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ON ITERATIVE ROOTS OF A HOMEOMORPHISM OF THE CIRCLE WITH AN IRRATIONAL ROTATION NUMBER

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Abstract: The aim of this paper is to give the general construction of continuous solutions of the equation $G^n = F$, where $n \geq 2$ is a fixed integer and $F : S^1 \longrightarrow S^1$ is a given homeomorphism. Our basic assumptions are that F has no periodic points and the iterative kernel of F has some algebraic property.

1. Introduction

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle with the positive orientation. Assume that a homeomorphism $F : S^1 \longrightarrow S^1$ is without periodic points. Then $\alpha(F)$, the rotation number of F, is irrational and F preserves orientation (see [5]).

Denote by L_F the set of all cluster points of the orbit $\{F^n(z),\ n \in \in \mathbb{Z}\}$ for a $z \in S^1$. This set does not depend on $z \in S^1$ and L_F either

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equals S^1 or is a nowhere dense perfect set (see [5]).

For every continuous mapping $F : S^1 \longrightarrow S^1$ there exist a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and an integer k such that $F(e^{2\pi ix}) = e^{2\pi if(x)}$, $x \in \mathbb{R}$ and $f(x+1) = f(x) + k$, $x \in \mathbb{R}$. The function f is said to be the lift of F and the integer k is called the degree of F, and is denoted by $\deg F$.

Proposition 1. ([4]) *Let $F, : G : S^1 \longrightarrow S^1$ be orientation-preserving homeomorphisms and suppose that there exists a continuous function $\Phi : S^1 \longrightarrow S^1$ such that $\Phi(F(z)) = G(\Phi(z))$, $z \in S^1$. Then $\alpha(G) = \alpha(F) \deg \Phi (\mathrm{mod}\ 1)$.*

Proposition 2. ([4], [9]) *Let F be a homeomorphism with no periodic points. Then the Schröder equation*

$$\varphi(F(z)) = s\varphi(z), \quad z \in S^1, \tag{1}$$

where $s = e^{2\pi i\alpha(F)}$ has a unique continuous solution $\varphi : S^1 \longrightarrow S^1$ such that $\varphi(1) = 1$. Moreover, $\deg \varphi = 1$ and φ is invertible iff $L_F = S^1$.

If $L_F \neq S^1$, then the set

$$K_F := \varphi[S^1 \setminus L_F],$$

where φ is the continuous solution of (1) such that $\varphi(1) = 1$, is said to be an iterative kernel of F (see [10]).

It was proved by M. C. Zdun [10] that if $L_F = S^1$, then the homeomorphism F has exactly n iterative roots of n-th order that is continuous solutions of the functional equation

$$G^n(z) = F(z), \quad z \in S^1. \tag{2}$$

However, if $L_F \neq S^1$, then F has iterative roots of n-th order with the rotation number $\frac{1}{n}(\alpha(F) + m)$ if and only if

$$\left(\sqrt[n]{s}\right)_m K_F = K_F, \tag{3}$$

where

$$\left(\sqrt[n]{s}\right)_m = e^{2\pi i \frac{1}{n}(\alpha(F)+m)}, \quad m \in \{0, \dots, n-1\}.$$

Moreover, in this case F has infinitely many iterative roots depending on an arbitrary function. In [10] M. C. Zdun also gave the construction of iterative roots. The problem of existence of iterative roots of homeomorphisms of the circle has also been worked out by J. H. Mai in [8].

In this paper we give a construction of iterative roots. We do this on the strength of the method given by M. Kuczma in [6] (see also [7]), i.e. we find an extension of a function defined on the set $S^1 \setminus L_F$.

2. Preliminaries

Let $u, w, z \in S^1$, then there exist unique $t_1, t_2 \in [0, 1)$ such that $we^{2\pi i t_1} = z$, and $we^{2\pi i t_2} = u$. Define

$$u \prec w \prec z \quad \text{iff} \quad 0 < t_1 < t_2$$

and

$$u \preceq w \preceq z \quad \text{iff} \quad t_1 \leq t_2 \quad \text{or} \quad t_2 = 0$$

(see [1]). The properties of these relations can be found in [3] (see also [2]). If $u, z \in S^1, u \neq z$, then there exist $t_u, t_z \in \mathbb{R}$ such that $t_u < t_z < t_u + 1$ and $e^{2\pi i t_u} = u$, $e^{2\pi i t_z} = z$. Put

$$(\overrightarrow{u, z}) = \{e^{2\pi i t} : t \in (t_u, t_z)\},$$

this set is said to be an open arc.

The following lemma is easy to check.

Lemma 1. *Let $L_1, L_2, L_3 \subset S^1$ be pairwise disjoint open arcs and $u, w, z \in S^1$ be such that $u \in L_1$, $w \in L_2$, $z \in L_3$. If $u \prec w \prec z$, then $u_1 \prec w_1 \prec z_1$ for every $u_1 \in L_1$, $w_1 \in L_2$, $z_1 \in L_3$.*

Let $A \subset S^1$ be such that $\text{card}A \geq 3$. We say that the function $\varphi :$ $: A \longrightarrow S^1$ is strictly increasing (respectively increasing) with respect to the cyclic order if for every u, w, z belonging to A such that $u \prec w \prec z$ we have $\varphi(u) \prec \varphi(w) \prec \varphi(z)$ (respectively $\varphi(u) \preceq \varphi(w) \preceq \varphi(z)$). It is easy to check that every strictly increasing mapping is an injection and if F, G are strictly increasing, then so are the mappings F^{-1} and $F \circ G$. Moreover, a homeomorphism $F : S^1 \longrightarrow S^1$ preserves orientation if and only if F is strictly increasing.

Let $F : S^1 \longrightarrow S^1$ be a homeomorphism without periodic points and $L_F \neq S^1$, then the set $S^1 \setminus L_F$ is a countable sum of pairwise disjoint open arcs. Denote the family of these arcs by $\mathcal{A}$. Let $\mathcal{M} :=$ $= \{c(I), I \in \mathcal{A}\}$, where $c(I)$ is the middle point of the arc $I \subsetneq S^1$. Put $I_p := c^{-1}(p)$ for $p \in \mathcal{M}$. Thus we have the decomposition

$$S^1 \setminus L_F = \bigcup_{p \in \mathcal{M}} I_p \quad \text{and } p \in I_p \ \text{ for } p \in \mathcal{M}.$$

Lemma 2. ([10], [3]) *Let F be a homeomorphism without periodic points, $L_F \neq S^1$ and φ be a continuous solution of* (1) *such that* $\varphi(1) =$

$= 1$. *Then for every* $p \in \mathcal{M}$ φ *is constant in* I_p *and there exists a* $q \in \mathcal{M}$ *such that* $F[I_p] = I_q$. *If, moreover, an integer* $n \geq 2$ *and an* $m \in \{0, \ldots, n-1\}$ *are such that* (3) *holds, then a function* $H : \mathcal{M} \longrightarrow \mathcal{M}$ *defined by*

$$H(p) := \Phi^{-1}\left(\left(\sqrt{[n]s}\right)_m \Phi(p)\right), \quad p \in \mathcal{M}, \tag{4}$$

where $\Phi := \varphi|_{\mathcal{M}}$, *is a strictly increasing bijection and* $F[I_p] = I_{H^n(p)}$ *for* $p \in \mathcal{M}$.

Let us note that $H^i(p) \neq p$ for $i \neq 0$, since $\alpha(F) \notin \mathbb{Q}$.

We introduce the following *relation* on $\mathcal{M}$

$p \sim q$ iff there exists an integer $k$such that $q = H^k(p)$.

It is clear that „$\sim$" is an equivalence relation on $\mathcal{M}$. Let E be an arbitrary subset of $\mathcal{M}$ which has exactly one point in common with every equivalence class with respect to the relation „$\sim$".

3. Main result

Theorem. *Let* $F : S^1 \longrightarrow S^1$ *be a homeomorphism with no periodic points such that* $L_F \neq S^1$. *Assume that* (3) *holds for an integer* $n \geq 2$ *and an* $m \in \{0, \ldots, n-1\}$. *If* $g_{p,k} : I_{H^k(p)} \longrightarrow I_{H^{k+1}(p)}$ *for* $p \in E$ *and* $k \in \{0, \ldots, n-2\}$ *are increasing homeomorphisms, then there exists a homeomorphism* $G : S^1 \longrightarrow S^1$ *such that* $G^n = F$ *and* $G|_{I_{H^k(p)}} = g_{p,k}$ *for* $p \in E$ *and* $k \in \{0, \ldots, n-2\}$. *Moreover,*

$$\alpha(G) = \frac{1}{n}(\alpha(F) + m). \tag{5}$$

Proof. Let us construct an auxiliary function $\hat{G}$, which we shall extend to the whole circle S^1. Fix a $p \in E$ and define

$$g_{p,n-1} := F \circ g_{p,0}^{-1} \circ g_{p,1}^{-1} \circ \ldots \circ g_{p,n-2}^{-1}. \tag{6}$$

For every integer i there exist a unique $l \in \mathbb{Z}$ and $r \in \{0, \ldots, n-1\}$ such that $i = ln + r$. Hence for every $i \in \mathbb{Z} \setminus \{0, \ldots, n-1\}$ define

$$g_{p,i} = g_{p,ln+r} := F^l \circ g_{p,r} \circ F^{-l}_{|I_{H^i(p)}}. \tag{7}$$

It follows by Lemma 2 that

$$g_{p,i}\left[I_{H^i(p)}\right] = I_{H^{i+1}(p)} \quad \text{for } i \in \mathbb{Z}. \tag{8}$$

In fact, $F^{-l}\left[I_{H^i(p)}\right] = F^{-l}\left[I_{H^{ln+r}(p)}\right] = I_{H^r(p)}$, $g_{p,r}\left[I_{H^r(p)}\right] = I_{H^{r+1}(p)}$ and $F^l\left[I_{H^{r+1}(p)}\right] = I_{H^{ln+r+1}(p)} = I_{H^{i+1}(p)}$. Consequently,

$$g_{p,i} : I_{H^i(p)} \longrightarrow I_{H^{i+1}(p)} \text{ for } p \in E,\ i \in \mathbb{Z} \text{ are increasing homeomorphisms.} \tag{9}$$

For every $q \in \mathcal{M}$ there exist a unique $p \in E$ and a unique integer i such that $q = H^i(p)$. Define

$$\hat{\mathrm{G}}(z) := g_{p,i}(z) \quad \text{for } z \in I_q,\ q \in \mathcal{M}. \tag{10}$$

It follows by (8) and (9) that the function $\hat{\mathrm{G}}\colon S^1 \setminus L_F \longrightarrow S^1 \setminus L_F$ is a bijection. We shall show that $\hat{\mathrm{G}}$ satisfies the equation

$$\hat{\mathrm{G}}^n(z) = F(z), \quad z \in S^1 \setminus L_F.$$

For this purpose we are going to show that

$$g_{p,i} = F \circ g^{-1}_{p,i-n+1} \circ g^{-1}_{p,i-n+2} \circ \ldots \circ g^{-1}_{p,i-1} \tag{11}$$

for all integers $i \geq n-1$ and

$$g_{p,i} = g^{-1}_{p,i+1} \circ g^{-1}_{p,i+2} \circ \ldots \circ g^{-1}_{p,i+n-1} \circ F_{|I_{H^i(p)}} \tag{12}$$

for $i < n-1$.

We prove this by induction. Obviously for $i = n-1$ we get (11) by (6). Assuming (11) to hold for a $k-1 \geq n-1$, we get

$$F_{|I_{H^{k-n}(p)}} = g_{p,k-1} \circ g_{p,k-2} \circ \ldots \circ g_{p,k-n}. \tag{13}$$

We may write the index k in the form $k = nl + r$, where $l \geq 1$, and $r \in \{0, 1, \ldots, n-1\}$. By (7) we have

$$\begin{aligned} g_{p,k} = g_{p,ln+r} &= F \circ F^{l-1} \circ g_{p,r} \circ F^{-l+1} \circ F^{-1}_{|I_{H^k(p)}} \\ &= F \circ g_{p,(l-1)n+r} \circ F^{-1}_{|I_{H^k(p)}} = F \circ g_{p,k-n} \circ F^{-1}_{|I_{H^k(p)}}. \end{aligned}$$

Using (13) we see that

$$g_{p,k} = F \circ g^{-1}_{p,k-n+1} \circ \ldots \circ g^{-1}_{p,k-1}.$$

Hence by induction (11) holds for $i \geq n-1$. Moreover, we also have (13) for all $k \geq n$. Fix an $r \in \{0, 1, \ldots, n-2\}$. From (13) for $k = n+r$ we have

$$g_{p,r} = g_{p,r} \circ F^{-1} \circ F_{|I_{H^r(p)}} = g_{p,r} \circ g_{p,r}^{-1} \circ g_{p,r+1}^{-1} \circ \ldots \circ g_{p,n+r-1}^{-1} \circ F_{|I_{H^r(p)}},$$

so $g_{p,r}$ satisfy (12). The proof of this part is completed by showing that $g_{p,i}$ satisfy (12) for $i \leq -1$. To do this note that by (7) and (6) we get

$$g_{p,-1} = g_{p,-n+(n-1)} = F^{-1} \circ g_{p,n-1} \circ F_{|I_{H^{-1}(p)}} = \\ = g_{p,0}^{-1} \circ g_{p,1}^{-1} \ldots \circ g_{p,n-2}^{-1} \circ F_{|I_{H^{-1}(p)}},$$

so $g_{p,-1}$ satisfies (12). Fix a $k \in \mathbb{Z}$ such that $k \leq -2$. Suppose that $g_{p,k+1}$ fulfils (12). Hence

$$F_{|I_{H^{k+1}(p)}} = g_{p,k+n} \circ g_{p,k+n-1} \circ \ldots \circ g_{p,k+1}. \tag{14}$$

We shall prove that $g_{p,k}$ satisfies (12). There exist an $l \in \mathbb{Z}$ and an $r \in \{0, 1, \ldots, n-1\}$ such that $k = ln + r$. By (7) and (14) we have

$$g_{p,k} = F^{-1} \circ F^{l+1} \circ g_{p,r} \circ F^{-l-1} \circ F_{|I_{H^k(p)}} = F^{-1} \circ g_{p,k+n} \circ F_{|I_{H^k(p)}} \\ = g_{p,k+1}^{-1} \circ \ldots \circ g_{p,k+n-1}^{-1} \circ F_{|I_{H^k(p)}},$$

which completes this part of the proof.

By (10) and(8) we have

$$\hat{G}^n(z) = g_{p,i+n-1} \circ \ldots \circ g_{p,i+1} \circ g_{p,i}(z) \quad \text{for } z \in I_{H^i(p)}.$$

Thus, for $i < n-1$ using (12) we obtain $\hat{G}^n(z) = F(z)$ for $z \in I_{H^i(p)}$. However, for $i \geq n-1$ it follows from (8) and (11) that

$$g_{p,i} \circ g_{p,i-1} \circ \ldots \circ g_{p,i-n+1}(z) = F(z) \quad \text{for } z \in I_{H^{i-n+1}(p)}.$$

Hence $\hat{G}^n(z) = F(z)$ for $z \in I_{H^j(p)}$, $j \geq 0$. Thus

$$\hat{G}^n(z) = F(z) \quad \text{for } z \in S^1 \setminus L_F.$$

Now we shall show that $\hat{G}$ is strictly increasing. To do this take $u, w, z \in \in S^1 \setminus L_F$ such that $u \prec w \prec z$ and consider three cases

1^o There exist a $p \in E$ and an $i \in \mathbb{Z}$ such that $\{u, w, z\} \subset I_{H^i(p)}$. By (9) and (10) it is clear that $\hat{G}(u) \prec \hat{G}(w) \prec \hat{G}(z)$.

2^o There exist $p, q \in E$, $i, j \in \mathbb{Z}$ such that $H^i(p) \neq H^j(q)$, and one of the following conditions is fulfiled:

(a) $\{u, z\} \subset I_{H^i(p)}, w \in I_{H^j(q)}$,
(b) $\{u, w\} \subset I_{H^i(p)}, z \in I_{H^j(q)}$,
(c) $\{z, w\} \subset I_{H^i(p)}, u \in I_{H^j(q)}$.

According to Lemma 2 in [3] it suffices to consider only case (a). Then $\overrightarrow{(z,u)} \subset I_{H^i(p)}$, whence by (9),

$$g_{p,i}[\overrightarrow{(z,u)}] = \overrightarrow{(g_{p,i}(z), g_{p,i}(u))} = \overrightarrow{(\hat{\mathrm{G}}(z), \hat{\mathrm{G}}(u))} \subset I_{H^{i+1}(p)}.$$

Since $\hat{\mathrm{G}}(w) \in I_{H^{j+1}(q)}$ and $I_{H^{j+1}(q)} \cap I_{H^{i+1}(p)} = \emptyset$, it follows that $\hat{\mathrm{G}}(u) \prec \prec \hat{\mathrm{G}}(w) \prec \hat{\mathrm{G}}(z)$.

3^o There exist $p, q, t \in E$, $i, j, k \in \mathbb{Z}$ such that $H^i(p) \neq H^j(q) \neq \neq H^k(t) \neq H^i(p)$ and $u \in I_{H^i(p)}, w \in I_{H^j(q)}$ and $z \in I_{H^k(t)}$. Hence by Lemma 1

$$H^i(p) \prec H^j(q) \prec H^k(t)$$

but H is strictly increasing so

$$H^{i+1}(p) \prec H^{j+1}(q) \prec H^{k+1}(t).$$

Using Lemma 1 once more we have

$$\hat{\mathrm{G}}(u) \prec \hat{\mathrm{G}}(w) \prec \hat{\mathrm{G}}(z).$$

Thus, we have shown that $\hat{\mathrm{G}}$ is a strictly increasing bijection.

Since the set $S^1 \setminus L_F$ is dense in S^1, it follows (see [2]) that the function $\hat{\mathrm{G}}$ has a unique homeomorphic extension $G : S^1 \longrightarrow S^1$. Moreover, G satisfies (2). It remains to prove (5). Let $z \in I_p$, $p \in \mathcal{M}$, then by (8) and (10), $G(z) \in I_{H(p)}$. By Lemma 2, we have

$$\varphi(G(z)) = \Phi\left(H(p)\right) = \left(\sqrt[n]{s}\right)_m \Phi(p) = \left(\sqrt[n]{s}\right)_m \varphi(z).$$

Hence G and the rotation $R(z) = \left(\sqrt[n]{s}\right)_m z$ are semi-conjugate. By Prop. 1 and Prop. 2 we get $\alpha(R) = \alpha(G)\mathrm{deg}\varphi (\mathrm{mod}\ 1)$ and $\mathrm{deg}\varphi = 1$, thus G and R have the same rotation number, which is our claim. ◊

Remark. *Under the assumptions of Theorem every solution of* (2) *satisfying* (5) *may be obtained in the manner described in the proof of Theorem.*

Proof. If G is a solution of (2) satisfying (5), then by Lemma 1 in [10] $L_F = L_G$, so

$$S^1 \setminus L_G = S^1 \setminus L_F = \bigcup_{p \in \mathcal{M}} I_p.$$

Define functions $p \in E$ $g_{p,i}$ for $p \in E$, $i \in \mathbb{Z}$ by $g_{p,i} = G|_{I_{H^i(p)}}$. It is clear that they are strictly increasing. Next we prove that

$$G[I_p] = I_{H(p)} \quad \text{for } p \in \mathcal{M}. \tag{15}$$

Let us observe that by Prop. 2 there exists a continuous solution of the equation

$$\psi(G(z)) = \left(\sqrt[n]{s}\right)_m \psi(z), \quad z \in S^1$$

such that $\psi : S^1 \longrightarrow S^1$, $\psi(1) = 1$. Moreover,

$$\psi(G^n(z)) = \left(\left(\sqrt[n]{s}\right)_m\right)^n \psi(z),$$

so ψ satisfies (1) and $\psi = \varphi$, by the uniqueness of the solution of the equation (1). Fix a $p \in \mathcal{M}$. By Lemma 2 there is a $q \in \mathcal{M}$ such that $G[I_p] = I_q$ and

$$\{\Phi(q)\} = \varphi\left[G[I_p]\right] = \psi\left[G[I_p]\right] = \left(\sqrt[n]{s}\right)_m \psi[I_p] =$$
$$= \left(\sqrt[n]{s}\right)_m \varphi[I_p] = \left\{\left(\sqrt[n]{s}\right)_m \Phi(p)\right\},$$

so by (4), $q = H(p)$, which gives (15). What is left is to show that $g_{p,i}$ for $p \in E$ satisfy (6) and (7). Observe that from (2) for $z \in I_p$ we get

$$g_{p,n-1} \circ \ldots \circ g_{p,1} \circ g_{p,0}(z) = F(z),$$

so

$$g_{p,n-1}(z) = F \circ g_{p,0}^{-1} \ldots \circ g_{p,n-2}^{-1}(z), \quad z \in I_{H^{n-1}(p)}$$

and we have (6). We conclude from (2) that $G \circ F = F \circ G$, hence

$$G \circ F^l = F^l \circ G, \quad l \in \mathbb{Z}. \tag{16}$$

Fix $z \in I_{H^r(p)}$, $r \in \{0, 1, \ldots, n-1\}$. Thus from (16), Lemma 2 and the definition of $g_{p,i}$, $i \in \mathbb{Z}$ we have

$$g_{p,ln+r} \circ F^l(z) = F^l \circ g_{p,r}(z),$$

which gives (7), and the proof is completed. ◊

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