RADICAL CLASSES OF DISTRIBUTIVE LATTICES HAVING THE LEAST ELEMENT

Ján Jakubík, Košice

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Abstract. Let $\mathcal{D}$ be the system of all distributive lattices and let $\mathcal{D}_0$ be the system of all $L \in \mathcal{D}$ such that L possesses the least element. Further, let $\mathcal{D}_1$ be the system of all infinitely distributive lattices belonging to $\mathcal{D}_0$. In the present paper we investigate the radical classes of the systems $\mathcal{D}$, $\mathcal{D}_0$ and $\mathcal{D}_1$.

Keywords: distributive lattice, infinite distributivity, radical class

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1. Introduction

Radical classes of generalized Boolean algebras have been studied in [15]. A nonempty subclass X of the class $\mathcal{B}_0$ of all generalized Boolean algebras is called a radical class if it is closed with respect to isomorphisms, convex subalgebras and joins of convex subalgebras.

Earlier, radical classes of other types of ordered algebraic structures have been dealt with (under the definitions analogous to that given above); namely, the radical classes of lattice ordered groups and linearly ordered groups (cf. [1], [3]–[13], [17], [21]; see also Section 9.5 of the monograph [19]), convergence lattice ordered groups [14], cyclically ordered groups [18] and MV-algebras [16].

The reviewer of the paper [15] (Math. Reviews, 99m:06024) proposed to investigate the question what are radical classes in the systems of distributive lattices, lattice ordered semi-rings and autometrized algebras.

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Let $\mathcal{D}$ be the system of all distributive lattices. Further, let $\mathcal{D}_0$ be the system of all $L \in \mathcal{D}$ such that L has the least element.

The aim of the present paper is to investigate the radical classes in the systems $\mathcal{D}$ and $\mathcal{D}_0$.

We show that there exists exactly one radical class in the system $\mathcal{D}$ (namely, $\mathcal{D}$) and that there are many radical classes of elements of $\mathcal{D}_0$ (namely, there exists an injective mapping of the class of all infinite cardinals into $R(\mathcal{D}_0)$, where $R(\mathcal{D}_0)$ is the collection of all radical classes of elements of $\mathcal{D}_0$.)

We deal also with the radical classes of the system $\mathcal{D}_1$ consisting of all infinitely distributive lattices which have the least element; here we generalize a result of [15].

2. Preliminaries

We start by considering the system $\mathcal{D}$. Let $L \in \mathcal{D}$ and $\emptyset \neq L_1 \subseteq L$. If L_1 is a sublattice of L, then it is said to be a subalgebra of L (with respect to $\mathcal{D}$). A subset $L_2 \subseteq L$ is *convex* in L if $z \in L_2$, whenever $z_1, z_2 \in L_2$, $x \in L$ and $z_1 \leqslant x \leqslant z_2$.

We denote by $c(L)$ the system of all convex subalgebras of L. Further, let $c'(L) = c(L) \cup \{\emptyset\}$. The system $c'(L)$ is partially ordered by the set-theoretical inclusion.

Let $\emptyset \neq \{L_i\}_{i\in I} \subseteq c'(L)$. Put $L^1 = \bigcap\limits_{i\in I} L_i$, $L^2 = \bigcap\limits_{j\in J} L'_j$, where $\{L'_j\}_{j\in J}$ is the system of all upper bounds of $\{L_i\}_{i\in I}$ in $c'(L)$. Then L^1 and L^2 is the meet and the join, respectively, of the system $\{L_i\}_{i\in I}$ in $c'(L)$. Hence $c'(L)$ is a complete lattice. We denote $L^1 = \bigwedge\limits_{i\in I}^{1} L_i$, $L^2 = \bigvee\limits_{i\in I}^{1} L_i$. It is clear that $L^2 \supseteq \bigcup\limits_{i\in I} L_i$.

2.1. Definition. A nonempty class X of distributive lattices is called a *radical class* (with respect to $\mathcal{D}$) if it satisfies the following conditions:

(i) X is closed with respect to isomorphisms;

(ii) whenever $L \in X$ and $L_1 \in c(L)$, then $L_1 \in X$;

(iii) whenever $L \in \mathcal{D}$ and $\emptyset \neq \{L_i\}_{i\in I} \subseteq c(L) \cap X$, then $\bigvee\limits_{i\in I}^{1} L_i \in X$.

2.2. Proposition. *Let X be a radical class of distributive lattices (with respect to $\mathcal{D}$). Then $X = \mathcal{D}$.*

Proof. There exists $L \in X$. Choose any $x \in L$ and let L_1 be an arbitrary element of $\mathcal{D}$. Then $\{x\} \in c(L)$, whence $\{x\} \in X$. Thus all one-element lattices belong to X. Let us express the lattice L_1 as $L_1 = \{y_i\}_{i\in I}$. We have $\bigvee\limits_{i\in I}^{1} \{y_i\} = L_1$, whence $L_1 \in X$. Therefore $X = \mathcal{D}$. $\square$

Now let us consider the system $\mathcal{D}_0$. Let $L \in \mathcal{D}_0$; the least element of L will be denoted by 0_L (or by 0, if no misunderstanding can occur). A sublattice L_1 of L will be called a subalgebra of L if $0_L \in L_1$. We denote by $c_0(L)$ the system of all convex subalgebras of L; this system is partially ordered by the set-theoretical inclusion.

Similarly as in the case of $c(L)$ for $L \in \mathcal{D}$, we can verify that for each $L \in \mathcal{D}_0$, $c_0(L)$ is a complete lattice. The lattice operations in $c_0(L)$ will be denoted by $\wedge^0$ and $\vee^0$. If $\{L_i\}_{i\in I}$ is a nonempty subsystem of $c_0(L)$, then $\overset{0}{\bigwedge\limits_{i\in I}} L_i = \bigcap\limits_{i\in I} L_i$.

2.3. Lemma. *Let $L \in \mathcal{D}_0$ and $\emptyset \neq \{L_i\}_{i\in I} \subseteq c_0(L)$. Let Z be the set of all $z \in L$ such that there exist $z_1, z_2, \ldots, z_n \in \bigcup\limits_{i\in I} L_i$ with $z = z_1 \vee z_2 \vee \ldots \vee z_n$. Then $Z = \overset{0}{\bigvee\limits_{i\in I}} L_i$.*

P r o o f. Let z and z' be elements of Z. For z we apply the notation as above; further, let $z' = z'_1 \vee z'_2 \vee \ldots \vee z'_m$ (under analogous assumptions). Then we have

$$z \vee z' = z_1 \vee \ldots \vee z_n \vee z'_1 \vee \ldots \vee z'_m,$$

whence $z \vee z' \in Z$. Further,

$$z \wedge z' = \bigvee (z_k \wedge z'_j) \quad (k \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\}).$$

For each $k \in \{1, 2, \ldots, n\}$ there is $i(k) \in I$ with $z_k \in L_{i(k)}$, thus $z_k \wedge z'_j \in L_{i(k)}$ for each $j \in \{1, 2, \ldots, m\}$. Therefore $z \wedge z' \in Z$ and hence Z is a sublattice of L.

Let $z \in Z$, $x \in L$, $x \leqslant z$. Then (under the notation as above)

$$x = x \wedge z = (x \wedge z_1) \vee \ldots \vee (x \wedge z_n)$$

yielding that $x \in Z$. Thus Z belongs to $c_0(L)$.

Clearly $L_i \leqslant Z$ for each $i \in I$. Assume that $Z_1 \in c_0(L)$ and $L_i \leqslant Z_1$ for each $i \in I$. Let z be as above. Then $z \in Z_1$, whence $Z \leqslant Z_1$. This shows that Z is the join of the system $\{L_i\}_{i\in I}$ in $c_0(L)$. $\square$

2.4. Definition. Let X be a nonempty class of lattices belonging to $\mathcal{D}_0$. Assume that X is closed with respect to isomorphisms and that it satisfies the following conditions:

(ii_0) Whenever $L \in X$ and $L_1 \in c_0(L)$, then $L_1 \in X$.

(iii_0) Whenever $L \in \mathcal{D}_0$ and $\{L_i\}_{i\in I} \subseteq c_0(L) \cap X$, then $\overset{0}{\bigvee\limits_{i\in I}} L_i \in X$.

Under these conditions we call X a *radical class* (in $\mathcal{D}_0$).

In what follows, mainly radical classes in $\mathcal{D}_0$ will be considered (thus the words 'in $\mathcal{D}_0$' will be omitted when no misunderstancing can occur).

We denote by $R(\mathcal{D}_0)$ the collection of all radical classes; for X_1 and X_2 belonging to $R(\mathcal{D}_0)$ we put $X_1 \leqslant X_2$ if X_1 is a subclass of X_2. Thus we can consider the partial order $\leqslant$ on $R(\mathcal{D}_0)$. Then we have

2.5. Lemma. *$\mathcal{D}_0$ is the greatest element of $R(\mathcal{D}_0)$. The class X_0 consisting of all one-element lattices is the least element of $R(\mathcal{D}_0)$.*

Let T be a nonempty subclass of $\mathcal{D}_0$ which is closed with respect to isomorphisms. We denote by c_0T the class of all lattices L having the property that there exists $L_1 \in T$ such that $L \in c_0(L_1)$.

Further, let j_0T be the class of all lattices L which can be written as $L = \bigvee_{i\in I}^{0} L_i$, where $L_i \in c_0(L) \cap T$ for each $i \in I$.

It is obvious that the relations

$$(*) \qquad c_0c_0T = c_0T, \quad j_0j_0T = j_0T$$

are satisfied. Moreover, both c_0T and j_0T are closed with respect to isomorphisms.

2.6. Lemma. *Let T be as above. Then $c_0j_0c_0T = j_0c_0T$.*

P r o o f. This is an easy consequence of 2.3. □

2.7. Lemma. *Let $\emptyset \neq T \subseteq \mathcal{D}_0$. Assume that T is closed with respect to isomorphisms.*

(i) *j_0c_0T is a radical class.*
(ii) *If X is a radical class and $T \subseteq X$, then $j_0c_0T \subseteq X$.*

P r o o f. The class j_0c_0T is closed with respect to isomorphisms. Thus from $(*)$ and 2.6 we obtain that (i) holds.

Let $T \subseteq X$, where X is a radical class. Then $j_0c_0T \subseteq j_0c_0X = X$. □

From 2.7 and from the definition 2.4 we conclude

2.8. Theorem. *Let X be a subclass of $\mathcal{D}_0$ which is closed with respect to isomorphisms. Then the following conditions are equivalent:*

(i) *$X \in R(\mathcal{D}_0)$.*
(ii) *There exists $\emptyset \neq T \subseteq \mathcal{D}_0$ such that T is closed with respect to isomorphisms and $X = j_0c_0T$.*

2.9. Example. Let T be the class of all chains having the least element. Then $c_0 j_0 T$ is a radical class. (E.g., the lattice L on Fig. 1 belongs to $j_0 c_0 T$, the lattice dual to L does not belong to $c_0 j_0 T$.)

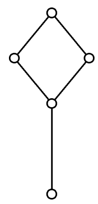

Fig. 1

In view of 2.7 and 2.8 we say that the radical class $j_0 c_0 T$ is generated by T.

3. Radical mappings

The notion of radical mapping of lattice ordered groups has been introduced in [5]. Analogously we proceed in the case of elements of $\mathcal{D}_0$.

3.1. Definition. A radical mapping in $\mathcal{D}_0$ is defined to be a rule that assigns to each element L of $\mathcal{D}_0$ an ideal $L = \varrho L$ of L such that the following conditions are satisfied:

(i) If $L_1, L_2 \in \mathcal{D}_0$ and if $\varphi : L_1 \to L_2$ is an isomorphism, then $\varphi(\varrho L_1) = \varrho\varphi(L_1)$.
(ii) If $L \in \mathcal{D}_0$ and $Z \in c_0(L)$, then $\varrho Z = Z \cap \varrho L$.

We denote by $R_1(\mathcal{D}_0)$ the collection of all radical mappings on $\mathcal{D}_1$. For $\varrho_1, \varrho_2 \in R_1(\mathcal{D}_0)$ we put $\varrho_1 \leqslant \varrho_2$ if $\varrho_1 L \leqslant \varrho_2 L$ is valid for each $L \in \mathcal{D}_0$.

3.2. Lemma. *Let $L \in \mathcal{D}_0$, $Z \in c_0(L)$, $\emptyset \neq \{L_i\}_{i\in I} \subseteq c_0(L)$. Then*

$$Z \wedge^0 \left(\overset{0}{\bigvee_{i\in I}} L_i \right) = \overset{0}{\bigvee_{i\in I}} (Z \wedge^0 L_i).$$

Proof. The relation $Z \wedge^0 \left(\overset{0}{\bigvee_{i\in I}} L_i \right) \geqslant \overset{0}{\bigvee_{i\in I}} (Z \wedge^0 L_i)$ is obvious. Let $x \in Z \wedge^0 \left(\overset{0}{\bigvee_{i\in I}} L_i \right)$. Hence $x \in Z$ and $x \in \overset{0}{\bigvee_{i\in I}} L_i$. Thus in view of 2.3 there are $i(1), i(2), \ldots, i(n) \in I$ and $z_1 \in L_{i(1)}, \ldots, z_n \in L_{i(n)}$ such that $x = z_1 \vee z_2 \vee \ldots \vee z_n$. Then we have

$$x = x \wedge (z_1 \vee z_2 \vee \ldots \vee z_n) = (x \wedge z_1) \vee \ldots \vee (z \wedge z_n).$$

Clearly $x \wedge z_1 \in Z \wedge^0 Z_{i(1)}, \ldots, x \wedge z_n \in Z \wedge^0 Z_{i(n)}$.

By using 2.3 again we conclude that

$$x \in \bigvee_{i\in I}^{0} (Z \wedge L_i), \quad Z \wedge^0 \left(\bigvee_{i\in I}^{0} L_i\right) \leqslant \bigvee_{i\in I}^{0} (Z \wedge^0 L_i).$$

□

Let $X \in R(\mathcal{D}_0)$. For $L \in \mathcal{D}_0$ let $\{L_i\}_{i\in I}$ be the set of all elements of $c_0(L)$ which belong to X. We put

$$\varrho_X L = \bigvee_{i\in I}^{0} L_i. \tag{1}$$

3.3. Lemma. *For each $X \in R(\mathcal{D}_0)$, ϱ_X belongs to $R_1(\mathcal{D}_0)$.*

Proof. a) Let $L \in \mathcal{D}_0$. According to the definition of the operation $\overset{0}{\bigvee}$, $\varrho_X L$ belongs to $c_0(L)$, whence it is an ideal of L.

b) Let $L_1, L_2 \in \mathcal{D}_0$ and let $\varphi : L_1 \to L_2$ be an isomorphism. Let $\{L_i\}_{i\in I}$ be the set of all elements of $c_0(L_1)$ which belong to the class X. Then $\{\varphi(L_i)\}_{i\in I}$ is the set of all elements of $c_0(L_2)$ which belong to X. Hence we have $\varphi(\varrho_X L_1) = \varrho_X \varphi(L_1)$.

c) Let $L \in \mathcal{D}_0$ and $Y \in c_0(L)$. We have to verify that $\varrho_X Z = Z \cap \varrho_X L$.

Let $\{L_1\}_{i\in I}$ be as in (1). Further, let $c_0(Z) \cap X = \{Z_j\}_{j\in J}$. Thus we have $\varrho_X Z = \overset{0}{\bigvee}_{j\in J} Z_j$.

Assume that $t \in \varrho_X Z$. Then in view of 2.3 there are $j(1), j(2), \ldots, j(n)$ in J and $z_1 \in Z_{j(1)}, \ldots, z_n \in Z_{j(n)}$ such that $t = z_1 \vee z_2 \vee \ldots \vee z_n$. Further, $t \in Z$. There are $i(1), \ldots, i(n) \in I$ such that $Z_{j(1)} = L_{i(1)}, \ldots, Z_{j(n)} = L_{i(n)}$. Thus $t \in \varrho_X L$ and $\varrho_X Z \subseteq Z \cap \varrho_X L$.

Conversely, assume that $p \in Z \cap \varrho_X L$. Again, by 2.3 and (1) there are $i(1), \ldots, i(n) \in I$ and $y_1 \in L_{i(1)}, \ldots, y_n \in L_{i(n)}$ such that $p = y_1 \vee y_2 \vee \ldots \vee y_n$. Thus

$$p = (y_1 \wedge p) \vee (y_2 \wedge p) \vee \ldots \vee (y_n \wedge p).$$

We have $y_1 \wedge p \in L_1 \cap Z = L_1 \wedge^0 Z$ and analogously for $y_2, \ldots, y_n$. Hence

$$p \in \bigvee_{i\in I}^{0} (L_i \wedge^0 Z).$$

Therefore according to 3.2,

$$p \in \left(\bigvee_{i\in I}^{0} L_i\right) \wedge^0 Z = Z \cap \varrho_X L,$$

whence $Z \cap \varrho_X L \subseteq \varrho_X Z$. □

Let $\varrho \in R_1(\mathcal{D}_0)$. We denote by X_ϱ the class of all $L \in \mathcal{D}_0$ such that $\varrho L = L$.

3.4. Lemma. *For each $\varrho \in R_1(\mathcal{D}_0)$, X_ϱ belongs to $R(\mathcal{D}_0)$.*

Proof. In view of the condition (i) from 3.1, the class X_ϱ is closed with respect to isomorphisms.

Let $L \in X_\varrho$ and $L_1 \in c_0(L)$. We have $\varrho L_1 = L_1 \cap \varrho L = L_1 \cap L = L_1$, whence $L_1 \in X_\varrho$.

Further, let $L \in \mathcal{D}_0$ and $\emptyset \neq \{L_i\}_{i\in I} \subseteq c_0(L) \cap X_\varrho$. Hence $\varrho L_i = L_i$ for each $i \in I$. Put $Z = \bigvee\limits_{i\in I}^{0} L_i$. Then $Z \geqslant \varrho Z \geqslant \varrho L_i = L_i$ for each $i \in I$, therefore

$$\varrho Z \geqslant \bigvee_{i\in I}^{0} L_i = Z$$

and thus $\varrho Z = Z$. Thus $Z \in X_\varrho$, which completes the proof. □

From the definitions of ϱ_X and X_ϱ we immediately obtain

3.5. Lemma. (i) *If $X(1), X(2) \in R(\mathcal{D}_0)$ and $X(1) \leqslant X(2)$, then $\varrho_{X(1)} \leqslant \varrho_{X(2)}$.*

(ii) *If $\varrho(1), \varrho(2) \in R_1(\mathcal{D}_0)$ and $\varrho(1) \leqslant \varrho(2)$, then $X_{\varrho(1)} \leqslant X_{\varrho(2)}$.*

(iii) *If $X(1) \in R(\mathcal{D}_0)$, $\varrho(1) \in R_1(\mathcal{D}_0)$ and $\varrho_{X(1)} = \varrho(2)$, $X_{\varrho(1)} = X(2)$, then $X_{\varrho(2)} = X(1)$, $\quad \varrho_{X(2)} = \varrho(1)$.*

For $X \in R(\mathcal{D}_0)$ we put $\psi(X) = \varrho_X$. In view of 3.3, 3.4 and 3.5 we conclude that

3.6. Proposition. *ψ is a bijective mapping of $R(\mathcal{D}_0)$ onto $R_1(\mathcal{D}_0)$. Moreover, if $X(1), X(2) \in R(\mathcal{D}_0)$, then*

$$X(1) \leqslant X(2) \Longleftrightarrow \psi(X(1)) \leqslant \psi(X(2)).$$

Let $\{\varrho_i\}_{i\in I}$ be a nonempty subcollection of the collection $R_1(\mathcal{D}_0)$. Further, let $L \in \mathcal{D}_0$. Consider the subset $\{\varrho_i L\}_{i\in I}$ of the set $c_0(L)$. We put

$$Z^1(L) = \bigwedge_{i\in I}^{0} \varrho_i L, \quad Z^2(L) = \bigvee_{i\in I}^{0} \varrho_i L.$$

For each $L \in \mathcal{D}_0$ we denote $\varrho^1 L = Z^1(L)$, $\varrho^2 L = Z^2(L)$. Then $\varrho^1(L)$ and $\varrho^2 L$ are elements of $c_0(L)$.

Let $L_1 \in c_0(L)$. We have

$$\begin{aligned}\varrho^1 L_1 = Z^1(L_1) &= \bigwedge_{i\in I}^{0} \varrho_i L_1 = \bigwedge_{i\in I}^{0} (L_1 \cap \varrho_i L)\\ &= \bigwedge_{i\in I}^{0} (L_1 \wedge^0 \varrho_i L) = L_1 \wedge^0 \left(\bigwedge_{i\in I}^{0} \varrho_i L\right) = L_1 \wedge^0 \varrho^1 L = L_1 \cap \varrho^1 L.\end{aligned}$$

Further, 3.2 yields

$$\begin{aligned}\varrho^2 L_1 = Z^2(L_1) &= \bigvee_{i\in I}^{0} \varrho_i L_1 = \bigvee_{i\in I}^{0} (L_1 \cap \varrho_i L)\\ &= \bigvee_{i\in I}^{0} (L_1 \wedge^0 \varrho_i L) = L_1 \wedge^0 \left(\bigvee_{i\in I}^{0} \varrho_i L\right) = L_1 \wedge^0 \varrho^2 L = L_1 \cap \varrho^2 L.\end{aligned}$$

Therefore we obtain

3.7. Lemma. *ϱ^1 and ϱ^2 are elements of $R_1(\mathcal{D}_0)$.*

From the definitions of ϱ^1, ϱ^2 and from 3.7 we infer that

3.8. Proposition. *Let $S = \{\varrho_i\}_{i\in I}$ be a nonempty subcollection of the collection $R_1(\mathcal{D}_0)$. Further, let ϱ^1 and ϱ^2 be as in 3.7. Then ϱ^1 is the meet of S and ϱ^2 is the join of S in the partially ordered collection $R_1(\mathcal{D}_0)$.*

In view of 3.8 we denote $\varrho^1 = \bigwedge_{i\in I} \varrho_i$, $\varrho^2 = \bigvee_{i\in I} \varrho_i$.

According to 3.2 and 3.8 we have

3.9. Proposition. *Let $\{\varrho_i\}_{i\in I}$ be a nonempty subcollection of $R_1(\mathcal{D}_0)$ and $\varrho \in R_1(\mathcal{D}_0)$. Then*

$$\varrho \wedge \left(\bigvee_{i\in I} \varrho_i\right) = \bigvee_{i\in I} (\varrho \wedge \varrho_i).$$

Now let $S_1 = \{X_i\}_{i\in I}$ be a nonempty subcollection of $R(\mathcal{D}_0)$ and let ψ be as in 3.6. Put

$$\begin{gathered}\varrho^1 = \bigwedge_{i\in I} \psi(X_i), \quad \varrho^2 = \bigvee_{i\in I} \psi(X_i),\\ X^1 = \psi^{-1}(\varrho^1), \quad X^2 = \psi^{-1}(\varrho^2).\end{gathered}$$

Then in view of 3.8 and 3.6 we conclude that X^1 is the meet of S_1 in $R(\mathcal{D}_0)$ and, analogously, X^1 is the join of S_1 in $R(\mathcal{D}_0)$. Further, a rule analogous to that given in 3.9 is valid in $R(\mathcal{D}_0)$.

An element $X \in \mathcal{D}_0$ is called a generalized Boolean algebra if for each $x \in X$, the interval $[0_x, x]$ is a Boolean algebra. The collection of all generalized Boolean algebras will be denoted by $\mathcal{A}(\mathcal{B})$.

The notion of a radical class in $\mathcal{A}(\mathcal{B})$ is defined analogously as in the case of $\mathcal{D}_0$.

Let X be a radical class in $\mathcal{D}_0$. If X is closed with respect to homomorphisms, then it is called a torsion class in $\mathcal{D}_0$. This terminology is in accordance with that used in the theory of lattice ordered groups (cf. [20]).

Analogously we define the notion of the torsion class in $\mathcal{A}(\mathcal{B})$.

Torsion classes in $\mathcal{D}_0$ have been investigated in the paper [2] with applications in the theory of lattice ordered groups. Let us denote by $T(\mathcal{D}_0)$ the collection of all torsion classes in $\mathcal{D}_0$.

4.1. Theorem. ([2], Theorem 2.3.) *The class of generalized Boolean algebras forms a torsion class of distributive lattices with the least element.*

In other words, we have $\mathcal{A}(\mathcal{B}) \in T(\mathcal{D}_0)$.

4.2. Corollary. $\mathcal{A}(\mathcal{B}) \in R(\mathcal{D}_0)$.

4.3. Corollary. *Each radical class of generalized Boolean algebras (i.e., a radical class in $\mathcal{A}(\mathcal{B})$) is a radical class in $\mathcal{D}_0$.*

4.4. Theorem. *There exists an injective mapping φ of the class of all infinite cardinals into the collection $R(\mathcal{D}_0)$ such that for each infinite cardinal α and each $L \in \varphi(\alpha)$, the lattice L is conditionally complete.*

P r o o f. This is a consequence of 4.3 and of Theorem 5.1 in [15]. □

Let α be an infinite cardinal. We denote by $A_{e(\alpha)}$ the class of all generalized Boolean algebras L such that for each interval $[a_1, a_2]$ of L the relation $\operatorname{card}[a_1, a_2] \leqslant \alpha$ is valid.

4.5. Theorem. ([15], Proposition 3.11.) *For each infinite cardinal α, $A_{e(\alpha)}$ is a radical class in $\mathcal{A}(\mathcal{B})$.*

Then in view of 4.3 we have

4.6. Corollary. *For each infinite cardinal α, $A_{e(\alpha)}$ is a radical class in $\mathcal{D}_0$.*

Again, let α be an infinite cardinal and let M_α be a set of cardinality M_α. Let $L(M_\alpha)$ be the system of all subsets M of M_α such that either M is finite or the set $M_\alpha \setminus M$ is finite. Further, let $L(M_\alpha)$ be partially ordered by the set-theoretical inclusion. Then $L(M_\alpha)$ is a Boolean algebra with $\operatorname{card} L(M_\alpha) = \alpha$. Hence we have

(i) $L(M_\alpha) \in A_{e(\alpha)}$;

(ii) if $\alpha(1)$ is a cardinal with $\alpha(1) > \alpha$, then $L(M_{\alpha(1)}) \notin A_{e(\alpha)}$.

Thus $A_{e(\alpha)} \neq \emptyset$ for each infinite cardinal α, and $A_{e(\alpha)} \neq A_{e(\alpha(1))}$ whenever $\alpha(1) > \alpha$.

4.7. Lemma. *For each infinite cardinal α, the class $A_{e(\alpha)}$ is closed with respect to homomorphisms.*

Proof. Let $L \in A_{e(\alpha)}$, $L' \in \mathcal{D}_0$ and let ψ be a homomorphism of L onto L'. Let $[b_1, b_2]$ be an interval in L'. Then there exist $a_1, a_2 \in L$ such that $a_1 \leqslant a_2$ and $\psi([a_1, a_2]) = [b_1, b_2]$. Hence $\operatorname{card}[b_1, b_2] \leqslant \operatorname{card}[a_1, a_2] \leqslant \alpha$. □

Thus in view of 4.6 we have

4.8. Corollary. *For each infinite cardinal α, $A_{e(\alpha)}$ is a torsion class in $\mathcal{D}_0$.*

According to the properties (i) and (ii) above we conclude

4.9. Theorem. *There exists an injective mapping of the class of all infinite cardinals into the collection $T(\mathcal{D}_0)$.*

5. The classes C_α and C_α^1

We denote by $R(\mathcal{B}_0)$ the collection of all radical classes of generalized Boolean algebras. We have already observed (see 4.5) that $R(\mathcal{B}_0) \subseteq R(\mathcal{D}_0)$ and from this we deduced that $R(\mathcal{D}_0)$ is a large collection (cf. 4.6). We can ask whether the collection $R(\mathcal{D}_0) \setminus R(\mathcal{B}_0)$ has this property as well.

5.1. Theorem. *There exists an injective mapping of the class of all infinite cardinals into the collection $R(\mathcal{D}_0) \setminus R(\mathcal{B}_0)$.*

We need some lemmas. The following assertion is obvious.

5.2. Lemma. *Let α be an infinite cardinal. Then there exists a linearly ordered set L_α such that* $\operatorname{card} L_\alpha = \alpha$.

In what follows, L_α is as in 5.2. We have $L_\alpha \notin \mathcal{B}_0$.

For each infinite cardinal α we denote by C_α the class of all lattices $L \in \mathcal{D}_0$ such that $\operatorname{card}[0, x] \leqslant \alpha$, whenever $x \in L$.

Let α be a fixed infinite cardinal. From the definition of C_α we immediately obtain

5.3. Lemma. *C_α satisfies the conditions* (i) *and* (ii) *from* 2.1.

5.4. Lemma. *Let $L \in \mathcal{D}_0$ and $a, b \in L$. Suppose that* $\operatorname{card}[0,a] \leqslant \alpha$ *and* $\operatorname{card}[0,b] \leqslant \alpha$. *Then* $\operatorname{card}[0,a \vee b] \leqslant \alpha$.

P r o o f. For each $x \in [0, a \vee b]$ we put $\varphi(x) = (a \wedge x, b \wedge x)$. In view of the distributivity of L we obtain $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$. Hence if $y \in [0, a \vee b]$ and $\varphi(x) = \varphi(y)$, then $x = y$. Thus φ is an injective mapping of the interval $[0, a \vee b]$ into the Cartesian product $[0,a] \times [0,b]$. Therefore

$$\operatorname{card}[0, a \vee b] \leqslant \operatorname{card}([0,a] \times [0,b]) \leqslant \alpha \cdot \alpha = \alpha.$$

$\square$

From 5.4 we obtain by induction

5.5. Lemma. *Let $L \in \mathcal{D}_0$ and $a_1, a_2, \ldots, a_n \in L$. Suppose that* $\operatorname{card}[0, a_i] \leqslant \alpha$ *for $i = 1, 2, \ldots, n$. Put $v = a_1 \vee a_2 \vee \ldots \vee a_n$. Then* $\operatorname{card}[0,v] \leqslant \alpha$.

5.6. Lemma. *Let $L \in \mathcal{D}_0$ and $\emptyset \neq \{L_i\}_{i \in I} \subseteq c_0(L) \cap C_\alpha$. Put $L = \bigvee\limits_{i \in I}^{0} L_i$. Then $L^0 \in C_\alpha$.*

P r o o f. This is a consequence of 5.5 and 2.3. $\square$

Now we apply 5.3 and 5.5; since $L_\alpha \in C_\alpha \setminus \mathcal{B}_0$ we conclude

5.7. Lemma. *C_α belongs to the collection $R(\mathcal{D}_0) \setminus R(\mathcal{B}_0)$.*

P r o o f o f 5.1. For each infinite cardinal α we put $f(\alpha) = C_\alpha$. If $\alpha(1), \alpha(2)$ are infinite cardinals with $\alpha(1) < \alpha(2)$, then $L_{\alpha(2)} \in C_{\alpha(2)} \setminus C_{\alpha(1)}$, whence $C_{\alpha(2)} \neq C_{\alpha(1)}$. Now it suffices to apply 5.7. $\square$

Consider the following condition for a lattice L:

(a) Whenever X is a nonempty upper bounded subset of L with $\operatorname{card} X \leqslant \alpha$, then $\sup X$ exists in L.

Further, let (b) be the condition dual to (a). We denote by C_α^a the class of all lattices which belong to $\mathcal{D}_0$ and satisfy the condition (a). Let C_α^b be defined analogously.

5.8. Lemma. *Let $L \in \mathcal{D}_0$ and let $a, b \in L$, $a \vee b = v$. Assume that both $[0, a]$ and $[0, b]$ satisfy the condition* (a). *Then the interval $[0, v]$ satisfies this condition as well.*

Proof. Let $\emptyset \neq X \subseteq [0, v]$, $\operatorname{card} X \leqslant \alpha$. For each $x \in X$ we put $x_1 = x \wedge a$, $x_2 = x \wedge b$.

Further, we set

$$X_1 = \{x_1;\ x \in X\}, \quad X_2 = \{x_2;\ x \in X\}.$$

Then we have $X_1 \subseteq [0, a], \operatorname{card} X_1 \leqslant \alpha$, $X_2 \subseteq [0, b], \operatorname{card} X_2 \leqslant \alpha$. Thus, in view of the assumption, there exist $x^1 = \sup X_1$, $x^2 = \sup X_2$ in L; clearly $x^1 \in [0, a]$, $x^2 \in [0, b]$. Put $x^0 = x^1 \vee x^2$. For each $x \in X$ we have $x = x_1 \vee x_2$, whence $x^0 \geqslant x$ for each $x \in X$.

Assume that t is an upper bound of the set X. Put $t' = t \vee v$ and $t'_1 = t' \wedge a$, $t'_2 = t' \wedge b$. Then we have $t'_1 \geqslant x_1$, $t'_2 \geqslant x_2$ for each $x \in X$, whence $t'_1 \geqslant x^1$ and $t'_2 \geqslant x^2$. We obtain $t \geqslant t' = t'_1 \vee t'_2 \geqslant x^1 \vee x^2 = x^0$. Therefore $x^0 = \sup X$ in L. $\square$

For the class C_α^a we can now apply analogous steps as we did for C_α above (cf. 5.3, 5.5 and 5.6) with the distinction that instead of 5.4 we use now 5.8. We obtain

5.9. Proposition. *C_α^a is a radical class in $\mathcal{D}_0$.*

Similarly we can verify that C_α^b is a radical class in $\mathcal{D}_0$.

A lattice L is said to be conditionally α-complete if it satisfies both conditions (a) and (b). We denote by C_α^1 the class of all lattices which belong to $\mathcal{D}_0$ and are conditionally α-complete.

From 5.9 and from the analogous result concerning C_α^b we conclude

5.10. Theorem. *For each infinite cardinal α, C_α^1 belongs to $R(\mathcal{D}_0)$.*

6. Infinite distributivity

We recall that a lattice L is called infinitely distributive if, whenever $x, y \in L$ and $\emptyset \neq \{x_i\}_{i \in I} \subseteq L$, then

(i) $x = \bigvee_{i \in I} x_i \Longrightarrow x \wedge y = \bigvee_{i \in I} (x_i \wedge y)$,

(ii) $x = \bigwedge_{i \in I} x_i \Longrightarrow x \vee y = \bigwedge_{i \in I} (x_i \vee y)$.

We denote by $\mathcal{D}_1$ the class of all lattices L such that L is infinitely distributive and has the least element. Hence $\mathcal{D}_1$ is a subclass of $\mathcal{D}_0$.

The question whether the relation $\mathcal{D}_1 \in R(\mathcal{D}_0)$ is valid remains open.

We define the radical class in $\mathcal{D}_1$ analogously as in Definition 2.4 with the distinction that we take $\mathcal{D}_1$ instead of $\mathcal{D}_0$. Let $R(\mathcal{D}_1)$ be the collection of all radical classes in $\mathcal{D}_1$.

We start by mentioning the following facts.

A) The results of Section 3 remain valid if $\mathcal{D}_0$ is replaced by $\mathcal{D}_1$.

B) It is well-known that every Boolean algebra is infinitely distributive. From this we conclude that the same is valid for generalized Boolean algebras and that $\mathcal{B}_0 \in R(\mathcal{D}_1)$, $R(\mathcal{B}_0) \subseteq R(\mathcal{D}_1)$.

C) Each linearly ordered set is infinitely distributive. Therefore the results of Section 5 remain valid if we replace $\mathcal{D}_0$ by $\mathcal{D}_1$.

Let L be a lattice and let α, β be nonzero cardinals. We say that L satisfies the condition $c_1(\alpha,\beta)$ if, whenever $u, v \in L$, $\{x_{ij}\}_{i\in I, j\in J} \subseteq L$ such that $\operatorname{card} I \leqslant \alpha$, $\operatorname{card} J \leqslant \beta$,

$$v = \bigwedge_{i\in I} \bigvee_{j\in J} x_{ij}, \tag{1}$$

$$u = \bigvee_{\varphi\in J^I} \bigwedge_{i\in I} x_{i,\varphi(i)}, \tag{2}$$

then $u = v$.

Further, let $c_2(\alpha,\beta)$ be the conditions dual to $c_1(\alpha,\beta)$.

The lattice L is called (α,β)-distributive if it satisfies both conditions $c_1(\alpha,\beta)$ and $c_2(\alpha,\beta)$.

If a lattice L is (α,β)-distributive for all nonzero cardinals α and β, then it is said to be completely distributive.

Let us remark that if (1) and (2) are valid then $u \leqslant v$.

Let $u, v \in L$, $u < v$. If there exist elements $x_{ij} (i \in I, j \in J)$ such that $\operatorname{card} I \leqslant \alpha$, $\operatorname{card} J \leqslant \beta$ and the relations (1), (2) are valid, then we say that the pair (u, v) violates the condition $c_1(\alpha,\beta)$.

6.1. Lemma. *Suppose that L is an infinitely distributive lattice. Let $u, v, u_1, v_1 \in L$, $u \leqslant u_1 < v_1 \leqslant v$. Assume that the pair (u, v) violates the condition $c_1(\alpha,\beta)$. Then the pair (u_1, v_1) violates this condition as well.*

P r o o f. In view of the assumption we can suppose that the conditions (1), (2) are satisfied and that $\operatorname{card} I \leqslant \alpha$, $\operatorname{card} J \leqslant \beta$.

Denote $(x_{ij} \wedge v_1) \vee u_1 = x'_{ij}$. We have

$$(v \wedge v_1) \vee u_1 = v_1, \quad (u \wedge v_1) \vee u_1 = u_1.$$

Hence by applying the infinite distributivity we obtain from (1) and (2) the relations

$$(1') \qquad v_1 = \bigwedge_{i \in I} \bigvee_{j \in J} x'_{ij},$$

$$(2') \qquad u_1 = \bigvee_{\varphi \in J^I} \bigwedge_{i \in I} x'_{i,\varphi(i)}.$$

Since $u_1 < v_1$, the pair (u_1, v_1) violates the condition $c_1(\alpha, \beta)$. □

If the elements x'_{ij} are as above, then we obviously have

$$x'_{ij} \in [u_1, v_1] \quad \text{for each } i \in I, j \in J.$$

Hence from $(1')$ and $(2')$ we conclude

6.2. Corollary. *Let the assumptions as in* 6.1 *be valid. Then the lattice* $[u_1, v_1]$ *fails to satisfy the condition* $c_1(\alpha, \beta)$.

6.3. Lemma. *Let* $L \in \mathcal{D}_1$ *and* $a, b \in L$. *Assume that the intervals* $[0, a]$ *and* $[0, b]$ *both satisfy the condition* $c_1(\alpha, \beta)$. *Then the interval* $[0, a \vee b]$ *satisfies this condition as well.*

Proof. By way of contradiction, assume that the interval $[0, a \vee b]$ does not satisfy the condition $c_1(\alpha, \beta)$. Then there are elements $u, v \in [0, a \vee b]$ such that $u < v$ and the pair (u, v) violates the condition $c_1(\alpha, \beta)$. Put $u_1 = u \wedge a$, $u_2 = u \wedge b$, $v_1 = v \wedge a$, $v_2 = v \wedge b$. Then we have $u_1 \leqslant v_1$, $u_2 \leqslant v_2$, $[u_1, v_1] \subseteq [0, a]$, $[u_2, v_2] \subseteq [0, b]$. Further, $u = u_1 \vee u_2$, $v = v_1 \vee v_2$. If $u_1 = v_1$ and $u_2 = v_2$, then $u = v$, which is a contradiction. Hence without loss of generality we can suppose that $u_1 < v_1$. In view of the distributivity of L, the interval $[u_1, v_1]$ is projective to the interval $[u, u \vee v_1]$ and $[u, u \vee v_1] \subseteq [u, v]$. Hence the intervals $[u_1, v_1]$ and $[u, u \vee v_1]$ are isomorphic. Thus $u < u \vee v_1$. Then in view of 6.2, the interval $[u, u \vee v_1]$ fails to satisfy the condition $c_1(\alpha, \beta)$. This yields that the interval $[u, v_1]$ also fails to satisfy this condition.

On the other hand, $[0, a]$ satisfies the condition $c_1(\alpha, \beta)$ and $[u_1, v_1]$ is a subinterval of $[0, a]$, thus $[u_1, v_1]$ satisfies $c_1(\alpha, \beta)$; we have arrived at a contradiction. □

Let us denote by $X_{(1,\alpha,\beta)}$ the class of all lattices $L \in \mathcal{D}_1$ such that L satisfies the condition $c_1(\alpha, \beta)$.

6.4. Theorem. $X_{(1,\alpha,\beta)}$ *belongs to the collection* $R(\mathcal{D}_1)$.

P r o o f. It is obvious that $X_{(1,\alpha,\beta)}$ is closed with respect to isomorphisms and with respect to convex subalgebras. Now it suffices to apply 6.3, 2.3 and to use analogous steps as in the proof of 5.7. □

Similarly we can deal with the condition $c_2(\alpha,\beta)$; we obtain the radical class $X_{(2,\alpha,\beta)}$ in $\mathcal{D}_1$.

Summarizing the results concerning $X_{(1,\alpha,\beta)}$ and $X_{(2,\alpha,\beta)}$ we conclude.

6.5. Theorem. *The class* $X_{\alpha,\beta}$ *of all lattices which belong to* $\mathcal{D}_1$ *and are* (α,β)*-distributive is a radical class in* $\mathcal{D}_1$.

6.6. Corollary. *The class of all lattices which are completely distributive and have the least element is a radical class in* $\mathcal{D}_1$.

Since each generalized Boolean algebra belongs to $\mathcal{D}_1$, 6.5 is a generalization of Proposition 3.9 in [15].

7. Atoms in $R(\mathcal{D}_1)$

The class X_0 of all one-element lattices is the least element in $R(\mathcal{D}_1)$. Let $X \in R(\mathcal{D}_1)$. Assume that $X \neq X_0$ and that, whenever $Y \in R(\mathcal{D}_1)$ and $X_0 < Y \leqslant X$, then $Y = X$. A radical class X in $R(\mathcal{D}_1)$ having this property will be called an *atom* in $R(\mathcal{D}_1)$.

Analogously we define an atom in $R(\mathcal{D}_0)$. It is easy to see that each atom in $R(\mathcal{D}_1)$ is, at the same time, an atom in $R(\mathcal{D}_0)$.

In this section we show that there exists a large collection of atoms in $R(\mathcal{D}_1)$.

Let us introduce the following notation. For each infinite cardinal α we denote by $\beta(\alpha)$ the first ordinal whose cardinality is equal to α. We put $\beta'(\alpha) = \beta(\alpha) \cup \{0\}$ and we consider 0 to be the greatest element of $\beta'(\alpha)$.

Further, let X_α be the class of all linearly ordered sets L such that L is dually isomorphic to $\beta'(\alpha)$.

From the definition of X_α we immediately obtain

7.1. Lemma. *Let* α *be an infinite cardinal. Let* $L \in X_\alpha$. *Then*

(i) *L is infinitely distributive;*

(ii) *if* $L_1 \in c_0(L)$ *and* $\operatorname{card} L_1 > 1$, *then* L_1 *is isomorphic to* L.

We slightly modify the construction from Section 2; instead of $\mathcal{D}_0$ we deal with the class $\mathcal{D}_1$.

Let T be a nonempty subclass of $\mathcal{D}_1$ which is closed with respect to isomorphisms. We define c_0T in the same way as in Section 2; then we have $c_0T \subseteq \mathcal{D}_1$.

Further, we define $j_0'T$ to be the class of all lattices $L \in \mathcal{D}_1$ which can be expressed in the form $L = \bigvee\limits_{i\in I}^{0} L_i$, where $L_i \in c_0(L) \cap T$ for each $i \in I$.

Similarly as in Section 2 we have (cf. 2.7)

7.2. Lemma. *Let $\emptyset \neq T \subseteq \mathcal{D}_1$. Assume that T is closed with respect to isomorphisms. Then we have*

(i) *$j_0'c_0T$ is a radical class in $\mathcal{D}_1$.*
(ii) *If X is a radical class in $\mathcal{D}_1$ and $T \subseteq X$, then $j_0'c_0T \subseteq X$.*

Let X_α be as above. It is clear that X_α is closed with respect to isomorphisms. We put $Y_\alpha = j_0'c_0X_\alpha$. Then in view of 7.2 we have

7.3. Lemma. *Let α be an infinite cardinal. Then $Y_\alpha \in R(\mathcal{D}_1)$.*

7.4. Lemma. *For each infinite cardinal α, Y_α is an atom in $R(\mathcal{D}_1)$.*

Proof. Since the lattice dual to $\beta'(\alpha)$ belongs to Y_α we infer that $Y_\alpha \neq X_0$.

Let $Y \in R(\mathcal{D}_1)$, $X_0 < Y \leqslant Y_\alpha$. Hence there exists $L \in Y$ with $\operatorname{card} Y > 1$. Then $L \in j_0'c_0X_\alpha$. Thus there exists a system $\emptyset \neq \{L_i\}_{i\in I} \subseteq (c_0X_\alpha) \cap c_0(L)$ such that $L = \bigvee\limits_{i\in I}^{0} L_i$. Without loss of generality we can suppose that $\operatorname{card} L_i > 1$ for each $i \in I$. Then according to 7.1, L_i is dually isomorphic to $\beta'(\alpha)$ for each $i \in I$. Thus $X_\alpha \subseteq Y$, whence $Y_\alpha = j_0'c_0X_\alpha \subseteq j_0'c_0Y = T$. Therefore $Y = Y_\alpha$. □

Put $f(\alpha) = Y_\alpha$. If $\alpha(1)$ and $\alpha(2)$ are distinct infinite cardinals, then clearly $Y_{\alpha(1)} \neq Y_{\alpha(2)}$. Hence from 7.4 we conclude

7.5. Theorem. *f is an injective mapping of the class of all infinite cardinals into the collection of all atoms of $R(\mathcal{D}_1)$.*

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Author's address: *Ján Jakubík*, Matematický ústav SAV, Grešákova 6, 04 01 Košice, Slovakia, e-mail: `musavke@saske.sk`.