_M^{1,p}(G) \oplus \mathbb{R}, \\ u = u_0 + m_u \end{gathered} \tag{5.4}$$

where

$$m_u := |M|^{-1} \int_M u, \qquad u_0 := u - m_u. \tag{5.5}$$

By

$$|u|_{1,p;G;M} := \|\nabla u\|_{p;G} + \left|\int_M u\right| \tag{5.6}$$

a norm is defined on $L^{1,p}(G)$ (see [9], Lemma 4.1) such that $L^{1,p}(G)$ equipped with this norm is a reflexive Banach space (see [9], Theorem 4.5).

Clearly, for $u_0 \in L^{1,p}_M(G)$ we have

$$|u_0|_{1,p;G;M} = \|\nabla u_0\|_{p;G}. \tag{5.7}$$

We assume now that $\partial G \in C^0$ and choose $R_0 = R_0(M,K) > 0$ so that $\overline{M} \subset B_{R_0}$ and $\overline{K} \subset B_{R_0}$ and we write $G_{R_0} := G \cap B_{R_0}$. By [11], Lemma 4.2, for $u \in L^{1,p}(G)$, we see that $u|_{G_{R_0}} \in L^p(G_{R_0})$ and there exist $G' \subset\subset G$ and a constant $C_{R_0} > 0$ such that

$$\|u\|_{p;G_{R_0}} \leqslant C_{R_0}(\|\nabla u\|_{p;G} + \|u\|_{p;G'}) \quad \forall u \in L^{1,p}(G). \tag{5.8}$$

Because of the Poincaré-type inequality

$$\|u_0\|_{p;G'} \leqslant C_{G'}\|\nabla u_0\|_{p;G} \quad \forall u_0 \in L^{1,p}_M(G) \tag{5.9}$$

(with $C_{G'} = C(G', G, p) > 0$, see [9], Theorem 5.1), by (5.8), (5.9) we get

$$\|u_0\|_{p;G_{R_0}} \leqslant C_1\|\nabla u_0\|_{p;G} \quad \forall u_0 \in L^{1,p}_M(G) \tag{5.10}$$

with $C_1 := C_{R_0}(1 + C_{G'}) > 0$, and so

$$\|u_0\|_{W^{1,p}(G_{R_0})} \leqslant (1 + C_1^p)^{\frac{1}{p}}\|\nabla u_0\|_{p;G} \quad \forall u_0 \in L^{1,p}_M(G). \tag{5.11}$$

Lemma 5.1. *Assume that $\partial G \in C^{0,1}$ (e.g. $\partial G = \partial K$ is a Lipschitz manifold). Then there exists a linear extension*

$$E\colon\ L^{1,p}_M(G) \to L^{1,p}_M(\mathbb{R}^N)$$

such that $Eu_0|_G = u_0 \ \ \forall u_0 \in L^{1,p}_M(G)$. In addition, there is a constant $C_E > 0$ such that

$$\|\nabla E u_0\|_{p;\mathbb{R}^N} \leqslant C_E\|\nabla u_0\|_{p;G} \quad \forall u_0 \in L^{1,p}_M(G). \tag{5.12}$$

P r o o f. a) Because of $\partial G \in C^{0,1}$, there exists a linear extension

$$\widetilde{E}\colon\ W^{1,p}(G_{R_0}) \to W^{1,p}_0(\mathbb{R}^N),$$
$$\widetilde{E}v|_{G_{R_0}} = v \quad \forall v \in W^{1,p}(G_{R_0})$$

and a constant $\widetilde{C} = \widetilde{C}(G_{R_0}, p) > 0$ such that

$$\|\widetilde{E}v\|_{W^{1,p}(\mathbb{R}^N)} \leqslant \widetilde{C}\|v\|_{W^{1,p}(G_{R_0})} \quad \forall v \in W^{1,p}(G_{R_0}) \tag{5.13}$$

(see e.g. [10], Thèoréme 3.9).

b) As we mentioned above, $u_0 \in L^{1,p}_M(G)$ implies $u_0|_{G_{R_0}} \in W^{1,p}(G_{R_0})$. With help of $\widetilde{E}$ we define

$$(5.14)\qquad (Eu_0)(x) := \begin{cases} u_0(x) & \text{for } x \in G, \\ \widetilde{E}(u_0|_{G_{R_0}})(x) & \text{for } x \in \mathbb{R}^N \setminus \overline{G} = K. \end{cases}$$

Since $M \subset\subset G$ it is clear that $Eu_0 \in L^{1,p}_M(\mathbb{R}^N)$ for $u_0 \in L^{1,p}_M(G)$ and $Eu_0|_G = u_0$. By (5.11) and (5.13) we see

$$\begin{aligned} \|\nabla Eu_0\|_{p;\mathbb{R}^N} &\leqslant \|\nabla u_0\|_{p;G} + \|\nabla \widetilde{E}(u_0|_{G_{R_0}})\|_{p;K} \\ &\leqslant \|\nabla u_0\|_{p;G} + \widetilde{C}\|u_0\|_{W^{1,p}(G_{R_0})} \leqslant C_E\|\nabla u_0\|_{p;G} \end{aligned}$$

with $C_E := 1 + \widetilde{C}(1 + C_1^p)^{\frac{1}{p}}$. □

Obviously we get

Corollary 5.1. *Let $\partial G \in C^{0,1}$. Then*

$$(5.15)\qquad L^{1,p}_M(G) = \{v|_G\,;\ v \in L^{1,p}_M(\mathbb{R}^N)\}.$$

Let $\partial G \in C^{0,1}$. Due to (5.4) any $u \in L^{1,p}(G)$ can be written as $u = u_0 + m_u$. Define a linear map $E_1\colon\ L^{1,p}(G) \to L^{1,p}(\mathbb{R}^N)$ by

$$(5.16)\qquad E_1 u := Eu_0 + m_u.$$

Then $E_1 u|_G = u \quad \forall u \in L^{1,p}(G)$.

This extension enables us to apply the result found for the whole space $\mathbb{R}^N$ to the underlying case. But the price we have to pay is the assumption $\partial G \in C^{0,1}$. On the other hand, without any regularity assumptions on ∂G we never may expect any imbedding theorems for G.

Let $1 < p < N$. We recall the decomposition (4.2) and the density property (4.3).

Lemma 5.2. *Let $\partial G \in C^{0,1}$ and*

$$(5.17)\qquad \widehat{H}^{1,p}(G) := \{u^* \in L^{1,p}(G);\ u^* \in L^{p^*}(G)\}.$$

Then in the sense of a direct decomposition

$$(5.18)\qquad \begin{gathered} L^{1,p}(G) = \widehat{H}^{1,p}(G) \oplus \mathbb{R}, \\ u = u^* + c_u \end{gathered}$$

where ($G_R := G \cap B_R$),

$$c_u := \lim_{\substack{R\to\infty\\ R>R_0}} \frac{1}{|G_R|} \int_{G_R} u. \tag{5.19}$$

Further, the map $J\colon L^{1,p}_M(G) \to \widehat{H}^{1,p}(G)$, $Ju := u^*$, *is an isometric isomorphism and* $(\widehat{H}^{1,p}(G), \|\nabla\cdot\|_p)$ *is a reflexive Banach space.*

With $C_{\mathrm{SOB}} > 0$ *(the constant for the Sobolev imbedding) and* $C_E > 0$ *from* (5.12), *we have*

$$\|u^*\|_{p^*;G} \leqslant C_{\mathrm{SOB}} C_E \|\nabla u^*\|_{p;G} \quad \forall u \in \widehat{H}^{1,p}(G). \tag{5.20}$$

Further, $\widehat{H}^{1,p}(G) = \{v^*|_G;\ v^* \in \widehat{H}^{1,p}_0(\mathbb{R}^N)\}$.

Let

$$C^\infty_0(\overline{G}) := \{\Phi \in C^\infty(\overline{G});\ \exists R_\Phi \geqslant R_0\colon\ \Phi(x) = 0 \text{ for } |x| \geqslant R_\Phi\}. \tag{5.21}$$

Then

$$\{\psi|_G;\ \psi \in C^\infty_0(\mathbb{R}^N)\} \subset C^\infty_0(\overline{G}) \subset \widehat{H}^{1,p}(G) \tag{5.22}$$

and

$$\widehat{H}^{1,p}(G) = \overline{C^\infty_0(\overline{G})}^{\|\nabla\cdot\|_{p;G}}. \tag{5.23}$$

P r o o f. a) If $u \in L^{1,p}(G)$, $u = u_0 + m_u$, then by virtue of (5.4), with $u_0 \in L^{1,p}_M(G)$ and $m_u \in \mathbb{R}$, we have $v := E_1 u = E u_0 + m_u \in L^{1,p}(\mathbb{R}^N)$. By (4.2), $v = v^* + c_v$ with $v^* \in \widehat{H}^{1,p}_0(\mathbb{R}^N)$ and $c_v \in \mathbb{R}$. Let $u^* := v^*|_G = (v - c_v)|_G = u - c_v = u_0 + m_u - c_v$. Therefore $u = u^* + c_v$. Since $u^* \in L^{p^*}(G)$, we get

$$\begin{aligned}\left| |G_R|^{-1} \int_{G_R} u - c_v \right| &= |G_R|^{-1} \left| \int_{G_R} (u(y) - c_v)\,\mathrm{d}y \right| \leqslant |G_R|^{-1} \|u^*\|_{p^*;G_R} |G_R|^{\frac{p^*-1}{p^*}} \\ &= \|u^*\|_{p^*;G}\ |G_R|^{-\frac{1}{p^*}} \to 0 \quad (R \to \infty).\end{aligned}$$

Hence $c_v = c_u = \lim\limits_{\substack{R\to\infty\\ R>R_0}} |G_R|^{-1} \int\limits_{G_R} u$.

If $u^* \in \widehat{H}^{1,p}(G) \cap \mathbb{R}$ then because of $|G| = \infty$ we have $u^* = 0$, proving (5.18), (5.19). If conversely $u^* \in \widehat{H}^{1,p}(G) \subset L^{1,p}(G)$ is given then $u^* = u_0 + m_u$, $u_0 \in L^{1,p}(G)$, $m_u \in \mathbb{R}$. Then $E_1 u^* = E_1 u_0 + m_u =: v$. Then $v = v^* + c_v$, $v^* \in \widehat{H}^{1,p}_0(\mathbb{R}^N)$, $c_v \in \mathbb{R}$. Further $u^* = v|_G = v^*|_G + c_v$. Then $c_v = (u^* - v^*|_G) \in L^{p^*}(G) \cap \mathbb{R}$ and again by

$|G| = \infty$ we see that $c_v = 0$, that is $u^* = v^*|_G$, proving $\widehat{H}^{1,p}(G) = \{v^*|_G\,;\ v^* \in \widehat{H}_0^{1,p}(\mathbb{R}^N)\}$.

Moreover, we derive (5.20) from

$$\begin{aligned}\|u^*\|_{p^*;G} &\leqslant \|v^*\|_{p^*;\mathbb{R}^N} \leqslant C_{\mathrm{SOB}}\|\nabla v^*\|_{p;\mathbb{R}^N}\\ &= C_{\mathrm{SOB}}\|\nabla v\|_{p;\mathbb{R}^N} = C_{\mathrm{SOB}}\|\nabla E u_0\|_{p;\mathbb{R}^N}\\ &\leqslant C_{\mathrm{SOB}}C_E\|\nabla u_0\|_{p;G} = C_{\mathrm{SOB}}C_E\|\nabla u^*\|_{p;G}\end{aligned}$$

and therefore completeness of $\widehat{H}^{1,p}(G)$ follows. If $u^* \in \widehat{H}^{1,p}(G)$, $u^* = v^*|_G$ with $v^* \in \widehat{H}^{1,p}(\mathbb{R}^N)$, then by (4.3) there exists a sequence $(v_k) \subset C_0^\infty(\mathbb{R}^N)$ with $\|\nabla v^* - \nabla v_k\|_{p,\mathbb{R}^N} \to 0$. Then $\Phi_k := v_k|_{\overline{G}} \in C_0^\infty(\overline{G})$ and

$$\|\nabla u^* - \nabla\Phi_k\|_{p;G} \leqslant \|\nabla u^* - \nabla v_k\|_{p;\mathbb{R}^N} \to 0,$$

which proves (5.23). Finally, the properties of the map $J\colon L_M^{1,p}(G) \to \widehat{H}^{1,p}(G)$ are obvious. □

Lemma 5.3. *Let $G \subset \mathbb{R}^N$ be a domain with $|G| = \infty$ and let $1 < p < N$. Let us suppose conversely that $\widehat{H}^{1,p}(G)$ defined by (5.17) is complete with respect to the $\|\nabla\cdot\|_{p;G}$-norm. Then there is a constant $C > 0$ such that*

$$\|u\|_{p^*;G} \leqslant C\|\nabla u\|_{p;G} \quad \forall u \in \widehat{H}^{1,p}(G). \tag{5.24}$$

Proof. Let $\mathcal{T}\colon \widehat{H}^{1,p}(G) \to L^{p^*}(G)$ be defined by $\mathcal{T}u^* := u^*$ $\forall u^* \in \widehat{H}^{1,p}(G)$. Suppose that $(u_j^*) \subset \widehat{H}^{1,p}(G)$ and $u^* \in \widehat{H}^{1,p}(G)$ with $\|\nabla u^* - \nabla u_j^*\|_{p;G} \to 0$. Suppose in addition that there is $v \in L^{p^*}(G)$ with

$$\|v - \mathcal{T}u_j^*\|_{p^*;G} = \|v - u_j^*\|_{p^*;G} \to 0.$$

Then for $\Phi \in C_0^\infty(G)$ and $i = 1, \dots, N$ we have

$$\int_G v\partial_i\Phi = \lim_{j\to\infty}\int_G u_j^*\partial_i\Phi = -\lim_{j\to\infty}\int \Phi\partial_i u_j^* = -\int_G \Phi\partial_i u^*,$$

proving that v has the weak derivatives $\partial_i u^*$. Then $\nabla v = \nabla u^*$ and therefore, since G is a domain, $u^* = v + c$. Since $u^*, v \in L^{p^*}(G)$ and $|G| = \infty$ we see that $c = 0$ and $v = u^*$. This proves closedness of $\mathcal{T}$ and since $D(\mathcal{T}) = \widehat{H}^{1,p}(G)$ by Banach's closed graph theorem the boundedness of $\mathcal{T}$ and therefore (5.24) follow. □

Theorem 5.3. *Let $G \subset \mathbb{R}^N$ be an exterior domain with $\partial G \in C^{0,1}$ and $X := \widehat{H}^{1,p}(G)$. Given $f \in L^{p^{*\prime}}(\mathbb{R}^N)$ there exists a unique $u_f \in X$ such that*

$$\int_G |\nabla u_f|^{p-2} \nabla u_f \cdot \nabla v = \int_G f v \quad \forall v \in X.$$

Proof. By (5.20), for $v \in X$ we have

$$\left| \int_G f v \right| \leqslant \|f\|_{p^{*\prime};G} C_{\mathrm{SOB}} C_E \|\nabla v\|_{p;G}.$$

$\square$

Let $p \geqslant N$. We recall (4.6). Then the following assertion holds.

Lemma 5.4. *Let $G \subset \mathbb{R}^N$ be an exterior domain with $\partial G \in C^{0,1}$. Let $\emptyset \neq M \subset\subset G$ and*

$$C^{\infty}_{0,M}(\overline{G}) := \Big\{ \Phi \in C^{\infty}(\overline{G});\ \int_M \Phi \,\mathrm{d}y = 0 \text{ and } \exists R_\Phi > 0\colon\ \Phi(x) = 0 \text{ for } |x| \geqslant R_\Phi \Big\}. \tag{5.25}$$

Then $\{\Phi|_G;\ \Phi \in C^{\infty}_{0,M}(\mathbb{R}^N)\} \subset C^{\infty}_{0,M}(\overline{G})$ and for $p \geqslant N$ we have

$$L^{1,p}_M(G) = \overline{\{\Phi|_G\colon\ \Phi \in C^{\infty}_{0,M}(\mathbb{R}^N)\}}^{\|\nabla\cdot\|_{p;G}} \tag{5.26}$$

and

$$L^{1,p}_M(G) = \{v|_G;\ v \in L^{1,p}_M(\mathbb{R}^N)\}. \tag{5.27}$$

Proof. If $u \in L^{1,p}_M(G)$ then $Eu \in L^{1,p}_M(\mathbb{R}^N)$ and by (4.6) there exists a sequence $(\Phi_k) \subset C^{\infty}_{0,M}(\mathbb{R}^N)$ with $\|\nabla Eu - \nabla \Phi_k\|_{p;\mathbb{R}^N} \to 0$. $\square$

Theorem 5.4. *Let $X := L^{1,p}_M(G)$. Let $R \geqslant R_0(G)$ and suppose that $f \in L^{p'}(G)$ satisfies (4.7) if $p > N$ or (4.8) if $p = N$. Then there exists a unique $u \in L^{1,p}_M(G)$ with*

$$\int_G |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_G f v \qquad \forall v \in L^{1,p}_M(G). \tag{5.28}$$

Further, (5.28) holds even for all $v \in C^{\infty}_0(\overline{G})$ if and only if $\int_G f = 0$.

P r o o f. a) Existence is clear.

b) If $\int f = 0$, then $v \in C_0^\infty(\overline{G})$ may be decomposed into $v = v_0 + m_v$, $v_0 \in L_M^{1,p}(G)$, $m_v \in \mathbb{R}$. Since $\int_G f m_v = 0$ and $\nabla m_v = 0$, (5.28) holds for $v \in C_0^\infty(\overline{G})$, too. Conversely, consider again the sequence $(\eta_k) \subset C_0^\infty(\mathbb{R}^N)$ with $\eta_k|_{B_R} \to 1$ $(k \to \infty)$ uniformly for every fixed $R > 0$ and $\|\nabla \eta_k\|_{p;\mathbb{R}^N} \to \infty$. Then with $v := \eta_k$ we conclude from (5.28) for $k \to \infty$: $\int_G f = 0$. □

In the case $N < p < \infty$ we have an additional "realization" of $L_M^{1,p}(G)$ corresponding to the case $G = \mathbb{R}^N$.

Lemma 5.5. *Let $G \subset \mathbb{R}^N$ be an exterior domain with $\partial G \in C^{0,1}$ and let $N < p < \infty$. Let $x_0 \in G$ be fixed and let*

$$\widehat{H}^{1,p}_{\{x_o\}}(G) := \{\tilde{u} \in L^{1,p}(G);\ |\tilde{u}(x) - \tilde{u}(y)| \leqslant C(N,p)|x-y|^{1-\frac{N}{p}} \|\nabla \tilde{u}\|_{p;G}\ \forall x, y \in \overline{G},\ \text{and } \tilde{u}(x_0) = 0\}. \tag{5.29}$$

Then $\widehat{H}^{1,p}_{\{x_o\}}(G)$ equipped with the norm $\|\nabla \tilde{u}\|_{p,G}$ is a reflexive Banach space,

$$\widehat{H}^{1,p}_{\{x_0\}}(G) = \{(\tilde{v} - \tilde{v}(x_o))|_{\overline{G}};\ \tilde{v} \in \widehat{H}^{1,p}_{\bullet}(\mathbb{R}^N_+)\}$$

(with $\widehat{H}^{1,p}_{\bullet}(\mathbb{R}^N_+)$ by (4.15)*), and there is an isometrically isomorphic map I_p: $L_M^{1,p}(G) \to \widehat{H}^{1,p}_{\{x_o\}}(G)$.*

P r o o f. If $u \in L_M^{1,p}(G)$ then $v := Eu \in L_M^{1,p}(\mathbb{R}^N)$. Denote by $\tilde{w}$ the unique Hölder continuous representative of v. Then $\tilde{v} := (\tilde{w} - \tilde{w}(0)) \in \widehat{H}^{1,p}_{\bullet}(\mathbb{R}^N)$ and $\tilde{u} := (\tilde{v} - \tilde{v}(x_0)) \in \widehat{H}^{1,p}_{\bullet}(G)$. Clearly, if $\tilde{u} \in \widehat{H}^{1,p}_{\{x_0\}}(G)$ then $E\tilde{u} \in L_M^{1,p}(\mathbb{R}^N)$ and

$$\tilde{v} := E\tilde{u} - (E\tilde{u})(0) \in \widehat{H}^{1,p}_{\bullet}(\mathbb{R}^N_+), \quad \tilde{u} = (\tilde{v} - \tilde{v}(x_0))|_G.$$

Further, the map $I_p u := (E\tilde{u} - E\tilde{u}(x_0))$, $I_p : L_M^{1,p}(G) \to \widehat{H}^{1,p}_{\{x_0\}}(G)$ is an isometric isomorphism. □

Theorem 5.5. *Let $G \subset \mathbb{R}^N$ be an exterior domain with $\partial G \in C^{0,1}$ and $0 \in \mathbb{R}^N \setminus \overline{G}$ and let $N < p < \infty$. Let $f \in L^1_{\mathrm{loc}}(G)$ and assume that for some $q > p$,*

$$\int_G |f(x)|^{q'} |x|^{q'} \,\mathrm{d}x < \infty.$$

Then there exists a unique family $u_{f,c} = u_f + c$ with $u_f \in X := \widehat{H}^{1,p}_{\{x_0\}}(G)$ and $c \in \mathbb{R}$ satisfying

$$\int_G |\nabla u_{f,c}|^{p-2}\, \nabla u_{f,c}.\nabla \varphi = \int_G f\varphi \qquad \forall \varphi \in C_0^\infty(\overline{G})$$

(see (5.21)*) if and only if $\int_G f = 0$.*

P r o o f. The proof is performed analogously to that of Theorem 4.3. □

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