CONNECTED RESOLVING DECOMPOSITIONS IN GRAPHS

Varaporn Saenpholphat, Ping Zhang, Kalamazoo

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Abstract. For an ordered k-decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G and an edge e of G, the $\mathcal{D}$-code of e is the k-tuple $c_{\mathcal{D}}(e) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k))$, where $d(e, G_i)$ is the distance from e to G_i. A decomposition $\mathcal{D}$ is resolving if every two distinct edges of G have distinct $\mathcal{D}$-codes. The minimum k for which G has a resolving k-decomposition is its decomposition dimension $\dim_{\rm d}(G)$. A resolving decomposition $\mathcal{D}$ of G is connected if each G_i is connected for $1 \leqslant i \leqslant k$. The minimum k for which G has a connected resolving k-decomposition is its connected decomposition number $\mathrm{cd}(G)$. Thus $2 \leqslant \dim_{\rm d}(G) \leqslant \mathrm{cd}(G) \leqslant m$ for every connected graph G of size $m \geqslant 2$. All nontrivial connected graphs with connected decomposition number 2 or m are characterized. We provide bounds for the connected decomposition number of a connected graph in terms of its size, diameter, girth, and other parameters. A formula for the connected decomposition number of a nonpath tree is established. It is shown that, for every pair a, b of integers with $3 \leqslant a \leqslant b$, there exists a connected graph G with $\dim_{\rm d}(G) = a$ and $\mathrm{cd}(G) = b$.

Keywords: distance, resolving decomposition, connected resolving decomposition

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1. Introduction

A *decomposition* of a graph G is a collection of subgraphs of G, none of which have isolated vertices, whose edge sets provide a partition of $E(G)$. A decomposition into k subgraphs is a k-*decomposition*. A decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ is *ordered* if the ordering $(G_1, G_2, \ldots, G_k)$ has been imposed on $\mathcal{D}$. If each subgraph G_i $(1 \leqslant i \leqslant k)$ is isomorphic to a graph H, then $\mathcal{D}$ is called an H-*decomposition* of G. Decompositions of graphs have been the subject of many studies. J. Bosák [1] has written a book devoted to the subject.

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For edges e and f in a connected graph G, the *distance* $d(e,f)$ between e and f is the minimum nonnegative integer k for which there exists a sequence $e = e_0, e_1, \ldots, e_k = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, k-1$. Thus $d(e,f) = 0$ if and only if $e = f$, $d(e,f) = 1$ if and only if e and f are adjacent, and $d(e,f) = 2$ if and only if e and f are nonadjacent edges that are adjacent to a common edge of G. Also, this distance equals the standard distance between vertices e and f in the line graph $L(G)$. For an edge e of G and a subgraph F of G, we define the distance between e and F as

$$d(e,F) = \min_{f \in E(F)} d(e,f).$$

Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ be an ordered k-decomposition of a connected graph G. For $e \in E(G)$, the *$\mathcal{D}$-code* (or simply the *code*) of e is the k-vector

$$c_{\mathcal{D}}(e) = (d(e,G_1), d(e,G_2), \ldots, d(e,G_k)).$$

Hence exactly one coordinate of $c_{\mathcal{D}}(e)$ is 0, namely the ith coordinate if $e \in E(G_i)$. The decomposition $\mathcal{D}$ is said to be a *resolving decomposition* for G if every two distinct edges of G have distinct $\mathcal{D}$-codes. The minimum k for which G has a resolving k-decomposition is its *decomposition dimension* $\dim_{\mathrm{d}}(G)$. A resolving decomposition of G with $\dim_{\mathrm{d}}(G)$ elements is a *minimum resolving decomposition* for G. Thus if G is a connected graph of size at least 2, then $\dim_{\mathrm{d}}(G) \geqslant 2$. The following result appeared in [2].

Theorem A. *Let G be a connected graph order $n \geqslant 3$.*
(a) *Then $\dim_{\mathrm{d}}(G) = 2$ if and only if $G = P_n$.*
(b) *If $n \geqslant 5$, then $\dim_{\mathrm{d}}(G) \leqslant n$.*

The concept of resolvability in graphs has appeared in the literature. Slater introduced and studied these ideas with different terminology in [9], [10]. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Harary and Melter [8] discovered these concepts independently. Recently, these concepts were rediscovered by Johnson [6], [7] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. Resolving decompositions in graphs were introduced and studied in [2] and further studied in [4], [5]. We refer to the book [3] for graph theory notation and terminology not described here.

A resolving decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G is *connected* if each subgraph G_i $(1 \leqslant i \leqslant k)$ is a connected subgraph in G. The minimum

k for which G has a connected resolving k-decomposition is its *connected decomposition number* $\mathrm{cd}(G)$. A connected resolving decomposition of G with $\mathrm{cd}(G)$ elements is called a *minimum connected resolving decomposition of* G. If G has $m \geqslant 2$ edges, then the m-decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_m\}$, where each G_i $(1 \leqslant i \leqslant m)$ contains a single edge, is a connected resolving decomposition of G. Thus $\mathrm{cd}(G)$ is defined for every connected graph G of size at least 2. Moreover, every connected resolving k-decomposition is a resolving k-decomposition, and so

$$2 \leqslant \dim_{\mathrm{d}}(G) \leqslant \mathrm{cd}(G) \leqslant m. \tag{1}$$

for every connected graph G of size $m \geqslant 2$.

To illustrate these concepts, consider the graph G of Figure 1. Let $\mathcal{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{e_1, e_5, f_1, f_5, f_4\}$, $E(G_2) = \{e_2, e_3, f_2\}$, and $E(G_3) = \{e_4, e_6, f_3, f_6, f_7\}$. The $\mathcal{D}$-codes of the edges of G are:

$$\begin{aligned}
&c_{\mathcal{D}}(e_1) = (0,1,2),\ c_{\mathcal{D}}(e_2) = (1,0,2),\ c_{\mathcal{D}}(e_3) = (2,0,1),\ c_{\mathcal{D}}(e_4) = (2,1,0),\\
&c_{\mathcal{D}}(e_5) = (0,4,1),\ c_{\mathcal{D}}(e_6) = (1,4,0),\ c_{\mathcal{D}}(f_1) = (0,1,1),\ c_{\mathcal{D}}(f_2) = (1,0,1),\\
&c_{\mathcal{D}}(f_3) = (1,1,0),\ c_{\mathcal{D}}(f_4) = (0,2,1),\ c_{\mathcal{D}}(f_5) = (0,3,1),\ c_{\mathcal{D}}(f_6) = (1,3,0),\\
&c_{\mathcal{D}}(f_7) = (1,2,0).
\end{aligned}$$

Thus $\mathcal{D}$ is a resolving decomposition of G. By Theorem A, $\dim_{\mathrm{d}}(G) = |\mathcal{D}| = 3$. However, $\mathcal{D}$ is not connected since G_1 and G_2 are not connected subgraphs in G. On the other hand, let $\mathcal{D}^* = \{G_1^*, G_2^*, G_3^*, G_4^*, G_5^*\}$, where $E(G_1^*) = \{e_1, f_1\}$, $E(G_2^*) = \{e_5, f_4, f_5\}$, $E(G_3^*) = \{e_2, e_3, f_2\}$, $E(G_4^*) = \{e_4, f_3\}$, and $E(G_5^*) = \{e_6, f_6, f_7\}$. Then $\mathcal{D}^*$ is a connected resolving decomposition of G. But $\mathcal{D}^*$ is not minimum since the decomposition $\mathcal{D}' = \{G_1', G_2', G_3', G_4'\}$, where $E(G_1') = \{e_1\}$, $E(G_2') = \{e_3\}$, $E(G_3') = \{e_5\}$, and $E(G_4') = E(G) - \{e_1, e_3, e_5\}$, is a connected resolving decomposition of G with fewer elements. Indeed, it can be verified that $\mathcal{D}'$ is a minimum connected resolving decomposition of G and so $\mathrm{cd}(G) = |\mathcal{D}'| = 4$.

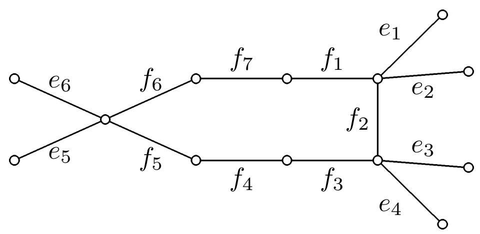

Figure 1. A graph G with $\dim_{\mathrm{d}}(G) = 3$ and $\mathrm{cd}(G) = 4$

The example just presented also illustrates an important point. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ be a resolving decomposition of G. If $e \in E(G_i)$ and $f \in E(G_j)$, where $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$ since $d(e, G_i) = 0$ and $d(e, G_j) \neq 0$. Thus, when determining whether a given decomposition $\mathcal{D}$ of a graph G is a resolving decomposition for G, we need only verify that the edges of G belonging to same element in $\mathcal{D}$ have distinct $\mathcal{D}$-codes. The following two observations are useful.

Observation 1.1. *Let $\mathcal{D}$ be a resolving decomposition of G and $e_1, e_2 \in E(G)$. If $d(e_1, f) = d(e_2, f)$ for all $f \in E(G) - \{e_1, e_2\}$, then e_1 and e_2 belong to distinct elements of $\mathcal{D}$.*

Observation 1.2. *Let G be a connected graph. Then $\dim_{\mathrm{d}}(G) = \mathrm{cd}(G)$ if and only if G contains a minimum resolving decomposition that is connected.*

2. Refinements of decompositions of a graph

Let $\mathcal{D}$ and $\mathcal{D}^*$ be two decompositions of a connected graph G. Then $\mathcal{D}^*$ is called a *refinement* of $\mathcal{D}$ if every element in $\mathcal{D}^*$ is a subgraph of some element of $\mathcal{D}$. A refinement $\mathcal{D}^*$ of $\mathcal{D}$ is *connected* if $\mathcal{D}^*$ is a connected decomposition of G. For the graph G of Figure 1, the decomposition $\mathcal{D}^*$ of G is a connected refinement of $\mathcal{D}$. We have seen that $\mathcal{D}$ is resolving and its refinement $\mathcal{D}^*$ is also resolving. This is not coincident, as we show now.

Theorem 2.1. *Let $\mathcal{D}$ and $\mathcal{D}^*$ be two decompositions of a connected graph G. If $\mathcal{D}$ is a resolving decomposition of G and $\mathcal{D}^*$ is a refinement of $\mathcal{D}$, then $\mathcal{D}^*$ is also a resolving decomposition of G.*

Proof. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ and $\mathcal{D}^* = \{H_1, H_2, \ldots, H_\ell\}$ be two decompositions of G, where $k \leqslant \ell$, such that each H_i $(1 \leqslant i \leqslant \ell)$ is a subgraph of G_j for some j with $1 \leqslant j \leqslant k$. Let e and f be distinct edges of G. We show that $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$. Since $\mathcal{D}$ is a resolving decomposition of G, it follows that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus $d(e, G_j) \neq d(f, G_j)$ for some j with $1 \leqslant j \leqslant k$, say $d(e, G_1) \neq d(f, G_1)$. If G_1 is an element of $\mathcal{D}^*$, then $d(e, G_1) \neq d(f, G_1)$ and so $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$. Thus we may assume that $G_1 = H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_s}$, where $1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant \ell$ and $s \geqslant 2$. Observe that at least one of e and f does not belong to G_1, for otherwise, $d(f, G_1) = 0 = d(f, G_1)$. We consider two cases.

Case 1. *Exactly one of e and f is in G_1, say $e \in E(G_1)$ and $f \notin E(G_1)$.* Thus $e \in E(H_{i_p})$ for some p with $1 \leqslant p \leqslant s$ and so $d(e, H_{i_p}) = 0$. Since $f \notin E(G_1)$, it follows that $f \notin E(H_{i_p})$ and so $d(v, H_{i_p}) \neq 0$. Hence $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$.

Case 2. $e, f \notin E(G_1)$. Let $e', f' \in E(G_1)$ such that $d(e, G_1) = d(e, e')$ and $d(f, G_1) = d(f, f')$, where say $d(e, e') < d(f, f')$. If $e', f' \in E(H_{i_p})$ for some p with $1 \leqslant p \leqslant s$, then $d(e, H_{i_p}) = d(e, e') < d(f, f') = d(f, H_{i_p})$, implying that $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$. If $e' \in E(H_{i_p})$ and $f \in E(H_{i_q})$, where $1 \leqslant p \neq q \leqslant s$, then $d(e, H_{i_p}) = d(e, e') < d(f, f') \leqslant d(f, H_{i_p})$, again, implying that $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$.

Therefore, $\mathcal{D}^*$ is a resolving decomposition of G. □

By Theorem 2.1, a connected resolving decomposition of a connected graph can be obtained from a resolving decomposition by means of refinement. However, a connected refinement of a resolving decomposition is not necessary to be minimum. Indeed, using an extensive case-by-case analysis, we can show that the graph G of Figure 1 has two distinct minimum resolving decompositions (up to isomorphic), namely, $\{G_1, G_2, G_3\}$ and $\{H_1, H_2, H_3\}$, where $G_1 = G_2 = P_3 \cup P_4$, $G_3 = P_4$, $H_1 = H_2 = P_2 \cup 2P_3$, and $H_3 = P_4$. For example, $\mathcal{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{e_1, e_5, f_1, f_5, f_4\}$, $E(G_2) = \{e_2, e_3, f_2\}$, and $E(G_3) = \{e_4, e_6, f_3, f_6, f_7\}$ and $\widetilde{\mathcal{D}} = \{H_1, H_2, H_3\}$, where $E(H_1) = \{e_1, e_6, f_1, f_4, f_6\}$, $E(H_2) = \{e_2, e_3, f_2\}$, and $E(H_3) = \{e_4, e_5, f_3, f_5, f_7\}$. The decompositions $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are shown in Figure 2. Since each connected refinement of $\mathcal{D}$ contains at least five elements, each connected refinement of $\widetilde{\mathcal{D}}$ contains at least seven elements, and $\operatorname{cd}(G) = 4$, it follows that no minimum connected resolving decomposition of G is a refinement of any minimum resolving decomposition of G.

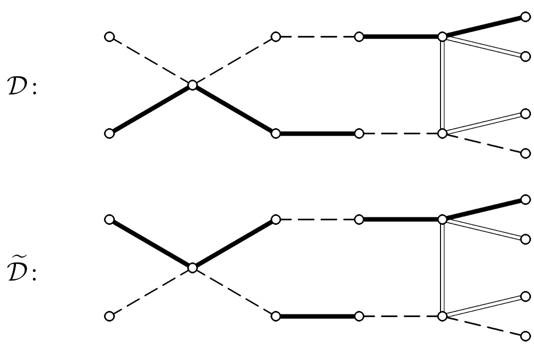

Figure 2. The two distinct minimum resolving decompositions $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ of G

3. Bounds for connected decomposition numbers of graphs

We have seen that if G is a connected graph of size $m \geqslant 2$, then $2 \leqslant \operatorname{cd}(G) \leqslant m$. In this section, we first characterize those connected graphs G of size $m \geqslant 2$ such that $\operatorname{cd}(G) = 2$ or $\operatorname{cd}(G) = m$.

Theorem 3.1. *Let G be a connected graph of order $n \geqslant 3$ and size m. Then*
(a) $\operatorname{cd}(G) = 2$ *if and only if* $G = P_n$, *and*
(b) $\operatorname{cd}(G) = m$ *if and only if* $G = K_3$ *or* $G = K_{1,n-1}$.

Proof. We first verify (a). Let $P_n\colon v_1, v_2, \ldots, v_n$ and let $\mathcal{D} = \{G_1, G_2\}$ be the decomposition of P_n in which $E(G_1) = \{v_1v_2\}$ and G_2 is the path $v_2, v_3, \ldots, v_n$. Thus $\mathcal{D}$ is connected. For $2 \leqslant i \leqslant n-1$, the edge v_iv_{i+1} is the unique edge of G_2 at distance $i-1$ from G_1. Therefore, $\mathcal{D}$ is a connected resolving decomposition of P_n and so $\operatorname{cd}(P_n) = 2$. For the converse, let G be a connected graph of order $n \geqslant 3$ and $\operatorname{cd}(G) = 2$. By (1) $\dim_{\mathrm{d}}(G) = 2$ as well. It then follows by Theorem A that $G = P_n$.

Next we verify (b). It is routine to show that $\operatorname{cd}(K_3) = 2$ and $\operatorname{cd}(K_{1,n-1}) = n-1$ and so the graphs described in (b) have $\operatorname{cd}(G) = m$. For the converse, let G be a connected graph of order $n \geqslant 3$ and size $m \geqslant 2$ such that $\operatorname{cd}(G) = m$. If $m = 2$, then $G = P_3$ and $\operatorname{cd}(P_3) = 2$ by (a). If $m = 3$, then $G \in \{P_4, K_3, K_{1,3}\}$. Since $\operatorname{cd}(P_4) = 2$ and $\operatorname{cd}(K_3) = \operatorname{cd}(K_{1,3}) = 3$, it follows that $G = K_3$ or $G = K_{1,3}$. Now let G be a connected graph of size $m \geqslant 4$ and let $E(G) = \{e_1, e_2, \ldots, e_m\}$. If $G \neq K_{1,n-1}$, then G contains a path P_4 of order 4 with three edges, say e_1, e_2, and e_3, such that $d(e_1, e_2) = 1$, $d(e_1, e_3) = 2$, and $d(e_2, e_3) = 1$. Then $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-1}\}$, where $E(G_1) = \{e_1, e_2\}$ and $E(G_i) = \{e_{i+1}\}$ for $2 \leqslant i \leqslant m-1$, is a connected resolving decomposition of G. Thus $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m-1$. □

It was shown in [2] that $\dim_{\mathrm{d}}(K_3) = 3$ and $\dim_{\mathrm{d}}(K_{1,n-1}) = n-1$. Thus the following corollary is a consequence of (1) and Theorem 3.1.

Corollary 3.2. *Let G be a connected graph of order $n \geqslant 3$ and of size m. Then* $\dim_{\mathrm{d}}(G) = m$ *if and only if* $G = K_3$ *or* $G = K_{1,n-1}$.

Next, we present bounds for $\operatorname{cd}(G)$ of a connected graph G in terms of its size and diameter.

Proposition 3.3. *If G is a connected graph of size $m \geqslant 2$ and diameter d, then*

$$2 \leqslant \operatorname{cd}(G) \leqslant m - d + 2.$$

Proof. We have seen that $\operatorname{cd}(G) \geqslant 2$ for every connected graph G of size $m \geqslant 2$. Thus it remains to verify the upper bound. Let $u, v \in V(G)$ such that $d(u, v) = d$

and let $P\colon u = v_1, v_2, \ldots, v_{d+1} = v$ be a $u-v$ path of length d in G. Also, let $E(G) - E(P) = \{e_1, e_2, \ldots, e_{m-d}\}$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-d+2}\}$, where $E(G_i) = \{e_i\}$ for $1 \leqslant i \leqslant m-d$, $E(G_{m-d+1}) = \{v_1v_2\}$, and $E(G_{m-d+2}) = E(P - v_1)$. Then $\mathcal{D}$ is a connected decomposition of G. Since $d(v_iv_{i+1}, G_{m-d+1}) = i-1$ for $2 \leqslant i \leqslant d$, it follows that $\mathcal{D}$ is a resolving decomposition of G. Therefore, $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m-d+2$. □

By Theorem 3.1, the lower bound in Proposition 3.3 is sharp. If $d = 1$, then $G = K_n$ for some $n \geqslant 3$. Since $\dim_{\mathrm{d}}(K_n) = \operatorname{cd}(K_n)$, it then follows by Theorem A that the upper bound in Proposition 3.3 is not sharp for $d = 1$. If $d = 2$, then $G = K_{1,m}$ is the only graph with $\operatorname{cd}(G) = m - d + 2 = m$ by Theorem 3.1. Thus we may assume that $m \geqslant d \geqslant 3$. If $m = d$, then $G = P_{m+1}$ and $\operatorname{cd}(G) = 2 = m - d + 2$. If $m \geqslant d + 1$, let G be the graph obtained from the path $P_{d+1}\colon u_1, u_2, \ldots, u_{d+1}$ by adding the $m - d \geqslant 1$ new vertices $v_1, v_2, \ldots, v_{m-d}$ and joining each of these vertices to u_d. Then the diameter of G is d and size of G is m. Moreover, it can be verified that $\operatorname{cd}(G) = m - d + 2$. Thus the upper bound in Proposition 3.3 is sharp for $d \geqslant 2$.

The *girth* of a graph is the length of its shortest cycle. Next, we provide bounds for the connected decomposition number of a connected graph in terms of its size and girth.

Theorem 3.4. *If G is a connected graph of size $m \geqslant 3$ and girth $\ell \geqslant 3$, then*

$$3 \leqslant \operatorname{cd}(G) \leqslant m - \ell + 3.$$

Moreover, $\operatorname{cd}(G) = m - \ell + 3$ if and only if G is a cycle of order at least 3.

P r o o f. Since $\ell \geqslant 3$, it follows that G is not a path and so $\operatorname{cd}(G) \geqslant 3$ by Theorem 3.1. It remains to verify the upper bound. If $\ell = 3$, then $\operatorname{cd}(G) \leqslant m$ by (1) and so the upper bound holds. Thus we may assume that $\ell \geqslant 4$. Let $C_\ell\colon v_1, v_2, \ldots, v_\ell, v_1$ be a cycle of length ℓ in G, let $d = \lfloor \ell/2 \rfloor$, and let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+3}\}$ be a decomposition of G, where $E(G_1) = \{v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4, \ldots, v_dv_{d+1}\}$, $E(G_3) = \{v_{d+1}v_{d+2}, v_{d+2}v_{d+3}, \ldots, v_{\ell-1}v_\ell, v_\ell v_1\}$, and each of G_i $(4 \leqslant i \leqslant m - \ell + 3)$ contains exactly one edge in $E(G) - E(C_\ell)$. Thus $\mathcal{D}$ is connected. Furthermore, $c_{\mathcal{D}}(v_1v_2) = (0, 1, 1, \ldots)$, $c_{\mathcal{D}}(v_iv_{i+1}) = (i-1, 0, \ \min\{i, d-i+1\}, \ldots)$ for $2 \leqslant i \leqslant d$, $c_{\mathcal{D}}(v_{d+1}v_{d+2}) = (d, 1, 0, \ldots)$, $c_{\mathcal{D}}(v_iv_{i+1}) = (\ell - i + 1, \ \min\{i - d, \ell - i + 2\}, 0, \ldots)$ for $d + 2 \leqslant i \leqslant \ell - 1$, and $c_{\mathcal{D}}(v_\ell v_1) = (1, 2, 0, \ldots)$, it follows that the $\mathcal{D}$-codes of vertices of G are distinct. Thus $\mathcal{D}$ is a connected resolving decomposition of G and so $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m - \ell + 3$.

If G is a cycle C_n of order $n \geqslant 3$, then $\ell = m = n$ and so $\operatorname{cd}(G) = 3$. For the converse, let $G \neq C_n$ be a connected graph of order $n \geqslant 3$, size $m \geqslant 3$, and girth $\ell \geqslant 3$ and let C_ℓ: $v_1, v_2, \ldots, v_\ell, v_1$ be a smallest cycle in G, where $\ell < n$. Since G is connected and $G \neq C_n$, it follows that $m \geqslant 4$ and there exists a vertex $v \in V(G) - V(C_\ell)$ such that v is adjacent to a vertex of C_ℓ, say $vv_1 \in E(G)$. We consider three cases.

Case 1. $\ell = 3$. Then G contains an induced subgraph H_1 of Figure 3(a), where dashed lines indicate that the given edges may or may not be present. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\}$, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, $E(G_3) = \{v_1v_3\}$, and each of G_i $(4 \leqslant i \leqslant m - \ell + 2)$ contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Since $d(vv_1, G_2) = 1$ and $d(v_1v_2, G_2) = 2$, it follows that $\mathcal{D}$ is a connected resolving decomposition of G and so $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m - \ell + 2$.

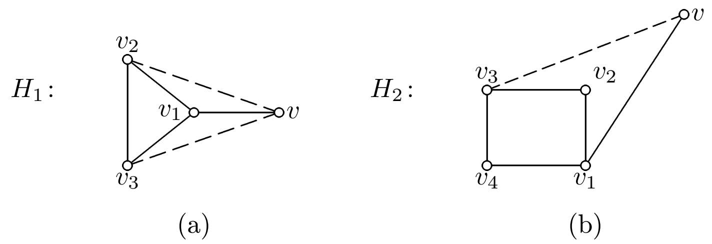

Figure 3. The subgraphs H_1 and H_2

Case 2. $\ell = 4$. Then G contains an induced subgraph H_2 of Figure 3(b), where the dashed line indicate that the given edge may or may not be present. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\}$, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, $E(G_3) = \{v_1v_4,\ v_3v_4\}$, and each of G_i $(4 \leqslant i \leqslant m - \ell + 2)$ contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Since $d(vv_1, G_2) = 2$, $d(v_1v_2, G_2) = 1$, $d(v_1v_4, G_2) = 2$, and $d(v_3v_4, G_2) = 1$, it follows that $\mathcal{D}$ is a connected resolving decomposition of G and so $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m - \ell + 2$.

Case 3. $\ell \geqslant 5$. Since C_ℓ is a smallest cycle in G, it follows that v is adjacent exactly one vertex of C_ℓ. Let $d = \lfloor \ell/2 \rfloor$ and let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\}$ be a decomposition of G, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4, \ldots, v_dv_{d+1}\}$, $E(G_3) = \{v_{d+1}v_{d+2}, v_{d+2}v_{d+3}, \ldots, v_{\ell-1}v_\ell, v_\ell v_1\}$, and each of G_i $(4 \leqslant i \leqslant m - \ell + 2)$ contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Thus $\mathcal{D}$ is connected. Since $c_{\mathcal{D}}(vv_1) = (0, 2, 2, \ldots)$, $c_{\mathcal{D}}(v_1v_2) = (0, 1, 1, \ldots)$, $c_{\mathcal{D}}(v_iv_{i+1}) = (i - 1, 0, \min\{i, d - i + 1\}, \ldots)$ for $2 \leqslant i \leqslant d$, $c_{\mathcal{D}}(v_{d+1}v_{d+2}) = (d, 1, 0, \ldots)$, $c_{\mathcal{D}}(v_iv_{i+1}) = (\ell - i + 1, \min\{i - d, \ell - i + 2\}, 0, \ldots)$ for $d + 2 \leqslant i \leqslant \ell - 1$, and $c_{\mathcal{D}}(v_\ell v_1) = (1, 2, 0, \ldots)$, it follows that $\mathcal{D}$ is a connected resolving decomposition of G. Thus $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m - \ell + 2$. $\square$

Next, we present an upper bound for $\operatorname{cd}(G)$ of a connected graph G in terms of its order. For a connected graph G, let

$$f(G) = \min\{k(G - E(T))\colon\ T \text{ is a spanning tree of } G\},$$

where $k(G - E(T))$ is the number of components of $G - E(T)$.

Theorem 3.5. *If G is a connected graph of order $n \geqslant 5$, then*

$$\operatorname{cd}(G) \leqslant n + f(G) - 1.$$

P r o o f. If G is a tree of order n, then $f(G) = 0$. Since the size of G is $n-1$, it follows by (1) that $\operatorname{cd}(G) \leqslant n-1$ and so the result is true for a tree. Thus we may assume that G is a connected graph that is not a tree. Suppose that $f(G) = k$. Let T be a spanning tree of G such that $k(G-E(T)) = k$, where $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ and $H_1, H_2, \ldots, H_k$ are k components of $G - E(T)$. Let

$$\mathcal{D} = \{G_1, G_2, \ldots, G_{n-1}, H_1, H_2, \ldots, H_k\},$$

where $E(G_i) = \{e_i\}$ for $1 \leqslant i \leqslant n-1$. Then $\mathcal{D}$ is a connected decomposition of G with $n+k-1$ elements.

We now show that $\mathcal{D}$ is a resolving decomposition of G. Let e and f be two edges of G. If e and f belongs to distinct elements of $\mathcal{D}$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus we may assume that e and f belong to the same element H_i in $\mathcal{D}$, where $1 \leqslant i \leqslant k$. We show that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Let $e = uv$ and let P be the unique $u - v$ path in T, and let u' and v' be the vertices on P adjacent to u and v, respectively. If f is adjacent to at most one of uu' and vv', then either $d(e, uu') \neq d(f, uu')$ or $d(e, vv') \neq d(f, vv')$, and so $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Hence we may assume that f is adjacent to both uu' and vv'. If $u' = v'$, then f is incident with the vertex u'. Since $n \geqslant 5$ and T is a spanning tree, there is a vertex $x \in V(G) - \{u, v, u'\}$ such that x is adjacent in T with exactly one of u, v and u'. If $u'x \in E(T)$, then $d(f, u'x) = 1 \neq 2 = d(e, u'x)$; otherwise, $d(e, ux) = 1 \neq 2 = d(f, ux)$ or $d(e, vx) = 1 \neq 2 = d(f, vx)$, according to whether ux or vx is an edge of T. So $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. If $u' \neq v'$, then we may assume that f is incident with u'. Let g be an edge of T distinct from uu' that is incident with u'. Then $d(e, g) = 2 \neq 1 = d(f, g)$. Therefore, $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Therefore, $\mathcal{D}$ is a connected resolving decomposition of G and so $\operatorname{cd}(G) \leqslant |\mathcal{D}| = n + k - 1 = n + f(G) - 1$. $\square$

Note that if $G = K_{1,n-1}$, where $n \geqslant 5$, then $f(G) = 0$ and $\operatorname{cd}(G) = n-1$. Thus the upper bound in Theorem 3.5 is attainable for stars. On the other hand, the inequality in Theorem 3.5 can be strict. For example, the graph G of Figure 4 has order $n = 8$

and $f(G) = 2$. Since $\mathcal{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{e_1, e_2, e_3, e_5, e_7, e_8, e_9\}$, $E(G_2) = \{e_4\}$, and $E(G_3) = \{e_6\}$, is a connected resolving decomposition of G, it then follows by Theorem 3.1 that $\mathrm{cd}(G) = 3$. Therefore, $\mathrm{cd}(G) < n + f(G) - 1$ for the graph of Figure 4.

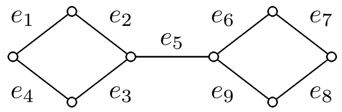

Figure 4. A graph G with $\mathrm{cd}(G) < n + f(G) - 1$

4. Connected decomposition numbers of trees

Although the decomposition dimensions of trees that are not paths have been studied in [2], [4], there is no general formula for the decomposition dimension of a tree that is not a path. However, we are able to establish a formula for the connected decomposition number of a tree that is not a path. First, we need some additional definitions.

A vertex of degree at least 3 in a connected graph G is called a *major vertex* of G. An end-vertex u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G and let $\mathrm{ex}(G)$ denote the number of exterior major vertices of G. If G is a tree that is not path, then $\sigma(G)$ is the number of end-vertices of G. For example, the tree T of Figure 5 has four major vertices, namely, v_1, v_2, v_3, v_4. The terminal vertices of v_1 are u_1 and u_2, the terminal vertices of v_3 are u_3, u_4, and u_5, and the terminal vertices of v_4 are u_6 and u_7. The major vertex v_2 has no terminal vertex and so v_2 is not an exterior major vertex of T. Therefore, $\sigma(T) = 7$ and $\mathrm{ex}(T) = 3$.

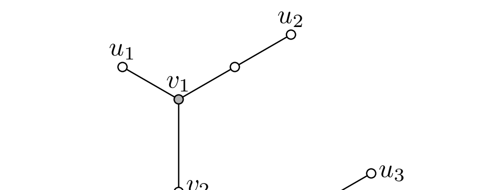

Figure 5. A tree with its exterior major vertices

In this section, we present a formula for the connected decomposition number of a tree T that is not a path in term of $\sigma(T)$ and $\mathrm{ex}(T)$. In order to do this, we first present a useful lemma. For an ordered set $W = \{e_1, e_2, \ldots, e_k\}$ of edges in a connected graph G and an edge e of G, the k-vector

$$c_W(e) = (d(e, e_1), d(e, e_2), \ldots, d(e, e_k))$$

is referred to as the *code of e with respect to W*. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ in G is called a *branch of G at v*. For a bridge e in a connected graph G and a component F of $G - e$, the subgraph F together the bridge e is called a *branch of G at e*. For two edges $e = u_1u_2$ and $f = v_1v_2$ in G, an *$e - f$ path* in G is a path with its initial edge e and terminal edge f.

Lemma 4.1. *Let T be a tree that is not a path, having order $n \geqslant 4$ and p exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of v_i, let P_{ij} be the $v_i - u_{ij}$ path $(1 \leqslant j \leqslant k_i)$, and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i. Let*

$$W = \{v_i x_{ij} \colon 1 \leqslant i \leqslant p \ \text{ and } \ 2 \leqslant j \leqslant k_i\}.$$

Then $c_W(e) \neq c_W(f)$ for each pair e, f of distinct edges of T that are not edges of P_{ij} for $1 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant k_i$.

Proof. Let e and f be two edges of T that are not edges of P_{ij} for $1 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant k_i$. We consider two cases.

Case 1. *e lies on some path P_{i1} for some i with $1 \leqslant i \leqslant p$.* There are two subcases.

Subcase 1.1. *There is an edge $w \in W$ such that f lies on the $e - w$ path or e lies on the $f - w$ path.* Then either $d(f, w) < d(e, w)$ or $d(e, w) < d(f, w)$. In either case, $c_W(e) \neq c_W(f)$.

Subcase 1.2. *Every path between f and an edge of W does not contain e and every path between e and an edge of W does not contain f.* Necessarily, then f lies on some path $P_{\ell 1}$ in T for some $1 \leqslant \ell \leqslant p$. Observe that $i \neq \ell$, for otherwise, f lies on $e - w$ path, where $w = v_i x_{i2} \in W$. Since v_i and v_ℓ are exterior major vertices, it follows that $\deg v_i \geqslant 3$ and $\deg v_\ell \geqslant 3$. Thus there exist a branch B_1 at v_i that does not contain x_{i1} and a branch B_2 at v_ℓ that does not contain $x_{\ell 1}$. Necessarily, each of B_1 and B_2 must contain an edge of W. Let w_1 and w_2 be two edges in W such that w_i belongs to B_i for $i = 1, 2$. If $d(e, w_2) \neq d(f, w_2)$, then $c_W(e) \neq c_W(f)$. Thus we may assume that $d(e, w_2) = d(f, w_2)$. However, then $d(e, w_1) < d(f, w_1)$, again implying that $c_W(e) \neq c_W(f)$.

C a s e 2. *e lies on no path* P_{i1} *for all* i *with* $1 \leqslant i \leqslant p$. Then there are at least two branches at e, say B_1^* and B_2^*, each of which contains some exterior major vertex of terminal degree at least 2. Thus each branch B_i^* $(i = 1, 2)$ contains an edge in W. Let $w_i^* \in W$ such that w_i^* belongs to B_i^* for $i = 1, 2$. First, assume that $f \in E(B_1^*)$. Then the $f - w_2^*$ path of T contains e. So $d(e, w_2^*) < d(f, w_2^*)$, implying that $c_W(e) \neq c_W(f)$. Next, assume that $f \notin E(B_1^*)$. Then the $f - w_1^*$ path of T contains e. Thus $d(e, w_1^*) < d(f, w_1^*)$ and so $c_W(e) \neq c_W(f)$. □

We are now prepared to establish a formula for the connected decomposition number of a tree that is not a path.

Theorem 4.2. *If* T *is a tree that is not a path, then*

$$\operatorname{cd}(T) = \sigma(T) - \operatorname{ex}(T) + 1.$$

P r o o f. Suppose that T contains p exterior major vertices $v_1, v_2, \ldots, v_p$. For each i with $1 \leqslant i \leqslant p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of v_i. For each pair i, j of integers with $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant k_i$, let P_{ij} be the $v_i - u_{ij}$ path in T and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i.

First, we claim that if $\mathcal{D}$ is a connected resolving decomposition of T, then, for each fixed exterior major vertex v_i $(1 \leqslant i \leqslant p)$, there is at least one edge, say e_{ij}, from each path P_{ij} $(1 \leqslant j \leqslant k_i)$ such that the k_i edges e_{ij} $(1 \leqslant j \leqslant k_i)$ of T belong to distinct elements in $\mathcal{D}$. To verify this claim, assume, to the contrary, that this is not the case. Since each element in $\mathcal{D}$ is connected, we assume, without loss of generality, that P_{i1} and P_{i2} are contained in the same element of $\mathcal{D}$. However, then, $d(v_i x_{i1}, e) = d(v_i x_{i2}, e)$ for all $e \in E(G - (P_{i1} \cup P_{i2}))$, and so $c_{\mathcal{D}}(v_i x_{i1}) = c_{\mathcal{D}}(v_i x_{i2})$, which is a contradiction. Therefore, for each fixed i with $1 \leqslant i \leqslant p$, the k_i edges $e_{ij} \in E(P_{ij})$ $(1 \leqslant j \leqslant k_i)$ belong to distinct elements in $\mathcal{D}$, as claimed.

First, we show that $\operatorname{cd}(T) \geqslant \sigma(T) - \operatorname{ex}(T) + 1$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_\ell\}$ be a minimum connected resolving decomposition of T. Let $V = \{v_1, v_2, \ldots, v_p\}$ be the set of the exterior major vertices of T. First, assume that $p = 1$. Since the k_1 edges $e_{1j} \in E(P_{1j})$ $(1 \leqslant j \leqslant k_1)$ belong to distinct elements in $\mathcal{D}$, it follows that $\operatorname{cd}(G) \geqslant k_1 = \sigma(T) - \operatorname{ex}(T) + 1$. Thus we may assume that $p \geqslant 2$. We proceed by the following steps:

S t e p 1. Since $p \geqslant 2$, there exists an exterior major vertex v_i with $1 \leqslant i \leqslant p$ such that $\deg v_i = k_i + 1$. Start with such an exterior major vertex, say v_1 with $\deg v_1 = k_1 + 1$. Since the k_1 edges $e_{1j} \in E(P_{1j})$ $(1 \leqslant j \leqslant k_1)$ belong to distinct elements in $\mathcal{D}$, we may assume, without loss of generality, that $e_{1j} \in E(G_j)$ for $1 \leqslant j \leqslant k_1$. Thus

$$\operatorname{cd}(G) = |\mathcal{D}| \geqslant k_1 = (k_1 - 1) + 1.$$

S t e p 2. Consider an exterior major vertex $v \in V - \{v_1\}$ such that the $v_1 - v$ path in T contains no other exterior major vertices in $V - \{v_1, v\}$. We may assume that $v = v_2$. Then the k_2 edges $e_{2j} \in E(P_{2j})$ $(1 \leqslant j \leqslant k_2)$ belong to distinct elements in $\mathcal{D}$. We claim that at most one of the edges e_{2j} $(1 \leqslant j \leqslant k_2)$ belongs to the elements $G_1, G_2, \ldots, G_{k_1}$ of $\mathcal{D}$. Assume, to the contrary, that two edges in $\{e_{2j} \colon 1 \leqslant j \leqslant k_2\}$ belong to $G_1, G_2, \ldots, G_{k_1}$, say e_{21} and e_{22} belong to $G_1, G_2, \ldots, G_{k_1}$. Since e_{21} and e_{22} belong to distinct elements in $\mathcal{D}$, it follows that e_{21} and e_{22} belong to two distinct elements of $G_1, G_2, \ldots, G_{k_1}$, say $e_{21} \in E(G_1)$ and $e_{22} \in E(G_2)$. However, then, either G_1 or G_2 must be disconnected, which is a contradiction. Hence, as claimed, at most one of the edges e_{2j} $(1 \leqslant j \leqslant k_2)$ belongs to the elements $G_1, G_2, \ldots, G_{k_1}$ in $\mathcal{D}$. Then assume, without loss of generality, that $e_{2j} \in E(G_{j+k_1})$ for $1 \leqslant j \leqslant k_2 - 1$. Thus $G_1, G_2, \ldots, G_{k_1}, G_{k_1+1}, \ldots, G_{k_1+k_2-1}$ must be distinct elements of $\mathcal{D}$, implying that

$$\mathrm{cd}(G) = |\mathcal{D}| \geqslant k_1 + k_2 - 1 = (k_1 - 1) + (k_2 - 1) + 1.$$

If $p = 2$, then $k_1 + k_2 - 1 = \sigma(T) - \mathrm{ex}(T) + 1$ and the proof is complete. Otherwise, we continue to the next step.

S t e p 3. Consider an exterior major vertex $v \in V - \{v_1, v_2\}$ such that the $v_1 - v$ path in T contains no other exterior major vertices in $V - \{v_1, v_2\}$. We may assume that $v = v_3$. Then the k_3 edges $e_{3j} \in E(P_{3j})$ $(1 \leqslant j \leqslant k_3)$ belong to distinct elements in $\mathcal{D}$. Again, we claim that at most one of the edges $e_{3j} \in E(P_{3j})$ $(1 \leqslant j \leqslant k_3)$ belongs to some element G_i of $\mathcal{D}$, where $1 \leqslant i \leqslant k_1 + k_2 - 1$. Assume, to the contrary, that two edges in $\{e_{3j} \colon 1 \leqslant j \leqslant k_2\}$ belong to G_s and G_t, respectively, where $1 \leqslant s < t \leqslant k_1 + k_2 - 1$, say $e_{31} \in E(G_s)$ and $e_{32} \in E(G_t)$. If $1 \leqslant s < t \leqslant k_1$ or $k_1 + 1 \leqslant s < t \leqslant k_1 + k_2 - 1$, then at least one of G_s and G_t must be disconnected, which is impossible. On the other hand, if $1 \leqslant s \leqslant k_1$ and $k_1 + 1 \leqslant t \leqslant k_1 + k_2 - 1$, then, since G_s and G_t are connected, there must be a cycle in T, which is again impossible. Thus, we may assume, without loss of generality, that $e_{3j} \in E(G_{k_1+k_2-1+j})$ for $1 \leqslant j \leqslant k_3 - 1$. Hence all subgraphs G_i $(1 \leqslant i \leqslant k_1 + k_2 + k_3 - 2)$ are distinct elements of $\mathcal{D}$ and so

$$\mathrm{cd}(G) = |\mathcal{D}| \geqslant k_1 + k_2 + k_3 - 2 = (k_1 - 1) + (k_2 - 1) + (k_3 - 1) + 1.$$

We continue this procedure to the remaining exterior major vertices in $V - \{v_1, v_2, v_3\}$ and repeat the argument similar to the one in the previous step until we exhaust all vertices in V. Then we obtain

$$\mathrm{cd}(G) = |\mathcal{D}| \geqslant \left(\sum_{i=1}^{p}(k_i - 1)\right) + 1 = \sigma(G) - \mathrm{ex}(G) + 1.$$

Next we show that $\mathrm{cd}(T) \leqslant \sigma(T) - \mathrm{ex}(T) + 1$. Let $k = \sigma(T) - \mathrm{ex}(T) + 1$. Let $f_{ij} = v_i x_{ij}$ for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant k_i$. Let $U = \{v_1, u_{11}, u_{21}, \ldots, u_{p1}\}$ and let T_0 be the subtree of T of smallest size such that T_0 contains U. Let

$$\mathcal{D} = \{T_0, P_{12}, P_{13}, \ldots, P_{1k_1}, P_{22}, P_{23}, \ldots, P_{2k_2}, \ldots, P_{p2}, P_{p3}, \ldots, P_{pk_p}\}.$$

Certainly, $\mathcal{D}$ is a connected k-decomposition of T. We show that $\mathcal{D}$ is a resolving decomposition of T. It suffices to show that the edges of T belonging to same element of $\mathcal{D}$ have distinct $\mathcal{D}$-codes. Let $e, f \in E(T)$. We consider two cases.

Case 1. $e, f \in E(T_0)$. Then $d(e, P_{ij}) = d(e, f_{ij})$ and $d(f, P_{ij}) = d(f, f_{ij})$ for all pairs i, j with $1 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant k_i$. Let

$$W = \{f_{ij}\colon\ 1 \leqslant i \leqslant p \text{ and } 2 \leqslant j \leqslant k_i\}.$$

By Lemma 4.1, $c_W(e) \neq c_W(f)$. Observe that the first coordinate in each of $c_{\mathcal{D}}(e)$ and $c_{\mathcal{D}}(f)$ is 0, the remaining $k-1$ coordinates of $c_{\mathcal{D}}(e)$ are exactly those of $c_W(e)$, and the remaining $k-1$ coordinates of $c_{\mathcal{D}}(f)$ are exactly those of $c_W(f)$. Since $c_W(e) \neq c_W(f)$, it follows that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Case 2. $e, f \in E(P_{ij})$, where $1 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant k_i$. Then $d(e, T_0) = d(e, f_{i1})$ and $d(f, T_0) = d(f, f_{i1})$. Since e and f are two distinct edges in the path P_{ij}, it follows that $d(e, f_{i1}) \neq d(f, f_{i1})$ and so $d(e, T_0) \neq d(f, T_0)$. Therefore, $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Therefore, $\mathcal{D}$ is a connected resolving k-decomposition of T and so $\mathrm{cd}(T) \leqslant k = \sigma(T) - \mathrm{ex}(T) + 1$, as desired. □

5. Graphs with prescribed decomposition dimension and connected decomposition number

We have seen that if G is a connected graph of size at least 2 with $\dim_{\mathrm{d}}(G) = a$ and $\mathrm{cd}(G) = b$, then $2 \leqslant a \leqslant b$. Furthermore, paths of order at least 3 are the only connected graphs G of size at least 2 with $\dim_{\mathrm{d}}(G) = \mathrm{cd}(G) = 2$. Thus there is no connected graph G with $\dim_{\mathrm{d}}(G) = 2$ and $\mathrm{cd}(G) > 2$. On the other hand, every pair a, b of integers with $3 \leqslant a \leqslant b$ is realizable as the decomposition dimension and connected decomposition number, respectively, of some graph. In order to show this, we first present a useful lemma.

Lemma 5.1. *Let G be a connected graph that is not a star. If G contains a vertex that is adjacent to $k \geqslant 1$ end-vertices, then $\dim_{\mathrm{d}}(G) \geqslant k+1$ and $\mathrm{cd}(G) \geqslant k+1$.*

Proof. By Observation 1.1, $\dim_{\mathrm{d}}(G) \geqslant k$. Next we show that $\dim_{\mathrm{d}}(G) \neq k$. Assume, to the contrary, that $\dim_{\mathrm{d}}(G) = k$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ be a resolving decomposition of G. Let v be a vertex of G that is adjacent to k end-vertices $v_1, v_2, \ldots, v_k$. Let $e_i = vv_i$, where $1 \leqslant i \leqslant k$. By Observation 1.1, the k edges e_i ($1 \leqslant i \leqslant k$) belong to distinct elements of $\mathcal{D}$. Without loss of generality, assume that $e_i \in E(G_i)$ for $1 \leqslant i \leqslant k$. Since G is not a star, there exists a vertex w distinct from v_i ($1 \leqslant i \leqslant k$) such that w is adjacent to v and w is not an end-vertex of G. We may assume the edge $e = vw$ belongs to G_1. However, then, $c_{\mathcal{D}}(e) = c_{\mathcal{D}}(e_1) = (0, 1, 1, \ldots, 1)$, which is a contradiction. Thus $\dim_{\mathrm{d}}(G) \geqslant k + 1$. The fact that $\mathrm{cd}(G) \geqslant k + 1$ follows by (1). □

Theorem 5.2. *For every pair a, b of integers with $3 \leqslant a \leqslant b$, there exists a connected graph G such that $\dim_{\mathrm{d}}(G) = a$ and $\mathrm{cd}(G) = b$.*

Proof. For $a = b \geqslant 3$, let $G = K_{1,a}$. Since $\dim_{\mathrm{d}}(K_{1,a}) = \mathrm{cd}(K_{1,a}) = a$, the result holds for $a = b$. Thus we may assume that $a < b$. We consider two cases, according to whether $a = 3$ or $a \geqslant 4$.

Case 1. $a = 3$. For each i with $1 \leqslant i \leqslant b-1$, let T_i be the tree obtained from the path $P_i\colon v_{i1}, v_{i2}, \ldots, v_{ii}$ of order i by adding two new vertices u_i and u_i^* and joining u_i and u_i^* to v_{ii}. Then the graph G is obtained from the graphs T_i ($1 \leqslant i \leqslant b-1$) by adding edges $v_{i1}v_{i+1,1}$ for $1 \leqslant i \leqslant b-2$. The graph G is shown in Figure 6 for $b = 5$. Since G is a tree with $\sigma(G) = 2(b-1)$ and $\mathrm{ex}(G) = b - 1$, it follows by Theorem 4.2 that $\mathrm{cd}(G) = b$. It remains to show that $\dim_{\mathrm{d}}(G) = 3$. Let $\mathcal{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{u_1v_{11}\}$, $E(G_2) = \{u_iv_{ii}\colon 2 \leqslant i \leqslant d-1\}$, and $E(G_3) = E(G) - (E(G_1) \cup E(G_2))$. We show that $\mathcal{D}$ is a resolving decomposition of G. Observe that $c_{\mathcal{D}}(u_iv_{ii}) = (2i-1, 0, 1)$ for $2 \leqslant i \leqslant b-1$, $c_{\mathcal{D}}(u_1^*v_{11}) = (1, 3, 0)$, $c_{\mathcal{D}}(v_{11}v_{21}) = (1, 2, 0)$, $c_{\mathcal{D}}(v_{i1}v_{i+1,1}) = (i, i, 0)$ for $2 \leqslant i \leqslant b-2$, $c_{\mathcal{D}}(v_{ij}v_{i,j+1}) = (i+j-1, i-j, 0)$ for $j \leqslant i$ and $2 \leqslant i \leqslant b-1$ and $1 \leqslant j \leqslant b-2$, and $c_{\mathcal{D}}(u_i^*v_{ii}) = (2i-1, 1, 0)$ for $2 \leqslant i \leqslant b-1$. Since all $\mathcal{D}$-codes of vertices G are distinct, $\mathcal{D}$ is a resolving decomposition of G and so $\dim_{\mathrm{d}}(G) \leqslant |\mathcal{D}| = 3$. By Theorem A, $\dim_{\mathrm{d}}(G) = 3$.

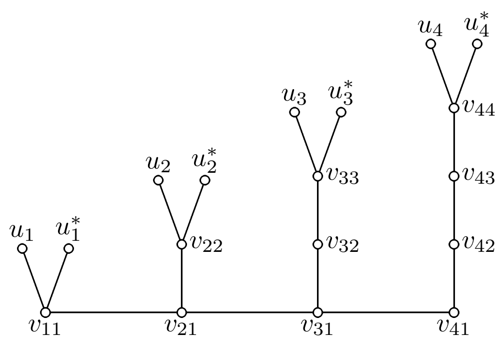

Figure 6. A graph G in Case 1 for $b = 5$

Case 2. $a \geqslant 4$. Let G be the graph obtained from the path P_{b-a+4}: $u_1, u_2, \ldots, u_{b-a+4}$ of order $b-a+4$ by (1) adding $a-2$ new vertices $v_1, v_2, \ldots, v_{a-2}$ and joining each vertex v_i $(1 \leqslant i \leqslant a-2)$ to u_2 (2) adding a new vertex v_{a-1} and joining v_{a-1} to u_{b-a+3}, and (2) adding $2(b-a)$ new vertices $w_3, w_3^*, w_4, w_4^*, \ldots, w_{b-a+2}, w_{b-a+2}^*$ and joining w_j and w_j^* to u_j for $3 \leqslant j \leqslant b-a+2$. Since G is a tree with $\sigma(G) = (a-1) + 2(b-a+1) = 2b-a+1$ and $\operatorname{ex}(G) = b-a+2$, it follows by Theorem 4.2 that $\operatorname{cd}(G) = b$. Next we show that $\dim_{\mathrm{d}}(G) = a$. Since u_2 is adjacent to $a-1$ end-vertices and T is not a star, it then follows by Lemma 5.1 that $\dim_{\mathrm{d}}(G) \geqslant a$. On the other hand, let $\mathcal{D} = \{G_1, G_2, \ldots, G_a\}$, where $E(G_1) = E(P_{b-a+4}) \cup \{u_i w_i\colon 3 \leqslant i \leqslant b-a+2\}$, $E(G_2) = \{u_2 v_1\} \cup \{u_i w_i^*\colon 3 \leqslant i \leqslant b-a+2\}$, $E(G_3) = \{u_{b-a+3} v_{a-1}\}$, and $E(G_i) = \{u_2 v_{i-2}\}$ for $4 \leqslant i \leqslant a$. It can be verified that $\mathcal{D}$ is a resolving decomposition of G, and so $\dim_{\mathrm{d}}(G) \leqslant |\mathcal{D}| = a$. Therefore, $\dim_{\mathrm{d}}(G) = a$, as desired. □

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Author's address: *Varaporn Saenpholphat*, *Ping Zhang*, Department of Mathematics and Statistics, Western Michigan University, Kalamazoo, MI 49008, USA, e-mails: phunoot@hotmail.com, ping.zhang@wmich.edu.