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# SINGLE VALUED EXTENSION PROPERTY AND GENERALIZED WEYL'S THEOREM

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*Abstract.* Let $T$ be an operator acting on a Banach space $X$, let $\sigma(T)$ and $\sigma_{BW}(T)$ be respectively the spectrum and the B-Weyl spectrum of $T$. We say that $T$ satisfies the generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, where $E(T)$ is the set of all isolated eigenvalues of $T$. The first goal of this paper is to show that if $T$ is an operator of topological uniform descent and 0 is an accumulation point of the point spectrum of $T$, then $T$ does not have the single valued extension property at 0, extending an earlier result of J. K. Finch and a recent result of Aiena and Monsalve. Our second goal is to give necessary and sufficient conditions under which an operator having the single valued extension property satisfies the generalized Weyl's theorem.

*Keywords*: single valued extension property, B-Weyl spectrum, generalized Weyl's theorem

*MSC 2000*: 47A53, 47A55

## 1. Introduction

For $T$ in the Banach algebra $L(X)$ of bounded linear operators acting on a Banach space $X$, we will denote by $N(T)$ its kernel and by $R(T)$ its range. The operator $T$ is called a B-Fredholm operator [2], if there is an integer $n$ such that the range $R(T^n)$ is closed and such that the operator $T_n\colon R(T^n) \to R(T^n)$ defined by $T_n(x) = T(x)$ for $x \in R(T^n)$ is a Fredholm operator. From [4, Theorem 3.1] it follows that $T$ is a B-Fredholm operator if and only if there exists an integer $n$ such that $c_n(T) < \infty$ and $c'_n(T) < \infty$, where $c_n(T) = \dim(R(T^n)/R(T^{n+1}))$ and $c'_n(T) = \dim(N(T^{n+1})/N(T^n))$. In this case, it follows from [4, Theorem 3.1] that the range $R(T^n)$ is closed. Then the index of $T$ is defined by $\operatorname{ind}(T) = c'_n(T) - c_n(T)$. From

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[2, Proposition 2.1], the definition of the index is independent of the choice of the integer $n$. Moreover, in the case of a Fredholm operator, we find the usual definition of the index.

Recall that $T$ is Drazin invertible if it has a finite ascent and descent (Definition 2.1); which is also equivalent to the fact that $T = T_0 \oplus T_1$, where $T_0$ is an invertible operator and $T_1$ is a nilpotent one. (See [14, Proposition 6], and [12, Corollary 2.2].) If $T \in L(X)$, then the Drazin spectrum of $T$ is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C}\colon\ T - \lambda I$ is not Drazin invertible$\}$. From [5, Corollary 2.4] we know that the Drazin spectrum $\sigma_D(T)$ of a bounded linear operator $T \in L(X)$ satisfies the spectral mapping theorem.

In [4] B-Weyl operators and the B-Weyl spectrum were defined as follows:

**Definition 1.1.** Let $T \in L(X)$. Then $T$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of $T$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C}\colon\ T - \lambda I$ is not a B-Weyl operator$\}$.

If we consider a normal operator $T$ acting on a Hilbert space $H$, Berkani proved in [4, Theorem 4.5] that $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, where $E(T)$ is the set of all isolated eigenvalues of $T$, which gives a generalization of the classical Weyl's Theorem. Recall that the classical Weyl's Theorem [16] asserts that if $T$ is a normal operator acting on a Hilbert space $H$, then the Weyl spectrum $\sigma_W(T)$ is exactly the set of all points in $\sigma(T)$ except the isolated eigenvalues of finite multiplicity, that is $\sigma_W(T) = \sigma(T) \setminus E_0(T)$. Here $E_0(T)$ is the set of isolated eigenvalues of finite multiplicity, that is $E_0(T) = \{\lambda \in \operatorname{iso}\sigma(T)\colon\ 0 < \dim N(T - \lambda I) < \infty\}$ where $\operatorname{iso}\sigma(T)$ is the set of isolated points of the spectrum of $T$ and $\sigma_W(T)$ is the Weyl spectrum of $T$, that is $\sigma_W(T) = \{\lambda \in \mathbb{C}\colon\ T - \lambda I$ is not a Fredholm operator of index $0\}$.

In [6, Theorem 3.9], it is shown that if $T$ satisfies the generalized Weyl's theorem: $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, then it satisfies Weyl's theorem: $\sigma_W(T) = \sigma(T) \setminus E_0(T)$, and if it satisfies the generalized Browder's theorem, $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$, then it satisfies Browder's theorem $\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$, where $\Pi(T)$ is the set of all the poles of the resolvent of $T$ and $\Pi_0(T)$ is the set of the poles of the resolvent of $T$ of finite rank, that's $\Pi_0(T) = \{\lambda \in \Pi(T)\colon\ 0 < \dim N(T - \lambda I) < \infty\}$. (See [6] for more details about the concepts introduced here.)

Moreover, we have the following theorem [7, Corollary 2.6] which characterizes operators satisfying the generalized Weyl's theorem:

**Theorem 1.2.** *Let $T \in L(X)$. Then $T$ satisfies the generalized Weyl's theorem if and only if $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$ and $E(T) = \Pi(T)$.*

## 2. The single valued extension property

**Definition 2.1.** For any $T \in L(X)$ we define sequences $(c_n(T))$, $(c'_n(T))$ and $(k_n(T))$ as follows:

(i) $c_n(T) = \dim(R(T^n)/R(T^{n+1}))$.

(ii) $c'_n(T) = \dim(N(T^{n+1})/N(T^n))$.

(iii) $k_n(T) = \dim\left[(R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T))\right]$.

The *descent* $\delta(T)$ and *ascent* $a(T)$ of $T$ are defined by

$$\delta(T) = \inf\{n\colon\ c_n(T) = 0\} = \inf\{n\colon\ R(T^n) = R(T^{n+1})\},$$
$$a(T) = \inf\{n\colon\ c'_n(T) = 0\} = \inf\{n\colon\ N(T^n) = N(T^{n+1})\}.$$

We set formally $\inf \emptyset = \infty$.

**Definition 2.2.** (See [9].) Let $T \in L(X)$ and let $d \in \mathbb{N}$. Then $T$ has a *uniform descent for* $n \geqslant d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for all $n \geqslant d$ (equivalently, $k_n(T) = 0$ for all $n \geqslant d$)). If, in addition, $R(T) + N(T^d)$ is closed, then $T$ is said to have a *topological uniform descent for* $n \geqslant d$.

**Definition 2.3.** We say that $T \in L(X)$ has the single valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$, if for an arbitrary open neighborhood $U$ of $\lambda_0$, $f = 0$ is the only analytic function $f\colon U \to X$ such that $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$. We will say that $T$ has the SVEP if $T$ has this property at every $\lambda \in \mathbb{C}$.

**Lemma 2.4.** *Let $(X, \|\cdot\|)$be a Banach space, let $T \in L(X)$ have SVEP at $\lambda_0$ and let $E \subset X$ be a subspace of $X$ invariant under $T$. If $E$ equipped with a norm $\|\cdot\|_1$ is a Banach space such the injection $i\colon E \to X$ is continuous, then $T|_E$ has SVEP at $\lambda_0$.*

P r o o f. Let $S = T|_E$, let $U_{\lambda_0}$ be an open neighborhood of $\lambda$ and let $f\colon U_{\lambda_0} \to E$ be an analytic function such that $(S - \mu I)f(\mu) = 0$ for every $\mu \in U_\lambda$. Define a function $F\colon U_\lambda \to X$ by $F(\mu) = (i \circ f)(\mu)$. Then $F$ is an analytic function such that $(T - \mu I)F(\mu) = 0$ for every $\mu \in U_{\lambda_0}$. Since $T$ has SVEP at $\lambda_0$, it follows that $F = 0$. Hence $f = 0$, and so $S$ has SVEP at $\lambda_0$. $\square$

Recall that the point spectrum of $T \in L(X)$ is defined by $\sigma_p(T) = \{\lambda \in \mathbb{C}\colon N(T - \lambda I) \neq \{0\}\}$.

**Theorem 2.5.** *Let $T \in L(X)$. If $T$ is an operator of topological uniform descent for $n \geqslant d$, then the following conditions are equivalent:*

(i) *$T$ does not have the single valued extension property at* 0.

(ii) 0 *is an accumulation point of the point spectrum of $T$.*

(iii) *The ascent $a(T)$ of $T$ is infinite.*

Proof. (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (iii) Assume that $a(T) = p < \infty$, then $c'_n(T) = 0$ for all $n \geqslant p$. Since $T$ is an operator of topological uniform descent, then by virtue of [9, Theorem 4.7] there exists an $\varepsilon > 0$ such that if $0 < |\lambda| < \varepsilon$, we have $c'_m(T - \lambda I) = 0$ for all $m$. In particular, $\alpha(T - \lambda I) = c'_0(T - \lambda I) = 0$. Therefore $\lambda$ is not in the point spectrum of $T$, and 0 is not an accumulation point of the point spectrum of $T$. This contradiction shows that $a(T) = \infty$.

(iii) $\Rightarrow$ (i) Assume that $a(T) = \infty$. Then $c'_n(T) > 0$ for all $n \geqslant d$. Let $Y = \bigcap\limits_{p \geqslant d} R(T^p)$. Then equipped with the topology induced by the operator range topology of $R(T^d)$, the space $Y$ is closed in $R(T^d)$, invariant under $T$. Moreover, by [9, Theorem 3.4], the restriction $T|_Y$ of $T$ to $Y$ is onto. Since $T$ is an operator of topological uniform descent, then by [9, Theorem 3.4] there exists an $\varepsilon > 0$ such that if $0 < |\lambda| < \varepsilon$, we have $c'_m(T - \lambda I) > 0$ for all $m$. In particular, $\alpha(T - \lambda I) = c'_0(T - \lambda I) > 0$. Therefore $\lambda$ is in the point spectrum of $T|_Y$ and also in the spectrum of $T|_Y$. As $\sigma(T|_Y)$ is closed, we have $0 \in \sigma(T|_Y)$. Using [8, Theorem 2], it follows that $T_{|Y}$ does not have the single valued extension property at 0. Since the injection $i\colon Y \to X$ of the Banach space $Y$ into $X$ is continuous, Lemma 2.4 yields that $T$ does not have SVEP at 0. □

Theorem 2.5 extends [8, Theorem 9 and 10] which establishes that $T$ has SVEP at 0 under the stronger assumption that $\sigma_p(T)$ contains a neighborhood of 0. It extends also [1, Theorem 2.6] which establishes that $T$ has SVEP at 0 under the stronger assumption that $T$ is a semi-Fredholm operator.

**Corollary 2.6.** *Let $T \in L(X)$ be an operator of topological uniform descent for $n \geqslant d$. If $c'_d(T) > c_d(T)$, then $T$ does not have SVEP at* 0.

Proof. Suppose that $T$ is an operator of topological uniform descent for $n \geqslant d$ and that $c'_d(T) > c_d(T)$. Therefore $c'_d(T) > 0$. If $\lambda \in \mathbb{C}, \lambda \neq 0$ and $|\lambda|$ is small enough, then from [9, Theorem 4.7] we have $\alpha(T - \lambda I) > 0$. Hence $\lambda$ is in the point spectrum of $T$ and 0 is an accumulation point of the point spectrum of $T$. Consequently $T$ does not have the SVEP at 0. □

Let $T \in L(X)$ be an operator of topological uniform descent for $n \geqslant d$ such that $R(T^{d+1})$ is closed (such operators are called in [13] quasi-Fredholm operators). By

[13, Lemma 12], for such operators $R(T^n)$ is closed for each integer $n \geqslant d$. Moreover, in this case it is easily seen that $T^*$ is also an operator of topological uniform descent for $n \geqslant d$ such that $R((T^{*d+1})$ is closed.

**Corollary 2.7.** *Let $T \in L(X)$ be an operator of topological uniform descent for $n \geqslant d$ such that $R(T^{d+1})$is closed. Then the following conditions are equivalent:*
(i) *$T^*$ does not have SVEP at 0;*
(ii) *The descent $\delta(T)$ of $T$ is infinite.*

Proof. Assume that $T^*$ does not have the SVEP at 0. If $\delta(T) < \infty$, then $R(T^d) = R(T^{d+1})$. Since both $R(T^d)$ and $R(T^{d+1})$ are closed, we have $N(T^{*d}) = N(T^{*d+1})$, and so $a(T^*) < \infty$. But this is a contradiction, since $T^*$ is an operator of topological uniform descent having the SVEP at 0. Hence $\delta(T) = \infty$.

Conversely, assume that $\delta(T) = \infty$. Then $c_d(T) > 0$. If $\lambda \in \mathbb{C}, \lambda \neq 0$ and $|\lambda|$ is small enough, then from [9, Theorem 4.7] we have $\beta(T - \lambda I) > 0$. Hence $(T - \lambda I)^d$ is not a surjective operator, and so $T^* - \lambda I$ is not injective. Therefore $\lambda$ is in the point spectrum of $T^*$ and 0 is an accumulation point of $T^*$. Consequently, $T^*$ does not have the single valued extension property at 0. □

**Corollary 2.8.** *Let $T \in L(X)$ be an operator of topological uniform descent for $n \geqslant d$ such that $R(T^{d+1})$ is closed. If $c_d(T) > c'_d(T)$, then $T^*$ does not have the single valued extension property at 0.*

Proof. Suppose that $T$ is an operator of topological uniform descent for $n \geqslant d$ such that $R(T^{d+1})$ is closed and $c_d(T) > c'_d(T)$. Therefore $c_d(T) > 0$. Hence $\delta(T) = \infty$. From the previous corollary it follows that $T^*$ does not have the single valued extension property at 0. □

**Corollary 2.9.** *Let $T \in L(X)$ be an operator of topological uniform descent for $n \geqslant d$ such that $R(T^{d+1})$ is closed. Then $T$ and $T^*$ have the single valued extension property at 0 if and only if $T$ is Drazin invertible, in other words if and only if 0 is a pole of the resolvent of $T$.*

Proof. If $T$ and $T^*$ have the SVEP at 0, then from Theorem 2.5 and Corollary 2.7 we have $a(T) < \infty$ and $\delta(T) < \infty$. From [12, Theorem 1.2] it follows that $a(T) = \delta(T) < \infty$. Hence $T$ is Drazin invertible.

Conversely, if $T$ is Drazin invertible, then $a(T) = \delta(T) < \infty$. From Theorem 2.5 and Corollary 2.7 it follows that $T$ and $T^*$ have SVEP at 0. □

**Corollary 2.10.** *Let $T \in L(X)$ be an operator of topological uniform descent. Then:*

(i) *If $S$ is a bounded linear operator commuting with $T$, such that $S - T$ is sufficiently small and invertible, then $T$ has SVEP at $0$ if and only if $S$ does.*

(ii) *If $S$ is an operator of topological uniform descent for $n \geqslant p$, commuting with $T$, such that $S - T$ is compact, then $T$ has SVEP at $0$ if and only if $S$ does.*

Proof. (i) In this case it follows from [9, Theorem 4.7] that $S$ is an operator of topological uniform descent for $n \geqslant 0$ and $T$ is of finite ascent if and only if $S$ is.

(ii) From [9, Theorem 5.8] it follows that $T$ is of finite ascent if and only if $S$ is.

Therefore the corollary is a direct consequence of Theorem 2.5. □

From this corollary we obtain the following perturbation result for semi-Fredholm operators having SVEP.

**Corollary 2.11.** *Let $T \in L(X)$ be a semi-Fredholm operator having SVEP at $0$, and let $K \in L(X)$ be a compact operator commuting with $T$. Then $T + K$ has SVEP at $0$.*

Proof. As $T$ is a semi-Fredholm operator, then $T + K$ is also a semi-Fredholm operator. Moreover, a semi-Fredholm operator is an operator of topological uniform descent. □

Let $A$ be an algebra with a unit $e$. In [11], Kordula and Müller defined the concept of regularity by

**Definition 2.12.** A non-empty subset $\mathbf{R} \subset A$ is called a regularity if it satisfies the following conditions:

(i) If $a \in A$ and $n \geqslant 1$ is an integer then $a \in \mathbf{R}$ if and only if $a^n \in \mathbf{R}$.

(ii) If $a, b, c, d \in L(X)$ are mutually commuting elements satisfying $ac + bd = e$, then $ab \in \mathbf{R}$ if and only if $a, b \in \mathbf{R}$.

A regularity $\mathbf{R}$ defines in a natural way a spectrum by $\sigma_{\mathbf{R}}(a) = \{\lambda \in \mathbb{C}\colon\ a - \lambda I \notin \mathbf{R}\}$ for every $a \in A$. Moreover, in the case of a Banach algebra $A$, the spectrum $\sigma_{\mathbf{R}}$ satisfies the spectral mapping theorem.

**Theorem 2.13.** *Let $X$ be a Banach space. Then the set $\mathcal{S} = \{T \in L(X)\colon T$ is an operator of topological uniform descent and $T$ has SVEP at $0\}$ is a regularity in the algebra $L(X)$.*

Proof. (i) Since every invertible element in $L(X)$ is an operator of topological uniform descent and has SVEP at 0, then $\mathcal{S}$ is a nonempty set. Let $T \in L(X)$ be an operator of topological uniform descent and let $n \geqslant 1$ be an integer. From

[3, Theorem 4.3] we know that $T$ is an operator of topological uniform descent if and only if $T^n$ is. Moreover, it is clear that $T$ is of finite ascent if and only if $T^n$ is. Therefore $T \in \mathcal{S}$ if and only if $T^n \in \mathcal{S}$.

(ii) Let $U$, $V$, $S$, $T$ be mutually commuting elements of $L(X)$ such that $US+VT = I$. From [3, Theorem 4.3] we know that $S$ and $T$ are operators of topological uniform descent if and only if $ST$ is an operator of topological uniform descent. Moreover, from [[13], p. 137] it follows that $a(ST)$ is finite if and only if $a(S)$ and $a(T)$ are finite. Hence $ST \in \mathcal{S}$ if and only $S$ and $T$ does. Therefore $\mathcal{S}$ is a regularity. □

For $T \in L(X)$, let $\sigma_{\mathcal{S}}(T) = \{\lambda \in \mathbb{C}\colon\ T - \lambda I \notin \mathcal{S}\}$ be the spectrum associated with the regularity $\mathcal{S}$. Using the properties of the regularities [11], we have immediately the following corollary:

**Corollary 2.14.** *Let $T \in L(X)$ and let $f$ be an analytic function in a neighborhood of the usual spectrum $\sigma(T)$ of $T$ which is non-constant on any connected component of the spectrum $\sigma(T)$. Then $f(\sigma_{\mathcal{S}}(T)) = \sigma_{\mathcal{S}}(f(T))$.*

## 3. Generalized Weyl's theorem and SVEP

It is natural to ask whether an operator $T$ having SVEP does satisfy the generalized Weyl's theorem. The following examples gives a negative answer to this question. Moreover, the first example shows that an operator having SVEP could satisfy Weyl's theorem but not the generalized Weyl's theorem.

Example 3.1 ([7], p. 602). Let $Q \in L(X)$ be any quasi-nilpotent operator acting on an infinite dimensional Banach space $X$ such that $R(Q^n)$ is non-closed for all $n$. Consider the operator $S = 0 \oplus Q$, defined on the Banach space $X \oplus X$. Since $R(S^n) = R(Q^n)$ is non-closed for all $n$, then $S$ is not a B-Fredholm operator. Moreover, $\sigma(S) = \{0\}$, $E(S) = \{0\}$, $E_0(S) = \emptyset$, $\sigma_W(S) = \{0\}$ and $\sigma_{BW}(S) = \{0\}$. Hence Weyl's theorem is satisfied by $S$, but the generalized Weyl's Theorem does not holds for $S$, while $S$ as a quasinilpotent operator satisfies SVEP.

Example 3.2. Let $T$ be defined on $l_2$ by: $T(x_1, x_2, x_3, \ldots) = (1/3x_3, 1/4x_4, 1/5x_5, \ldots)$. As $T$ is quasinilpotent, $T$ has the SVEP. As $\sigma(T) = \sigma_W(T) = \{0\}$ and $E_0(T) = \{0\}$, $T$ does not satisfy Weyl's theorem. From [6, Theorem 3.9], $T$ does not satisfy the generalized Weyl's theorem.

Hence, it is natural to seek for necessary and sufficient conditions for an operator $T$ having SVEP to satisfy the generalized Weyl's theorem. We begin with the following result:

**Theorem 3.3.** *Let $T \in L(X)$. If $T$ has the single valued extension property, then $\sigma_D(T) = \sigma_{BW}(T)$.*

Proof. Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda I$ is a B-Fredholm operator of index 0. Therefore for $n$ large enough, we have $c_n(T - \lambda I) = c'_n(T - \lambda I) < \infty$. Since a B-Fredholm operator is an operator of topological uniform descent and since $T$ has the SVEP, then $a(T - \lambda I) < \infty$. As we have for $n$ large enough $c_n(T - \lambda I) = c'_n(T - \lambda I) < \infty$, then we have also $\delta(T - \lambda I) < \infty$. Therefore $\lambda$ is a pole of the resolvent of $T$, and $\lambda \notin \sigma_D(T)$. Hence $\sigma_D(T) \subset \sigma_{BW}(T)$. As $\sigma_{BW}(T) \subset \sigma_D(T)$ always holds, we have $\sigma_D(T) = \sigma_{BW}(T)$. □

**Corollary 3.4.** *Let $T \in B(X)$ and let $H(\sigma(T))$ denote the set of functions $f$ which are analytic on an open neighborhood of $\sigma(T)$. If $T$ has the single valued extension property, then $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for every $f \in H(\sigma(T))$ which is non-constant on any connected component of $\sigma(T)$.*

Proof. Since $T$ has the SVEP, also $f(T)$ has the SVEP. From the previous theorem we have $\sigma_{BW}(f(T)) = \sigma_D(f(T))$. From [5, Corollary 2.4] we have $\sigma_D(f(T)) = f(\sigma_D(T))$. Therefore $\sigma_{BW}(f(T)) = f(\sigma_D(T)) = f(\sigma_{BW}(T))$. □

Let $T \in L(X)$, let $H_0(T) = \{x \in X\colon \|T x^n\|^{1/n} \to 0\}$, and let $K(T) = \{x \in X\colon$ there exist $c > 0$ and a sequence $(x_n)_{n\geqslant 1} \subset X$ such that $Tx_1 = x, Tx_{n+1} = x_n$ for all $n \in \mathbb{N}$ and $\|x_n\| \leqslant c^n\|x\|$ for all $n \in \mathbb{N}\}$.

If $\lambda$ is isolated in $\sigma(T)$, then it is known [15, Proposition 4] that $H_0(T - \lambda I)$ and $K(T - \lambda I)$ are closed subspaces of $X$, $X = H_0(T - \lambda I) \oplus K(T - \lambda I)$, $T_0 = (T - \lambda I)_{|K_0(T-\lambda I)}$ is an invertible operator and $T_1 = (T - \lambda I)_{|H_0(T-\lambda I)}$ is a quasi-nilpotent operator. Here $\oplus$ means the topological direct sum.

Recall also that for $T \in L(X)$ and a closed subset $F$ of $\mathbb{C}$, the spectral manifold is $\chi_T(F) = \{x \in X\colon$ there exists an analytic $X$-valued function $f\colon \mathbb{C} \setminus F \to X$ such that $(T - \lambda I)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F\}$.

**Theorem 3.5.** *If $T \in L(X)$ has SVEP, then the following properties are equivalent:*

(i) *$T$ satisfies the generalized Weyl's theorem.*
(ii) *$\sigma_{BW}(T) \cap E(T) = \emptyset$.*
(iii) *$E(T) = \Pi(T)$.*
(iv) *For every $\lambda \in E(T)$ there exist an integer $n$ such that $\chi_T(\lambda) = N((T - \lambda I)^n)$.*
(v) *For each $\lambda \in E(T)$, $T - \lambda I$ is an operator of topological uniform descent.*

Proof. (i) ⇒ (ii) If $T$ satisfies the generalized Weyl's theorem, then $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$. Therefore $\sigma_{BW}(T) \cap E(T) = \emptyset$.

(ii) ⇒ (iii) Assume that $\sigma_{BW}(T) \cap E(T) = \emptyset$. As $T$ has the single valued extension property, it follows from Theorem 3.3 that $\sigma_D(T) = \sigma_{BW}(T)$. Let $\lambda \in E(T)$. Then $\lambda \notin \sigma_{BW}(T)$. Therefore $\lambda \notin \sigma_D(T)$, and so $\lambda \in \Pi(T)$. As always $\Pi(T) \subset E(T)$, we have $E(T) = \Pi(T)$.

(iii) ⇒ (iv) Let $\lambda \in E(T) = \Pi(T)$. Then $\lambda$ is a pole of the resolvent of $T$, and by [15, Theorem 5] there exists an integer $n$ such that $H_0(T - \lambda I) = N((T - \lambda I)^n)$. Hence we have $\chi_T(\lambda) = H_0(T - \lambda I) = N((T - \lambda I)^n)$.

(iv) ⇒ (v) Let $\lambda \in E(T)$. Then there exists an integer $n$ such that $\chi_T(\lambda) = N((T - \lambda I)^n)$. Then we have $H_0(T - \lambda I) = \chi_T(\lambda) = N((T - \lambda I)^n)$ and $X = N((T - \lambda I)^n) \oplus K(T - \lambda)$. Therefore $\lambda \in \Pi(T)$, and $T - \lambda I$ is Drazin invertible. So it is an operator of topological uniform descent.

(v) ⇒ (i) Let $\lambda \in E(T)$. Then $T - \lambda I$ is an operator of topological uniform descent. From [9, Theorem 4.7] it follows that for $n$ large enough we have $c_n(T - \lambda I) = 0$ and $c'_n(T - \lambda I) = 0$. Therefore $\lambda$ is a pole of $T$. From Theorem 3.3 we have already $\sigma_D(T - \lambda I) = \sigma_{BW}(T - \lambda I)$. Then it follows from Theorem 1.2 that $T$ satisfies the generalized Weyl's theorem. □

Let $T \in L(X)$. We say that $T$ satisfies the growth condition $G_m$ if there exists an integer $m$ such that

$$\sup_{\lambda \notin \sigma(T)} \|(T - \lambda I)^{-1}\| \operatorname{dist}(\lambda, \sigma(T))^m < \infty.$$

**Lemma 3.6.** *If $T \in B(X)$ satisfies the growth condition $G_m$, then $E(T) = \Pi(T)$.*

P r o o f. Let $\alpha \in E(T)$, then $\alpha$ is isolated in $\sigma(T)$. Then we have $X = X_0 \oplus X_1$, where $X_0, X_1$ are closed subspaces of $X$, $T_0 = (T - \alpha I)_{|X_0}$ is an invertible operator and $T_1 = (T - \alpha I)_{|X_1}$ is a quasi-nilpotent operator.

Without loss of generality we can assume that $\alpha = 0$. Let $0 < \varepsilon < \frac{1}{3} d(0, \sigma(T) \backslash \{0\})$. Then we have $T_1^m = \frac{1}{2\pi i} \int_{|z|=\varepsilon} z^m (T - zI)^{-1} \, dz$. Since

$$\sup_{\lambda \notin \sigma(T)} \|(T - \lambda I)^{-1}\| \operatorname{dist}(\lambda, \sigma(T))^m < \infty,$$

it is easily seen that $T_1^m = 0$. Therefore 0 is a pole of $T$ and so $0 \in \Pi(T)$. As it is always true that $\Pi(T) \subset E(T)$, we have $E(T) = \Pi(T)$. □

**Corollary 3.7.** *If $T \in B(X)$ has SVEP and satisfies the growth condition $G_m$, G-Weyl's theorem holds for $T$.*

P r o o f. This is a direct consequence of Lemma 3.6 and Theorem 3.5 □

Since an operator satisfying the generalized Weyl's theorem satisfies also Weyl's theorem, and an operator having the Dunford property (C) has SVEP, from the previous corollary we obtain the result of Jeon [10, Theorem 1].

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