MICRO TANGENT SETS OF CONTINUOUS FUNCTIONS

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(Received December 31, 2001)

Abstract. Motivated by the concept of tangent measures and by H. Fürstenberg's definition of microsets of a compact set A we introduce micro tangent sets and central micro tangent sets of continuous functions. It turns out that the typical continuous function has a rich (universal) micro tangent set structure at many points. The Brownian motion, on the other hand, with probability one does not have graph like, or central graph like micro tangent sets at all. Finally we show that at almost all points Takagi's function is graph like, and Weierstrass's nowhere differentiable function is central graph like.

Keywords: typical continuous function, Brownian motion, Takagi's function, Weierstrass's function

MSC 2000: 26A27, 26A24, 28A78, 60J65

1. Introduction

In Mathematical Reviews 97j:28009, the reviewer (Joan Verdera) wrote the following: "Tangent measures play, with respect to measures, the same role that derivatives play with respect to functions. Given a measure μ (locally finite Borel measure on $\mathbb{R}^n$) and a point, one looks at the measure in a small neighborhood of the point, blows it up, normalizes suitably and takes a weak star limit in the space of measures. The result is a tangent measure for μ at the given point."

In this paper we return from measures to continuous functions and we see that this concept of blowing up and taking suitable limits, this time in the Hausdorff

This work was completed while the author visited Department of Mathematics of Michigan State University. Results of this paper were announced at meeting # 970 of the American Mathematical Society, October 5–6, 2001. The author also received partial support from the Hungarian National Foundation for Scientific Research Grant No. T 032042.

metric, might be useful to obtain information about tangential "regularity" of irregular functions. To do so we will introduce the concept of micro tangent sets of a continuous function. If the function is differentiable then the only micro tangent set we can obtain at a point is a straight line segment with the slope of the derivative. This shows that we have a generalization of the derivative.

For more irregular functions a much wider class of micro tangent sets may exist for many points. First we consider the typical continuous functions, then the Brownian motion.

It turns out that the micro tangent structure of the typical (in the sense of Baire category) $f \in C[0,1]$ for almost every $x \in [0,1]$ is universal. Nonetheless, $UMT(f)$, the set of points $(x; f(x))$ which are universal, has some paradoxical properties. Its projection onto the y-axis is of measure zero and it is of σ-finite one dimensional Hausdorff measure while the graph of the typical continuous function is of non-σ-finite one dimensional Hausdorff measure. We also show that the packing dimension of $UMT(f)$ is two, which is also the packing dimension of the graph of the typical continuous function. We remark that differentiability properties of typical continuous functions were studied for a long time and still are subject of current research, see for example Zajíček [20], Zajíček and Preiss [18] and the references in these papers.

The Brownian motion turns out to be wilder than the typical continuous function. With probability one the Brownian path $W(t)$ has no universal micro tangent points. If vertical distortion is allowed in taking the weak limits of tangent spaces of random processes, see also the recent papers of K. Falconer [4], [5]. We introduce some weaker conditions than universality, called graph like, or central graph like points, but the Brownian motion path is not behaving well with respect to these concepts either.

In Theorem 2 we see that on the graph of any continuous function the set of graph or central graph like points is of σ-finite $\mathscr{H}^1$-measure.

Finally, we look at two examples of "irregular functions": Takagi's function and Weierstrass's nowhere differentiable function. We see that these nowhere differentiable functions are tamer than the Brownian motion and almost every point is graph like for Takagi's function and central graph like for Weierstrass's nowhere differentiable function. Takagi's function has also a property which we call "micro-self similarity". This means that at almost every point the graph of the original function is a micro tangent set. The universality of typical continuous functions implies that they are also "micro-self similar".

We denote by $Q^{(m)}$ the closed unit cube of $\mathbb{R}^m$.

Working with a compact subset $A \subset \mathbb{R}^m$ in his talk [6] H. Fürstenberg defined the microsets of A in the following way: A' is a microset of A if there exist sequences of scaling constants $\gamma_n \in \mathbb{R}$ and translation vectors $t_n \in \mathbb{R}^m$ such that $\gamma_n A + t_n \cap Q^{(m)}$ converges to A' in the Hausdorff metric.

In this paper we will deal with special compact sets of $\mathbb{R}^2$, namely with graphs of continuous functions defined on $[0,1]$. This definition and the definition of tangent measures (see Preiss [17], Mattila [15], and Falconer [3] Ch. 9) have motivated our concept of micro tangent (M-tangent) sets of continuous functions.

2. Notation and basic definitions

Points in $\mathbb{R}^2$ will be denoted by $(x;y)$ while the open interval with endpoints x and y will be denoted by (x,y).

Given $A \subset \mathbb{R}^2$, by $|A|$, $\mathrm{int}(A)$, and $\mathrm{cl}(A)$ we mean its diameter, interior, and closure, respectively.

The closed cube of side length $2\delta > 0$ centered at $(x;y)$ will be denoted by $Q((x;y),\delta)$, that is, $Q((x;y),\delta) = \{(x';y') \colon |x'-x| \leqslant \delta \text{ and } |y'-y| \leqslant \delta\}$. Let Q^2 be the closed cube of side length 2, centered at $(0;0)$, that is, $Q((0;0),1)$.

If $F \subset \mathbb{R}^2$ then we denote by $CENT(F)$ the connected component of $F \cap Q^2$ which contains $(0;0)$, this component is the central component of F in Q^2.

The projections of the coordinate plane onto the x, or y axis are denoted by π_x, or π_y, respectively.

By $\mathrm{dist}_{\mathscr{H}}(A,B)$ we mean the Hausdorff distance of two compact sets A and B.

The one-dimensional Hausdorff measure in $\mathbb{R}^2$ will be denoted by $\mathscr{H}^1$, the Lebesgue measure on $\mathbb{R}$ will be denoted by λ.

It is not difficult to see that Vitali's covering theorem is also valid for coverings by closed squares, that is, the following variant of Theorem 2.8 of [14] holds.

Theorem 1. *Let μ be a Radon measure on $\mathbb{R}^2$, $A \subset \mathbb{R}^2$ and $\mathscr{Q}$ a family of closed squares such that each point of A is the centre of arbitrarily small squares of $\mathscr{Q}$, that is,*

$$\inf\{r \colon Q((x;y),r) \in \mathscr{Q}\} = 0 \text{ for } (x;y) \in A.$$

Then there are disjoint squares $Q_i \in \mathscr{Q}$ such that

$$\mu\Big(A \setminus \bigcup_i Q_i\Big) = 0.$$

By $C[-1,1]_0$ we mean the set of those functions g in $C[-1,1]$ for which $g(0)=0$.

Definition 1. The micro tangent (M-tangent) set system of $f \in C[0,1]$ at the point $x_0 \in (0,1)$ will be denoted by $f_{MT}(x_0)$ and it is defined as follows.

For $\delta_n > 0$ we put

$$F(f, x_0, \delta_n) = \frac{1}{\delta_n}((\text{graph}(f) \cap Q((x_0; f(x_0)), \delta_n)) - (x_0; f(x_0))), \tag{1}$$

that is, $F(f, x_0, \delta_n)$ is the $1/\delta_n$-times enlarged part of graph(f) belonging to $Q((x_0; f(x_0)), \delta_n)$ translated into Q^2.

The set F is a *micro tangent set (M-tangent set)* of f at x_0, that is, $F \in f_{MT}(x_0)$ if there exists $\delta_n \searrow 0$ such that $F(f, x_0, \delta_n)$ converges to F in the Hausdorff metric. The set F is a *central-micro tangent set* (CM-tangent set) of f at x_0, that is $F \in f_{CMT}(x_0)$ if there exists $\delta_n \searrow 0$ such that $CENT(F(f, x_0, \delta_n))$ converges to F in the Hausdorff metric.

It is easy to see that if f is differentiable at x_0 then $f_{MT}(x_0) = f_{CMT}(x_0)$ consists of one line segment of slope $f'(x_0)$ passing through the origin.

Definition 2. We say that x_0 is a *graph like*, or a *central graph like* MT-point for f if there exists $g \in C[-1,1]_0$ such that graph(g)$\cap Q^2 \in f_{MT}(x_0)$, or graph(g)$\cap Q^2 \in f_{CMT}(x_0)$, respectively. We denote by $GLMT(f)$, or by $CGLMT(f)$ the set of graph like micro tangent points, or the set of central graph like micro tangent points of f, that is, the set of those $(x_0; f(x_0))$ for which x_0 is a graph like, or central graph like MT-point of f, respectively.

Clearly, $CGLMT(f) \supset GLMT(f)$.

Definition 3. We say that x_0 is a *universal MT-point* for f if graph(g)$\cap Q^2 \in f_{MT}(x_0)$ for every $g \in C[-1,1]_0$.

The collection of those points $(x_0; f(x_0))$ for which x_0 is a universal MT-point of f will be denoted by $UMT(f)$.

Definition 4. We denote by $GLMT_g(f)$, or by $CGLMT_g(f)$ for a fixed $g \in C[-1,1]_0$ the set of those $(x_0; f(x_0))$ for which graph(g)$\cap Q^2$ belongs to $f_{MT}(x_0)$, or $CENT$(graph(g)$\cap Q^2$) belongs to $f_{CMT}(x_0)$, respectively.

Clearly, $GLMT_g(f) \subset GLMT(f)$, $GLMT_g(f) \subset CGLMT_g(f)$. Using the fact that we can always find a $g_1 \in C[-1,1]_0$ for which graph(g_1)$\cap Q^2 = CENT$(graph(g)$\cap Q^2$) we also have $CGLMT_g(f) \subset CGLMT(f)$.

We start with a result which is valid for an arbitrary continuous function. Theorem 2 shows that on the graph of f the set of graph like or central graph like points cannot be too large.

Theorem 2. *For any function $f \in C[0,1]$ the sets $GLMT(f)$ and $CGLMT(f)$ are of σ-finite $\mathscr{H}^1$-measure.*

Proof. By $CGLMT(f) \supset GLMT(f)$ it is enough to prove the theorem for $CGLMT(f)$. Given $\delta, \varepsilon > 0$ denote by $E_{\delta,\varepsilon}$ the set of those $(x_0; f(x_0))$ for which $CENT(F(f, x_0, \delta))$ does not intersect the line segments $L_{1,\varepsilon} = \{(t; 1) \colon |t| \leqslant \varepsilon\}$ and $L_{-1,\varepsilon} = \{(t; -1) \colon |t| \leqslant \varepsilon\}$.

Using the facts that f is continuous and that $L_{1,\varepsilon} \cup L_{-1,\varepsilon}$ is a closed set one can easily see that $E_{\delta,\varepsilon}$ is a relatively open subset of graph(f).

Next, assume that $g \in C[-1,1]_0$ and $\text{graph}(g) \cap Q^2 \in f_{CMT}(x_0)$. Then there exists $\varepsilon > 0$ such that $\text{graph}(g) \cap (L_{1,\varepsilon} \cup L_{-1,\varepsilon}) = \emptyset$. Since $\text{graph}(g) \cap Q^2$ and $L_{1,\varepsilon} \cup L_{-1,\varepsilon}$ are both compact, by the triangle inequality there exists $\varepsilon' > 0$ such that if $\text{dist}_{\mathscr{H}}(\text{graph}(g) \cap Q^2, CENT(F(f, x_0, \delta))) < \varepsilon'$ then $CENT(F(f, x_0, \delta))$ does not intersect $L_{1,\varepsilon} \cup L_{-1,\varepsilon}$. Using the definition of $f_{CMT}(x_0)$ we can choose $\delta_n \searrow 0$ such that $CENT(F(f, x_0, \delta_n))$ does not intersect $L_{1,\varepsilon} \cup L_{-1,\varepsilon}$.

Hence,

$$(x_0; f(x_0)) \in \bigcap_{n=1}^{\infty} \bigcup_{0<\delta<1/n} E_{\delta,\varepsilon}.$$

Therefore, from $(x_0; f(x_0)) \in CGLMT(f)$ it follows that

$$(x_0; f(x_0)) \in \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{0<\delta<1/n} E_{\delta,1/m} = H_0.$$

Next, we verify that $H = \bigcap_{n=1}^{\infty} \bigcup_{0<\delta<1/n} E_{\delta,1/m}$ is of σ-finite $\mathscr{H}^1$-measure for every $m \in \mathbb{N}$. This will imply that H_0, and hence $CGLMT(f)$, is also of σ-finite $\mathscr{H}^1$-measure.

Clearly, H_0 and H are Borel sets.

Proceeding towards a contradiction assume that $\mathscr{H}^1(H) = \infty$. Then by [2] Theorem 4.10 and Exercise 4.8 for any fixed c we can choose a Borel subset $H_c \subset H$ such that $\mathscr{H}^1(H_c) = c$. Assume $\eta > 0$ is given. Now, considering for each $(x_0; f(x_0)) \in H_c$ those squares $Q((x_0; f(x_0)), \delta_{n,x_0})$ for which $CENT(F(f, x_0, \delta_{n,x_0}))$ does not intersect $L_{1,1/m} \cup L_{-1,1/m}$ and $\delta_{n,x_0} < \eta$ we obtain a Vitali covering $\mathscr{Q}$ of H_c. Hence by Vitali's covering theorem used with the Radon measure $\mu(A) = \mathscr{H}^1(A \cap H_c)$, one can choose a system $(x_k; f(x_k)) \in H_c$ with a $\delta_k \in (0, \eta)$ such that the squares $Q_k = Q((x_k; f(x_k)), \delta_k)$ are disjoint, are of diameter less than $2\sqrt{2}\eta$, and $\mathscr{H}^1\Big(H_c \setminus \bigcup_k Q_k\Big) = 0$.

We have $(x; f(x)) \in Q_k = Q((x_k; f(x_k)), \delta_k)$ for $x \in [x_k - (1/m)\delta_k, x_k + (1/m)\delta_k]$. Hence, for $k \neq k'$ the intervals $[x_k - (1/m)\delta_k, x_k + (1/m)\delta_k]$ and $[x_{k'} - (1/m)\delta_{k'}, x_{k'} + (1/m)\delta_{k'}]$ are disjoint subintervals of $[0,1]$. This implies $\sum_k (2/m)\delta_k \leqslant 1$ and $\sum_k |Q_k| = \sum_k 2\sqrt{2}\delta_k \leqslant \sqrt{2}m$ and our cover of $H_c \cap \bigcup_k Q_k$ by $\bigcup_k Q_k$ consists of squares of diameter less than $2\sqrt{2}\eta$. On the other hand, $\mathscr{H}^1\Big(H_c \setminus \bigcup_k Q_k\Big) = 0$ implies that $\sum_k |Q_k| > c/2$ for small values of η, which is impossible when $c > 2\sqrt{2}m$. □

By a result of R. D. Mauldin and S. C. Williams ([16] Theorem 2) the graph of the typical continuous function is of Hausdorff dimension one, but is not of σ-finite $\mathscr{H}^1$-measure. So, Theorem 2 says that most points in the sense of $\mathscr{H}^1$-measure on the graph of the typical continuous function are neither graph nor central graph like. The next lemma implies that despite this relative smallness of $GLMT(f)$ and $CGLMT(f)$ the projection of these sets onto the x-axis is of full measure and, as we will see in Theorem 5 below, the projection of $UMT(f)$ onto the x-axis is also of full measure.

Lemma 3. *For a fixed $g \in C[-1,1]_0$ the set of those functions $f \in C[0,1]$ for which $\lambda(\pi_x(GLMT_g(f))) = \lambda(\pi_x(CGLMT_g(f))) = 1 = \lambda([0,1])$ is a dense G_δ set in $C[0,1]$.*

P r o o f. From $GLMT_g(f) \subset CGLMT_g(f)$ it follows that it is sufficient to show $\lambda(\pi_x(GLMT_g(f))) = 1$ for any $f \in \mathscr{G}$ for a dense G_δ set $\mathscr{G}$ of $C[0,1]$.

First choose and fix a countable dense subset $\{f_m\}_{m=1}^{\infty}$ in $C[0,1]$.

Assume $n \in \mathbb{N}$ is fixed. Our goal is to choose functions $\widehat{f}_{m,n}$ and numbers $\widehat{\eta}_{m,n} > 0$ such that if $f \in B(\widehat{f}_{m,n}, \widehat{\eta}_{m,n})$ then there exists a set $X_{m,n} \subset [0,1]$ such that

- $\lambda([0,1] \setminus X_{m,n}) < 2^{-n}$,
- for any $x_0 \in X_{m,n}$ there exists $\delta_{x_0} \in (0, 1/n)$ for which $\mathrm{dist}_{\mathscr{H}}(F(f, x_0, \delta_{x_0}), \mathrm{graph}(g) \cap Q^2) < 1/n$, and
- $B(\widehat{f}_{m,n}, \widehat{\eta}_{m,n}) \subset B(f_m, 1/mn)$.

Then we set $\mathscr{G}_n = \bigcup_m B(\widehat{f}_{m,n}, \widehat{\eta}_{m,n})$ and $\mathscr{G} = \bigcap_{n=1}^{\infty} \mathscr{G}_n$.

Now, $\mathscr{G}_n$ is a dense open set in $C[0,1]$ and if $f \in \mathscr{G}$ then there exist sequences $\{\widehat{f}_{m_n,n}\}_{n=1}^{\infty}$, $\{\widehat{\eta}_{m_n,n}\}_{n=1}^{\infty}$ such that $f \in B(\widehat{f}_{m_n,n}, \widehat{\eta}_{m_n,n})$. Since $\lambda([0,1] \setminus X_{m_n,n}) < 2^{-n}$, by the Borel-Cantelli lemma almost every $x_0 \in [0,1]$ belongs to finitely many sets of $[0,1] \setminus X_{m_n,n}$. Hence for almost every $x_0 \in [0,1]$ there exists an N_{x_0} such that $x_0 \in X_{m_n,n}$ for $n \geqslant N_{x_0}$. This implies $\operatorname{graph}(g) \cap Q^2 \in f_{MT}(x_0)$, that is, $(x_0; f(x_0)) \in GLMT_g(f)$.

Therefore, to complete the proof of this lemma it is sufficient to show how to find $\widehat{f}_{m,n}$ and $\widehat{\eta}_{m,n}$ for fixed $m, n \in \mathbb{N}$.

Set $f_1^* = f_{m,n}$, $\eta_1 = 1/mn$, $X_1^* = \emptyset$ and $\tau_1 = 1$.

Put

$$g_1(x) = \begin{cases} g(x) & \text{if } -1 \leqslant x \leqslant 1; \\ g(-1) & \text{if } x \leqslant -1; \\ g(1) & \text{if } x \geqslant 1. \end{cases}$$

Since $g \in C[-1,1]_0$ we can choose and fix M such that $|g|, |g_1| < M$.

Without loss of generality we can also suppose that $M \geqslant 1$.

As a small perturbation of g_1 choose a function $g_2 \in C(\mathbb{R})$ such that

- $g_2(0) = 0$,
- $|g_2| < M$,
- g_2 is continuously differentiable,
- there is no interval on which g_2 is constant,
- we have

$$\begin{aligned} &\operatorname{dist}_{\mathscr{H}}(\operatorname{graph}(g_2) \cap Q^2,\ \operatorname{graph}(g) \cap Q^2) \\ &\qquad = \operatorname{dist}_{\mathscr{H}}(F(g_2, 0, 1),\ \operatorname{graph}(g) \cap Q^2) < \frac{1}{2n}, \end{aligned} \tag{2}$$

- g_2 has no extrema on the boundary of Q^2, and
- g_2 does not go through any vertex of Q^2.

Using the above properties and the uniform continuity of g_2 on bounded intervals choose $\gamma \in (0, 1/2)$ and $\eta^* > 0$ such that for all $g^* \in B(g_2, \eta^*)$

$$\operatorname{dist}_{\mathscr{H}}(F(g^*, x, 1), F(g_2, 0, 1)) < \frac{1}{2n} \quad \text{if } |x| \leqslant \gamma.$$

(We need the properties of g_2 because intervals of constancy, local extrema etc. on the boundary of Q^2 might cause "jumps" in the Hausdorff metric when we move from g_2 to a nearby function g^* or move from 0 to x.) This by (2) implies for all $g^* \in B(g_2, \eta^*)$

$$\operatorname{dist}_{\mathscr{H}}(F(g^*, x, 1),\ \operatorname{graph}(g) \cap Q^2) < \frac{1}{n} \quad \text{if } |x| \leqslant \gamma. \tag{3}$$

Suppose f_j^*, η_j, X_j^* are given for a $j \geqslant 1$.

We will assume that X_j^* when $j \geqslant 2$ is the union of finitely many disjoint closed intervals and each maximal subinterval of $[0,1] \setminus X_j$ is of length at least τ_j.

Next, choose a large natural number κ_j such that it is divisible by four,

$$1/\kappa_j < \min(\tau_j/8, \eta_j/6M), \tag{4}$$

and the oscillation of f_j^* on an interval of length $4/\kappa_j$ is less than $\eta_j/3$. Assume k is an integer and $(4k+2)/\kappa_j \in [0,1]$. For $x \in [(4k+1)/\kappa_j, (4k+3)/\kappa_j]$ set

$$f_{j+1}^*(x) = f_j^*\Big(\frac{4k+2}{\kappa_j}\Big) + \frac{1}{\kappa_j(1+\gamma)} g_2\Big(\kappa_j(1+\gamma)\Big(x - \frac{4k+2}{\kappa_j}\Big)\Big). \tag{5}$$

Then our assumptions on M and κ_j imply that

$$|f_{j+1}^*(x) - f_j^*(x)| < \eta_j \text{ for } x \in \Big[\frac{4k+1}{\kappa_j}, \frac{4k+3}{\kappa_j}\Big]. \tag{6}$$

Next, extend the definition of f_{j+1}^* onto intervals of the form $[4k/\kappa_j, (4k+1)/\kappa_j)$ and $((4k+3)/\kappa_j, 4k/\kappa_j]$ so that (6) holds on these intervals as well and f_{j+1}^* is continuous on $[0,1]$.

Using (3) and (5) choose η_{j+1} such that for any $f \in B(f_{j+1}^*, \eta_{j+1})$ we have

$$\operatorname{dist}_{\mathscr{H}}\Big(F\Big(f, x, \frac{1}{\kappa_j(1+\gamma)}\Big),\ \operatorname{graph}(g) \cap Q^2\Big) < \frac{1}{n}$$
$$\text{if } x \in \Big[\frac{4k+2}{\kappa_j} - \frac{\gamma}{\kappa_j(1+\gamma)}, \frac{4k+2}{\kappa_j} + \frac{\gamma}{\kappa_j(1+\gamma)}\Big]$$

and $B(f_{j+1}^*, \eta_{j+1}) \subset B(f_j^*, \eta_j)$.

Set

$$X_{j+1}^* = X_j^* \cup \bigcup_{k \in \mathbb{Z}} \Big[\frac{4k+2}{\kappa_j} - \frac{\gamma}{\kappa_j(1+\gamma)}, \frac{4k+2}{\kappa_j} + \frac{\gamma}{\kappa_j(1+\gamma)}\Big] \cap [0,1].$$

From (4) and the definition of τ_j it follows that any interval contiguous to X_j contains at least one complete interval of the form $[4k/\kappa_j, 4(k+1)/\kappa_j]$. Thus there exists a constant γ^* depending only on γ and not depending on j such that

$$\mathscr{H}^1([0,1] \setminus X_{j+1}^*) < (1-\gamma^*)\mathscr{H}^1([0,1] \setminus X_j^*).$$

Hence, repeating the above procedure, there exists j such that $\mathscr{H}^1([0,1] \setminus X_j^*) < 2^{-n}$. Then we stop and set $\widehat{f}_{m,n} = f_j^*$, $\widehat{\eta}_{m,n} = \eta_j$ and $X_{m,n} = X_j^*$. □

Lemma 4. *If $g \in C[-1,1]_0$ and $f \in C[0,1]$ then $GLMT_g(f)$ is a G_δ set in the relative topology of* graph(f).

P r o o f. For a given $\varepsilon > 0$ set

$$E'_{q,\varepsilon} = \{(x_0; f(x_0))\colon \operatorname{dist}_{\mathscr{H}}(F(f,x_0,q),\ \operatorname{graph}(g)\cap Q^2) < \varepsilon\}.$$

Denote by $E_{q,\varepsilon}$ the interior of $E'_{q,\varepsilon}$ in the relative topology of the graph of f. Then clearly

$$GLMT_g(f) \supset \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{q<1/m} E_{q,1/n}.$$

On the other hand, if $(x_0; f(x_0)) \in GLMT_g(f)$ and $n, m \in \mathbb{N}$ are given using the definition of $GLMT_g(f)$, choose $\delta < 1/m$ such that

$$\operatorname{dist}_{\mathscr{H}}(F(f,x_0,\delta), \operatorname{graph}(g)\cap Q^2) < \frac{1}{n}.$$

Recall that f has only countably many intervals of constancy and countably many strict local extreme values. By choosing a slightly larger $q \in (\delta, 1/m)$, for which we still have

$$\operatorname{dist}_{\mathscr{H}}(F(f,x_0,q),\ \operatorname{graph}(g)\cap Q^2) < \frac{1}{n}, \tag{7}$$

we can assume that f is not constant and has no extreme values on the boundary of $Q((x_0; f(x_0)), q)$. Then by the continuity of f at x_0 and by (7) one can choose a $\delta' \in (0, q-\delta)$ such that for any $x' \in (x_0-\delta', x_0+\delta')$ we have $\operatorname{dist}_{\mathscr{H}}(F(f,x',q),\ \operatorname{graph}(g)\cap Q^2) < 1/n$ and hence $(x_0; f(x_0))$ belongs to $E_{q,1/n}$. Therefore,

$$GLMT_g(f) = \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{q<1/m} E_{q,1/n}.$$

□

Theorem 5. *There is a dense G_δ set $\mathscr{G}$ of $C[0,1]$ such that $\lambda(\pi_x(UMT(f))) = 1$ for all $f \in \mathscr{G}$. Furthermore, $UMT(f)$ is a dense G_δ subset in the relative topology of* graph(f). *Hence, for the typical continuous function in $C[0,1]$ almost every $x \in [0,1]$ is a universal MT-point and a typical point on the graph of f is in $UMT(f)$.*

P r o o f. Choose a countable dense system $\{g_n\}_{n=1}^\infty$ in $C[-1,1]_0$. By Lemma 3 for each g_n there exists a dense G_δ set $\mathscr{G}^n$ in $C[0,1]$ such that $\lambda(\pi_x(GLMT_{g_n}(f))) = 1$ for any $f \in \mathscr{G}^n$. Set $\mathscr{G} = \bigcap\limits_{n=1}^{\infty} \mathscr{G}^n$.

Assume $f \in \mathscr{G}$ and $g \in C[-1,1]_0$ are given. We need to show that $\mathrm{graph}(g) \cap Q^2 \in f_{MT}(x_0)$ for almost every $x_0 \in [0,1]$. Set $X = \bigcap_{n=1}^{\infty} \pi_x(GLMT_{g_n}(f)) = \pi_x\Big(\bigcap_{n=1}^{\infty} GLMT_{g_n}(f)\Big)$. Then $\lambda(X) = 1$.

Suppose $x_0 \in X$ and $\varepsilon > 0$ are given. Choose g_n such that

$$\mathrm{dist}_{\mathscr{H}}(\mathrm{graph}(g_n) \cap Q^2,\ \mathrm{graph}(g) \cap Q^2) < \frac{\varepsilon}{2}. \tag{8}$$

Since $x_0 \in X$ and $f \in \bigcap_{m=1}^{\infty} \mathscr{G}^m \subset \mathscr{G}^n$ we can choose $\delta \in (0,\varepsilon)$ such that

$$\mathrm{dist}_{\mathscr{H}}(F(f,x_0,\delta),\ \mathrm{graph}(g_n) \cap Q^2) < \frac{\varepsilon}{2}. \tag{9}$$

Hence, for any $\varepsilon > 0$ there exists $\delta \in (0,\varepsilon)$ such that

$$\mathrm{dist}_{\mathscr{H}}(F(f,x_0,\delta),\ \mathrm{graph}(g) \cap Q^2) < \varepsilon.$$

Therefore, $\mathrm{graph}(g) \cap Q^2 \in f_{MT}(x_0)$.

In fact, above we have shown that if $(x_0; f(x_0)) \in \bigcap_{n=1}^{\infty} GLMT_{g_n}(f)$ then $(x_0; f(x_0)) \in UMT(f)$. The other inclusion being obvious, we have

$$\bigcap_{n=1}^{\infty} GLMT_{g_n}(f) = UMT(f).$$

Since f is continuous, by virtue of $\lambda(X) = 1$ the set $\bigcap_{n=1}^{\infty} GLMT_{g_n}(f)$ is dense on the graph of f. From Lemma 4 it follows that $\bigcap_{n=1}^{\infty} GLMT_{g_n}(f)$ is a dense G_δ subset of $\mathrm{graph}(f)$. Hence, $UMT(f)$ is also a dense G_δ subset of $\mathrm{graph}(f)$. □

Let g_0 denote the identically zero function in $[-1,1]$. Then $CGLMT_{g_0}(f) = GLMT_{g_0}(f)$.

Lemma 6. *We have $\lambda(\pi_y(GLMT_{g_0}(f))) = 0$ for any $f \in C[0,1]$.*

Proof. By Lemma 4, $GLMT_{g_0}(f)$ is a Borel set. We use the notation introduced in the proof of Theorem 2. For g_0 we choose $\varepsilon = 1$ and observe that for $\varepsilon' \in (0,1]$ it follows from $\mathrm{dist}_{\mathscr{H}}(F(f,x_0,\delta), \mathrm{graph}(g_0) \cap Q^2) < \varepsilon'$ that $F(f,x_0,\delta)$ does not intersect $L_{1,1} \cup L_{-1,1}$, moreover $|f(x) - f(x_0)| < \varepsilon'\delta$ holds for $x \in [x_0-\delta, x_0+\delta]$. It follows that $GLMT_{g_0}(f) \subset H = \bigcap_{n=1}^{\infty} \bigcup_{0<\delta<1/n} E_{\delta,1}$ where $E_{\delta,\varepsilon}$ was defined at the beginning of the proof of Theorem 2. Hence, by the argument of this proof H is of finite $\mathscr{H}^1$-measure, which implies that $GLMT_{g_0}(f)$ is also of finite $\mathscr{H}^1$-measure.

Next, for any fixed $\varepsilon' > 0$, considering for each $(x_0; f(x_0)) \in GLMT_{g_0}(f)$ those cubes $Q((x_0; f(x_0)), \delta)$ for which

$$|f(x) - f(x_0)| < \varepsilon'\delta \quad \text{for } x \in [x_0 - \delta, x_0 + \delta], \tag{10}$$

we obtain a Vitali cover of $GLMT_{g_0}(f)$. By the Vitali covering theorem one can choose $(x_k, f(x_k)) \in GLMT_{g_0}(f)$ and $\delta_k > 0$ such that the cubes $Q_k = Q((x_k; f(x_k)), \delta_k)$ are disjoint and $\mathscr{H}^1(GLMT_{g_0}(f) \setminus \bigcup_k Q_k) = 0$. Then

$$\lambda\Big(\pi_y\Big(GLMT_{g_0}(f) \setminus \bigcup_k Q_k\Big)\Big) = 0 \tag{11}$$

holds as well. Now the disjointness of Q_k and $Q_{k'}$ for $k \neq k'$ implies that $[x_k - \delta_k, x_k + \delta_k]$ and $[x_{k'} - \delta_{k'}, x_{k'} + \delta_{k'}]$ are also disjoint. Hence $\sum_k 2\delta_k < 1$ and by (10) we obtain $\lambda\Big(\pi_y\Big(\operatorname{graph}(f) \cap \bigcup_k Q_k\Big)\Big) < \varepsilon gr' \sum_k 2\delta_k < \varepsilon'$. Using this and (11) we obtain that $\lambda(\pi_y(GLMT_{g_0}(f))) < \varepsilon'$ holds for any $\varepsilon' > 0$ and this concludes the proof. □

The next theorem shows that though $UMT(f)$ has large x-projection, it has small y-projection.

Theorem 7. *There is a dense G_δ set $\mathscr{G}$ of $C[0,1]$ such that $\lambda(\pi_y(UMT(f))) = 0$ for all $f \in \mathscr{G}$. Hence any preimage of almost every y in the range of the typical continuous function is not a UMT-point.*

P r o o f. Since $UMT(f) \subset GLMT_{g_0}(f)$ the theorem follows from Lemma 6. □

For the definition of the packing dimension we will use the notation introduced in [2] 3.4 p. 47, or [3] 2.1 pp. 22–23 and we recall that

$$\mathscr{P}^s_\delta(F_i) = \sup\Big\{\sum_j |B_j|^s \colon \{B_j\} \text{ is a collection of disjoint balls of radii at most } \delta \text{ with centers in } F_i\Big\}$$

and $\mathscr{P}^s_0(F_i) = \lim_{\delta\to 0} \mathscr{P}^s_\delta(F_i)$. Finally, in the definition of $\mathscr{P}^s(F)$, the s-dimensional packing measure of the Borel set F, one needs to set

$$\mathscr{P}^s(F) = \inf\Big\{\sum_i \mathscr{P}^s_0(F_i) \colon F \subset \bigcup_i F_i\Big\}. \tag{12}$$

We also recall that by a result of P. Humke and G. Petruska ([9]) the packing dimension of the typical continuous function equals two. Next we see that from the packing dimension point of view $UMT(f)$ is sufficiently large for the typical continuous function.

By considering typical restrictions of functions in $C[0,1]$ onto intervals $[a,b] \subset [0,1]$ with rational endpoints one can easily see that the graph of the typical continuous function $f \in C[0,1]$ restricted onto any interval $[a,b]$, $(a<b)$ is of packing dimension two.

Theorem 8. *For the typical continuous function $f \in C[0,1]$ the packing dimension of $UMT(f)$ equals two.*

Proof. We will work in the relative topology of graph(f) for an $f \in C[0,1]$.

Assume that $UMT(f)$ is a dense G_δ subset of graph(f) and $\mathscr{P}^s(UMT(f)) < \infty$ for an $s<2$.

Next we show that there exists an interval $[a,b]$ on which the graph of f is of packing dimension less than or equal to s. From (12) and Baire's Category Theorem applied to the graph of f it follows that there exists an F_i in a countable covering of F for which $\mathscr{P}^s_0(F_i) < \infty$ and F_i is dense in a portion of graph(f). From now on we assume that this F_i is fixed. Choose an interval $[a,b] \subset [0,1]$ $(a<b)$ such that F_i is dense in the set $S = \text{graph}(f|_{[a,b]})$. Since $F_i \cap S$ is dense in S we have $\mathscr{P}^s_0(F_i \cap S) = \mathscr{P}^s_0(S)$. Thus $\mathscr{P}^s_0(S) < \infty$, which implies $\mathscr{P}^s(S) < \infty$ and hence the packing dimension of S is less than two, but by the Humke-Petruska result for the typical continuous function graph$(f|_{[a,b]})$ is of packing dimension two. This implies the statement of Theorem 8. □

4. Brownian motion

In this section instead of working with $C[0,1]$ we will work with $C[0,+\infty]$, our definitions concerning micro tangent sets being generalized to this case in the obvious way. We use the notation of [1] Chapter 7.

Assume that $[W(t)\colon\ t \geqslant 0]$ denotes the Brownian motion. By [1] 37.14, p. 505 if

$$X_{n,k} = \max\left\{\left|W\Big(\frac{k+1}{2^n}\Big) - W\Big(\frac{k}{2^n}\Big)\right|,\ \left|W\Big(\frac{k+2}{2^n}\Big) - W\Big(\frac{k+1}{2^n}\Big)\right|,\right.$$
$$\left.\left|W\Big(\frac{k+3}{2^n}\Big) - W\Big(\frac{k+2}{2^n}\Big)\right|\right\}$$

then $P[X_{n,k} \leqslant \varepsilon] \leqslant (2 \cdot 2^{n/2} \cdot \varepsilon)^3$ (where $P[X_{n,k} \leqslant \varepsilon]$ denotes the probability that $X_{n,k} \leqslant \varepsilon$). Hence if $Y_n = \min\limits_{k \leqslant n \cdot 2^n} X_{n,k}$ then

$$P[Y_n \leqslant \varepsilon] \leqslant n \cdot 2^n (2 \cdot 2^{n/2} \varepsilon)^3. \tag{13}$$

Now we can formulate the main theorem of this section.

Theorem 9. *For almost every Brownian motion path $W(t)$, from $F \in W_{CMT}(t)$ $(t > 0)$ it follows that $F \subset S_0 \stackrel{\text{def}}{=} \{(0; y)\colon\ |y| \leqslant 1\}$. Therefore, $CGLMT(W) = \emptyset$ and $UMT(W) = \emptyset$ with probability one.*

Proof. We want to show that for any $\eta \in (0,1)$ with probability one for the Brownian motion path at any $t > 0$ there exists $\delta_{t,\eta} > 0$ such that for any $\delta \in (0, \delta_{t,\eta})$

$$CENT(F(W,t,\delta)) \subset S_\eta \stackrel{\text{def}}{=} \{(x; y)\colon\ |x| \leqslant \eta,\ |y| \leqslant 1\}. \tag{14}$$

This will imply that if $F \in W_{CMT}(t)$ then $F \subset S_\eta$ for all $\eta > 0$, that is, $F \subset S_0 = \bigcap\limits_{\eta > 0} S_\eta$.

To verify (14) it is sufficient to show that for any $t > 0$ there exists $\delta_{t,\eta} > 0$ such that for any $\delta \in (0, \delta_{t,\eta})$ one can find $t_- \in [t - \eta\delta, t]$ and $t_+ \in [t, t + \eta\delta]$ satisfying

$$|f(t_-) - f(t)| > \delta \quad \text{and} \quad |f(t_+) - f(t)| > \delta. \tag{15}$$

Set

$$K_\eta = 32/\eta \quad \text{and} \quad \varepsilon_n = K_\eta \cdot 2^{-n}. \tag{16}$$

From (13) applied with $\varepsilon = \varepsilon_n$ it follows that

$$P[Y_n \leqslant K_\eta 2^{-n}] \leqslant n \cdot 2^n (2 \cdot 2^{n/2} K_\eta \cdot 2^{-n})^3 = n \cdot 2^n (2K_\eta)^3 2^{-3n/2}.$$

Thus $\sum\limits_n P[Y_n \leqslant K_\eta 2^{-n}] < \infty$ and by the Borel-Cantelli lemma with probability one we have $Y_n \leqslant K_\eta 2^{-n}$ for only finitely many n's for a Brownian motion path W.

Assume that for W under consideration N_0 is chosen such that $Y_n > K_\eta 2^{-n}$ if $n > N_0$. For a fixed $t > 0$ we can assume that N_0 is chosen to be so large that $t \in (0, N_0/2)$.

Choose $\delta_{t,\eta} > 0$ such that

$$2^{N_0} < \frac{4}{\eta \delta_{t,\eta}} \quad \text{and} \quad \delta_{t,\eta} < t. \tag{17}$$

Then for a $\delta \in (0, \delta_{t,\eta})$ choose n such that

$$4 \cdot 2^{-n} < \eta\delta \leqslant 8 \cdot 2^{-n}. \tag{18}$$

This implies $4/\eta\delta_{t,\eta} < 4/\eta\delta < 2^n$ and by (17) we have $n > N_0$. By the definition of Y_n in any subinterval of length $4 \cdot 2^{-n}$ in $[0, n]$ one can choose two points t_1, t_2 such that

$$|f(t_1) - f(t_2)| > K_\eta 2^{-n} \geqslant (K_\eta/8)\eta\delta > 2\delta \tag{19}$$

where the last inequality follows from (16). By virtue of (18) from (19) we conclude that one can find t_1, t_2 either in $[t - \eta\delta, t]$, or in $[t, t + \eta\delta]$ such that (19) holds. This implies (14). □

5. Specific functions

The behavior experienced at the Brownian motion is in a certain aspect the worst possible, the function is central graph like at no point. In this section we want to illustrate that there are other examples of non-differentiable functions for which one can find a lot of points where $GLMT(f)$ and/or $CGLMT(f)$ is non-trivial. To illustrate the applicability of micro tangent sets here we discuss two such examples. (Of course, exact determination of the micro tangent properties of other functions and classes of functions can be subject of further research.) The first example is *Takagi's function*, $\mathscr{T}(x)$.

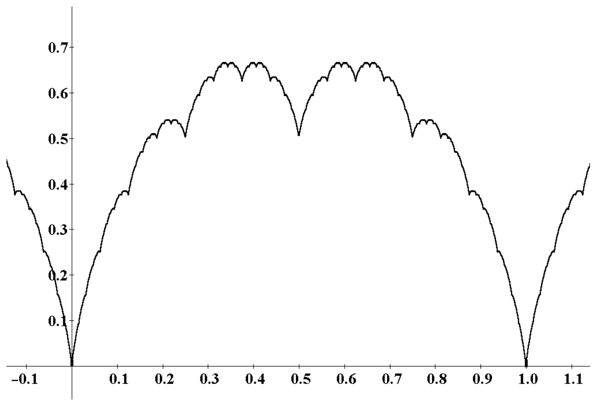

Figure 1. Takagi's function

Let $\Phi(x)\overset{\text{def}}{=}\operatorname{dist}(x,\mathbb{Z})$ and set

$$\mathscr{T}(x)=\sum_{n=0}^{\infty}2^{-n}\Phi(2^n x).$$

This is one of the well-known examples of nowhere differentiable functions, however, its Hölder spectrum is very simple, it is a monofractal, [10] Section 6.

Theorem 10. *For almost every $x_0\in\mathbb{R}$, $(x_0,\mathscr{T}(x_0))$ is a graph like micro tangent point of Takagi's function, $\mathscr{T}(x)$. In fact, this function is "micro self-similar" in the sense that if we take $g=\mathscr{T}|_{[-1,1]}$ then $\operatorname{graph}(g)\in\mathscr{T}_{MT}(x_0)$ for almost every $x_0\in\mathbb{R}$. (We remind the reader that $f_{MT}(x_0)$ was defined in Definition 1.)*

Proof. By inspecting the graph of $\mathscr{T}(x)$ (see Figure 1) and doing some elementary estimates of the first few terms of the sum defining $\mathscr{T}(x)$ one can see that if $x_0\in[0.49,0.51]$ and $\delta=0.25$ then

$$(x;\mathscr{T}(x))\in Q((x_0;\mathscr{T}(x_0)),\delta)\ \text{for}\ x\in[x_0-\delta,x_0+\delta]. \tag{20}$$

It is well-known that for almost every $x\in[0,1]$ the number of zeros and ones in the dyadic expansion of x is the same infinitely often. (The corresponding symmetric random walk model, where an n'th digit 0 means a unit step in the negative and an n'th digit 1 means a unit step in the positive direction, is persistent by Pólya's theorem (p. 118 of [1]), that is, the particle doing the random walk returns infinitely often to the origin.)

For an $x\in\mathbb{R}$ we will consider the dyadic "expansion"

$$x=[x]+\sum_{j=1}^{\infty}r_j(x)2^{-j},\ \text{where}\ r_j(x)\in\{0,1\} \tag{21}$$

and $[x]$ is the integer part of x. Since we work with almost every x we can exclude the dyadic rationals and hence the $r_j(x)$'s are unique. Denote by X_∞ the set of those $x\in\mathbb{R}$ for which the number of zeros and ones in the above dyadic expansion is infinitely often the same. To make this more precise, set $E(x,0)=0$, and if $E(x,k)$ for a $k\geqslant 0$ is given then let $E(x,k+1)$ be the least $n>E(x,k)$ for which

$$\#\{j\colon\ r_j(x)=0,\ 1\leqslant j\leqslant n\}=\#\{j\colon\ r_j(x)=1,\ 1\leqslant j\leqslant n\}.$$

For an $\eta\in(0,0.001)$ we will denote by $X^*_{\infty,\eta}$ the set of those $x\in X_\infty$ for which there are infinitely many $k(j,x,\eta)$'s ($j=1,2,\ldots$) such that

$$\left|\operatorname{dist}\big(x,2^{-E(x,k(j,x,\eta))}\mathbb{Z}\big)-\frac{1}{2}\cdot 2^{-E(x,k(j,x,\eta))}\right|<\eta\cdot 2^{-E(x,k(j,x,\eta))}. \tag{22}$$

Next we show that for any $\eta \in (0, 0.001)$ almost every $x \in X_\infty$ belongs to $X^*_{\infty,\eta}$. Set

$$\Psi_m = \Big\{x\colon \Big|\mathrm{dist}(x, 2^{-m}\mathbb{Z}) - \frac{1}{2}2^{-m}\Big| < \eta 2^{-m}, \text{ and } \exists k \in \mathbb{N},\ E(x,k) = m\Big\}.$$

If $E(x,k) = m$ and the interval $I = [i2^{-m}, (i+1)2^{-m}]$, $i \in \mathbb{Z}$ contains x then

$$\lambda(I \cap \Psi_m)/\lambda(I) = 2\eta. \tag{23}$$

Set $L_M = \bigcup_{m=M}^{\infty} \Psi_m$. From (23) it follows that the upper Lebesgue density of L_M at any $x \in X_\infty$ is positive and hence by Lebesgue's density theorem almost every $x \in X_\infty$ is a density point of L_M, which implies $\lambda(X_\infty \setminus L_M) = 0$. Therefore, $\lambda\Big(X_\infty \setminus \bigcap_{M=1}^{\infty} L_M\Big) = 0$ and $X^*_{\infty,\eta} = \bigcap_{M=1}^{\infty} L_M$ is of full measure and almost every $x \in X_\infty$ belongs to $X^*_{\infty,\eta}$.

We claim that if $x_0 \in X^*_{\infty,\eta}$ then there exists $\tau^*_\eta \in [-4\eta, 4\eta]$ such that if $g_\eta(x)$ is the restriction of $\mathscr{T}(x+\tau^*_\eta) - \mathscr{T}(\tau^*_\eta)$ onto $[-1,1]$ then $\mathrm{graph}(g_\eta) = \mathrm{graph}(g_\eta) \cap Q^2 \in f_{MT}(x_0)$.

Denote by I_j the interval of length $l_j \overset{\mathrm{def}}{=} 2^{-E(x_0,k(j,x_0,\eta))}$ containing x_0 and with endpoints in $l_j\mathbb{Z}$. If m_j equals the midpoint of I_j then by (22)

$$|x_0 - m_j| < \eta \cdot l_j. \tag{24}$$

Put $\mathscr{T}_N(x) = \sum_{n=0}^{N} 2^{-n}\Phi(2^n x)$ and $\mathscr{T}^*_N(x) = \sum_{n=N+1}^{\infty} 2^{-n}\Phi(2^n x)$. Set $N_{j,\eta} = E(x_0, k(j,x_0,\eta)) - 1$ and observe that $\mathscr{T}_{N_{j,\eta}}(x)$ is constant on I_j and $\mathscr{T}^*_{N_{j,\eta}}(x)$ is an l_j-times rescaled (in both x and y directions) copy of $\mathscr{T}(x)$.

Hence, it follows from (20) and (24) that setting $\delta_j = 0.25 l_j$ we have $(x; \mathscr{T}(x)) \in Q((x_0; \mathscr{T}(x_0)), \delta_j)$ for $x \in [x_0 - \delta_j, x_0 + \delta_j]$ and

$$F(\mathscr{T}, x_0, \delta_j) = \mathrm{graph}(\mathscr{T}(x+\tau_j) - \mathscr{T}(\tau_j)) \cap Q^2$$

where the translation vector $\tau_j \in [-4\eta, 4\eta]$. By compactness there exists $\tau^*_\eta \in [-4\eta, 4\eta]$ to which a suitable subsequence of $\{\tau_j\}$ converges. Then for this subsequence $F(\mathscr{T}, x_0, \delta_j)$ converges to $\mathrm{graph}(g_\eta) \cap Q^2$ in the Hausdorff metric. This implies that $\mathrm{graph}(g_\eta) \in f_{MT}(x_0)$, as we have claimed.

Next, letting $\eta_K = 1/K$, clearly $\tau^*_{\eta_K} \to 0$ and almost every $x_0 \in \mathbb{R}$ belongs to $X^*_\infty \overset{\mathrm{def}}{=} \bigcap_{K=1}^{\infty} X^*_{\infty,\eta_K}$. If $x_0 \in X^*_\infty$ then one can easily choose a sequence $\delta'_K \to 0$ such that $F(\mathscr{T}, x_0, \delta'_K)$ converges in the Hausdorff metric to the graph of $g(x) = \mathscr{T}(x)|_{[-1,1]} = \lim_{K\to\infty} g_{\eta_K}(x)$. Hence $\mathrm{graph}(g)$ belongs to $\mathscr{T}_{MT}(x_0)$. $\square$

Our final example will be one of the simplest cases of Weierstrass's nowhere differentiable function. Probably the most famous detailed study of this type of functions is Hardy's paper [8]. We will take $\Psi(x) = \sin(2\pi x)$ and consider

$$\mathscr{W}(x) = \sum_{n=0}^{\infty} 2^{-n}\Psi(2^n x).$$

A similar, but a little more complicated argument works if one takes $\Psi(x) = \cos(2\pi x)$. For the partial and tail sums we will again use the notation

$$\mathscr{W}_N(x) = \sum_{n=0}^{N} 2^{-n}\Psi(2^n x) \text{ and } \mathscr{W}_N^*(x) = \sum_{n=N+1}^{\infty} 2^{-n}\Psi(2^n x).$$

Theorem 11. *For almost every $x_0 \in \mathbb{R}$, $(x_0, \mathscr{W}(x_0))$ is a central graph like micro tangent point of Weierstrass's function $\mathscr{W}(x)$.*

To prove this theorem we need the following lemma, which seems to be quite natural.

Lemma 12. *For almost every $x_0 \in \mathbb{R}$ we can find a strictly monotone increasing sequence $\{N(j,x_0)\}_{j=1}^{\infty}$ such that $\mathscr{W}'_{N(j,x_0)-1}(x_0)$ and $\mathscr{W}'_{N(j,x_0)}(x_0)$ are of opposite signs, which implies*

$$|\mathscr{W}'_{N(j,x_0)}(x_0)| = \left|\sum_{n=0}^{N(j,x_0)} 2\pi\cos(2\pi 2^n x_0)\right| \leqslant 2\pi. \tag{25}$$

First we will prove Theorem 11 based on this lemma and finally we will provide a proof of Lemma 12.

P r o o f o f T h e o r e m 11. Since $\mathscr{W}''_{N(j,x_0)}(x) = \sum_{n=0}^{N(j,x_0)} -4\pi^2 2^n \sin(2\pi 2^n x)$ we have $|\mathscr{W}''_{N(j,x_0)}(x)| < 8\pi^2 2^{N(j,x_0)}$. Hence, setting $\delta_j = 2^{-N(j,x_0)-1}$ and $I_j = [x_0 - \delta_j, x_0 + \delta_j]$, by using Lagrange's mean value theorem and (25) we obtain

$$|\mathscr{W}'_{N(j,x_0)}(x)| < 5\pi^2 \text{ for } x \in I_j. \tag{26}$$

The "Q^2 rescaled" copies of the partial and tail sums of $\mathscr{W}$ will be denoted by

$$U_{N(j,x_0)}(x) = \frac{1}{\delta_j}(\mathscr{W}_{N(j,x_0)}(\delta_j x + x_0) - \mathscr{W}_{N(j,x_0)}(x_0))$$

and

$$U^*_{N(j,x_0)}(x) = \frac{1}{\delta_j}(\mathscr{W}^*_{N(j,x_0)}(\delta_j x + x_0) - \mathscr{W}^*_{N(j,x_0)}(x_0)).$$

Clearly,

$$\mathscr{W}_{x_0,\delta_j}(x) \stackrel{\text{def}}{=} \frac{1}{\delta_j}(\mathscr{W}(\delta_j x + x_0) - \mathscr{W}(x_0)) = U_{N(j,x_0)}(x) + U^*_{N(j,x_0)}(x)$$

and

$$F(\mathscr{W}, x_0, \delta_j) = \text{graph}(\mathscr{W}_{x_0,\delta_j}) \cap Q^2.$$

From (26) it follows that

$$|U'_{N(j,x_0)}(x)| \leqslant 5\pi^2 \text{ for } x \in [-1,1]. \tag{27}$$

For each j there exists $\tau_j \in [-1,1]$ such that $U^*_{N(j,x_0)}(x) = \mathscr{W}(x+\tau_j) - \mathscr{W}(\tau_j)$. By $U_{N(j,x_0)}(0) = 0$ and (27) the family of functions $U_{N(j,x_0)}(x)$ is uniformly bounded and equicontinuous, so by the Arzela-Ascoli theorem (see, for example, [7] 1.6.9 p. 37) there exists a subsequence $\{U_{N(j_k,x_0)}(x)\}$ which uniformly converges to a function $U_{x_0}(x)$. From (27) it also follows that

$$|U_{x_0}(x) - U_{x_0}(y)| \leqslant 5\pi^2 \text{ for } x, y \in \mathbb{R}. \tag{28}$$

By turning to a subsequence, if necessary, we can also assume that $\tau_{j_k} \to \tau^* \in [-1,1]$. Hence,

$$\mathscr{W}_{x_0,\delta_{j_k}}(x) \text{ converges uniformly to } g(x) \stackrel{\text{def}}{=} U_{x_0}(x) + \mathscr{W}(x+\tau^*) - \mathscr{W}(\tau^*). \tag{29}$$

Since $\mathscr{W}$ is nowhere differentiable, by (28) there is no interval on which g is constant. Local extrema of g on the boundary of Q^2 might cause some minor problems, this is why we introduce g_1 below.

It follows also from (28) and (29) that we can choose $g_1 \in C[-1,1]_0$ such that

- $CENT(\text{graph}(g)) \supset CENT(\text{graph}(g_1)) \supset \text{cl}(CENT(\text{int}(Q^2) \cap \text{graph}(g)))$,
- $CENT(\text{graph}(g_1)) \in \mathscr{W}_{CMT}(x_0)$, and
- $|g_1(x)| > 1$ if $(x; g_1(x)) \notin CENT(\text{graph}(g_1))$.

This shows that x_0 is a central graph like micro tangent point of f. □

Finally, we prove Lemma 12.

P r o o f o f L e m m a 12. Denote by $X_\pm$ the set of those x's in $\mathbb{R}$ for which the sequence $\{\mathscr{W}'_N(x)\}_{N=1}^{\infty}$ changes its sign infinitely often. We need to show that $\lambda(X^c_\pm) = 0$, where we use the notation A^c for the complement of $A \subset \mathbb{R}$.

Proceeding towards a contradiction assume that $\lambda(X_{\pm}^c) > 0$. Set $\lambda_{\pm} = \lambda(X_{\pm}^c \cap [0,1])$. Since for all N's $\mathscr{W}_N'$ is periodic by one we have $\lambda_{\pm} > 0$.

For $x \in \mathbb{R}$ we use the dyadic expansion of the form (21). Denote by $X_{0,\infty}$ the set of those x in $\mathbb{R} \setminus \mathbb{Q}$ for which we have arbitrarily long blocks of 0's in the sequence $\{r_j(x)\}_{j=1}^{\infty}$. It is well-known (and not difficult to see) that $\lambda(X_{0,\infty}^c) = 0$.

Clearly, $\{\mathscr{W}_N'(x)\}_{N=1}^{\infty}$ is not bounded if $x \in X_{0,\infty}$.

Set

$$X_a = \{x \in \mathbb{R}\colon\ \{\mathscr{W}_N'(x)\}_{N=1}^{\infty} \text{ is bounded from above}\}.$$

By periodicity it follows that for any $k \in \mathbb{N}$ from $x \in X_a$, $x \pm 2^{-k}$ is also in X_a. Hence X_a is periodic by 2^{-k} for all $k \in \mathbb{N}$. Clearly, X_a is measurable and it is a consequence of the Lebesgue density theorem that there is a zero-one law, that is, $\lambda(X_a) = 0$ or $\lambda(X_a^c) = 0$. Similarly, letting

$$X_b = \{x \in \mathbb{R}\colon\ \{\mathscr{W}_N'(x)\}_{N=1}^{\infty} \text{ is bounded from below}\}$$

one can see that $\lambda(X_b) = 0$ or $\lambda(X_b^c) = 0$.

From $\lambda(X_{0,\infty}^c) = 0$ it follows that $\lambda(X_a^c) = 0$ and hence $\lambda(X_b^c) = 0$ is impossible.

If $\lambda(X_a) = 0$ and $\lambda(X_b) = 0$ then $\lambda(X_{\pm}^c) = 0$ and this contradicts $\lambda_{\pm} > 0$.

Assume

$$\lambda(X_a) = 0 \text{ and } \lambda(X_b^c) = 0 \tag{30}$$

(a similar argument works if $\lambda(X_a^c) = 0$ and $\lambda(X_b) = 0$). Our goal is again to obtain a contradiction.

Set

$$X_b^K = \{x \in \mathbb{R}\colon\ \mathscr{W}_N'(x) > -K \text{ for all } N \in \mathbb{N}\}.$$

Then X_b^K is periodic by one, measurable, $\bigcup_{K=1}^{\infty} X_b^K = X_b$ and $\lambda(X_b \cap [0,1)) = 1$. Hence there exists K such that $\lambda(X_b^K \cap [0,1)) > 0.9$. For $j = 0,1,2$ put

$$X_{b,j}^K = \{x \in \mathbb{R}\colon\ x - \tfrac{j}{3} \in X_b^K\}.$$

Then $X_{b,j}^K$ is periodic by one and $\lambda(X_{b,j}^K \cap [0,1)) > 0.9$. Set $Y = \bigcap_{j=1}^{3} X_{b,j}^K$. Then Y is also periodic by one, $\lambda(Y \cap [0,1)) > 0.7$ and $\mathscr{W}_N'(x - (j/3)) > -K$ for every $x \in Y$ and $N \in \mathbb{N}$. Recall that $\sum_{j=0}^{2} \cos(2\pi(\theta - (-1)^n(j/3))) = 0$ for any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$.

Therefore,

$$\sum_{j=0}^{2} \mathscr{W}_N'\Big(x-\frac{j}{3}\Big) = \sum_{j=0}^{2}\sum_{n=0}^{N} 2\pi\cos\Big(2\pi 2^n\Big(x-\frac{j}{3}\Big)\Big)$$

$$= \sum_{n=0}^{N} 2\pi \sum_{j=0}^{2} \cos\Big(2\pi 2^n x - 2\pi\frac{(3-1)^n}{3}j\Big) = \sum_{n=0}^{N} 2\pi \sum_{j=0}^{2} \cos\Big(2\pi 2^n x - 2\pi\frac{(-1)^n}{3}j\Big) = 0.$$

Hence for $x \in Y$ we have

$$\mathscr{W}_N'(x) = -\Big(\mathscr{W}_N'\Big(x-\frac{1}{3}\Big) + \mathscr{W}_N'\Big(x-\frac{2}{3}\Big)\Big) < 2K.$$

This would imply $Y \subset X_a$ and $\lambda(Y) \neq 0$, which contradicts (30). □

For further information about distribution of values of trigonometric polynomials considered in Lemma 12 we refer to [11] and [12]; one can prove this lemma basing on these results as well, but the treatment given here seemed to be more elementary.

The author would like to thank S. Konyagin for his suggestion of a simplified version of the proof of Lemma 12 and for pointing out references [11], [12] and [13]. Our original "real analysis" version of the proof was based on the idea that if the sequence $\mathscr{W}_N'(x)$ is not changing for almost every x its sign infinitely often then $\mathscr{W}'(x)$ would equal $+\infty$ or $-\infty$ almost everywhere, which contradicts [19] Ch. IX. (4.4) Theorem (the first version of this result, valid for continuous functions, is due to N. N. Luzin [13]).

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