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N-GROUPS WITH ACC ON ANNIHILATORS AND SOME TOPOLOGICAL PROPERTIES

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Abstract: Here we try to investigate some properties of topological structure on some generalization of what has been achieved on near-rings with acc on annihilators as by A. Oswald and K. C. Chowdhury et al. The so-called pseudo character on nilpotency and strongly semi-primeness, lead us satisfactorily towards our goal giving elegant results on N-groups rather than the near-ring structure. Topological structure on such N-groups gives some interesting fruitful results. Extending the idea of boundedness of Beidleman and Cox on topological near-rings, E-boundedness etc. together with the notion of topologically nilpotent sets are playing an important role on an N-group E having finite number of elements (e's) belonging to E with zero annihilators [$\text{Ann}(e) = 0$], which occur as a necessity of the N-group E in the above context. It is interesting to note the relevancy and elegancy of the result obtained, as the same may be determined with accommodating justification on such topological N-groups that their discrete character is in association with the E-boundedness with zero radical or open character of the radical with E-boundedness. Moreover all these lead us to the later type of the radical of the N-group. Together with all these, some interesting results regarding the discrete character of the N-group is observed in case of locally compact groups if the near-ring is without unity.

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1. Introduction

The results due to A. W. Goldie on the structure of semiprime rings [11, 12] seem to be still relevant due to its elegancy. As remarked by Meldrum there is not a great deal of works being done in near-ring theory on algebraic structures with acc on annihilators. Same may be stated so far on its topological structure, too. Chowdhury et al. attempted rigorously on what have been meant above [4, 7]. Here we try to investigate some properties of topological structure on some generalization of what has been achieved on near-rings with acc on annihilators [17] by Chowdhury et al. [5, 7]. The so-called pseudo character on nilpotency and strongly semi-primeness, lead us satisfactorily towards our goal giving elegant results on N-groups rather than the near-ring structure. Topological structure on such N-groups gives some interesting fruitful results.

An N-group E with so-called Goldie character is well behaved so far as pseudo quality on nilpotency as well as strongly semi-primeness are involved for the proper development of what we have attempted for.

Extending the idea of boundedness of Beidleman and Cox [3], E-boundedness etc. together with the notion of topologically nilpotent set are playing an important role on an N-group E having finite number of elements (e's) belonging to E with zero annihilators [$\text{Ann}(e) = 0$], which occur as a necessity of the N-group E in above context. It is interesting to note the relevancy and elegancy of the result obtained, as the same may be determined with accommodating justification on such topological N-groups that their discrete character is in association with the E-boundedness with zero radical or openness of the radical with E-boundedness.

For the sake of completeness of the idea we are dealing with, once more we like to mention that any near-ring (near-ring group) with ascending chain condition (acc) on its left near-ring subgroups obviously satisfies what Oswald has chosen (viz., no infinite direct sum of left ideals and satisfies the acc on left annihilators). But rings like $Z[X_i \mid \mid i = 1, 2, \ldots, X_iX_j = X_jX_i]$ satisfy the acc on left annihilators having no infinite direct sum of left ideals, though it may have a strictly ascending infinite chain of ideals viz.,

$$(X_1) \subset (X_1, X_2) \subset \ldots .$$

Thus near-rings with the conditions described in [17], need not satisfy the acc on its sub-algebraic structures. This has lead Chowdhury et al. towards the idea of the so-called strictly Artinian radical [7].

With N, a (right) topological near-ring, a topological N-group is a pair (E, μ), where E is a topological group under addition and μ is a continuous map from $N \times E$ to E such that $\mu(a+b, e) = \mu(a, e) + \mu(b, e)$ and $\mu(ab, e) = \mu(a, \mu(b, e))$, for all $a, b \in N$ and $e \in E$ (G. Pilz [18], Clay [8], Magill [16]). As seen [16], in case of the topological group $\mathbb{R}$ of real numbers under addition and $\mathbb{Z}$, the discrete group of integers under addition, $T = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus. On the other hand, $\mathbb{R}^n$ denotes the Euclidean N-group and T^n the n-dimensional torus. Also it is interesting to note that in case of any topological group E and a locally compact Hausdorff group H with compact open topology on $N(H)$ (the topological near-ring of all continuous self maps of H under pointwise addition and composition), E can be made a topological $N(H)$-group, where $\mu(f, x) = 0$, for all $f \in N(H)$ and $x \in E$. And Magill shows the existence of such μ in case of E to be $\mathbb{R}^n$ or T^n.

As insisted by Kaplansky [15] regarding continuity of addition and multiplication on the product space, in case of a ring, it is found that the coordinate wise continuity is all that is necessary in what we have attempted for.

To be more explicit, our proposed continuity regarding addition and multiplication is as follows: We stick to the definition of a topological (right) near-ring N as suggested by Beidleman and Cox [3] where coordinate wise continuities of the respective mappings are imposed. Keeping this in note, we define topological N-group E as an N-group E in which a Hausdorff topology is given with four continuous mappings μ_1, μ_2, μ_3, μ_4 such that for given $e \in E$ (i) $\mu_1(x) = x + e$, (ii) $\mu_2(x) = e + x$, for all $x \in E$, (iii) $\mu_3(n) = ne$, for all $n \in N$, and given $m \in N$, (iv) $\mu_4(e) = me$, for all $e \in E$.

Note. It is clear that if $1 \in N$, then for a given $e \in E$, as $-e = (-1)e$, the map $x \to e - x$ is continuous, for all $x \in E$. Hence if V is open in E and $e \in E$, then each of $V + e$, $e + V$ and $-V$ is open in E.

Definitions and notations

Unless otherwise specified, throughout this paper N will mean a zero symmetric right near-ring with unity 1; E will denote the left N-

group of ${}_NE$, a left N-subgroup L of N will mean an N-subgroup of ${}_NN$ and a left ideal of N will mean an ideal of ${}_NN$.

If L and B are two N-subgroups of E with $L \subseteq B$, then L is strictly essential N-subgroup of B when any non-zero N-subgroup C $(\subseteq B)$ has a non-zero intersection with L. We denote this by $L \subseteq_e B$.

A strictly essential left N-subgroup L of N is a strictly essential N-subgroup of ${}_NN$. Moreover for $L \subseteq B \subseteq E$, then $L \subseteq_e E$ if and only if $L \subseteq_e B \subseteq_e E$.

An ideal I of E is an essential ideal of E when it has a non-zero intersection with any non-zero ideal of E. If a left ideal I of N is an essential ideal of ${}_NN$, then I is an essential left ideal of N.

It is clear that a strictly essential N-subgroup of E is always essential as an ideal of it. We note that in the symmetric group $N(= S_3)$, [19 (37), p. 342] $\{0, a\}, \{0, b\}, \{0, c\}$ and $\{0, x, y\}$ are proper non-zero N-subgroups of ${}_NN$ where $\{0, x, y\}$ is an ideal of ${}_NN$. So this is not a strictly essential as an N-subgroup of it though it is an essential ideal. Thus an essential ideal of E need not be a strictly essential as an N-subgroup of it.

We define the set $Z_l(E) = \{x \in E \mid Lx = 0$, for some strictly essential N-subgroup L of ${}_NN\}$.

An annihilator of S $(\subseteq E)$ in N is defined as $\mathrm{Ann}_N(S) = \{n \in N \mid$ $\mid nS = 0\}$ (or denoted simply by $\mathrm{Ann}(S)$), if $S \subseteq N$, then it is denoted by $l(S)$, the left annihilator of S in N.

Near-ring N is a duo near-ring if for every $a, b \in N$, $ab = bc = da$, for some $c, d \in N$ [9].

An element $q \in N$ is quasi-regular [1] if there is an element $n \in N$ such that $n(1-q) = 1$. The set Q will denote the set of all quasi-regular elements of N.

The intersection of all maximal N-subgroups of ${}_NN$ is the radical subgroup [1] of N and will be denoted by A.

The radical $J(E)$ [2] of E is the intersection of all ideals of E that are maximal as N-subgroups of E. Similarly we define the radical $J(N)$ of N.

The content of almost all newly introduced concepts and some results is supported by a good number of examples from a sweeping point of view, viz. whether one intends to attempt with his liberty regarding the presence or absence of the unity 1 in the near-ring.

Availability or the concepts seem to be meaningful in case of near-rings even without unity is a refined of the same results from a broader angle, which may unveil the importance of generalization of the results obtained.

Discussion. Examples and some observations

1. The near-ring $N(=D_8)$ [19 (24) p. 345] is a non-zero symmetric near-ring without unity having non-zero proper left N-subsets viz.,

$\{0,b\},\{0,b,a+b\},\{0,b,2a+b\},\{0,2a,b,2a+b\}$ and $\{0,2a,b,2a+b,3a+b\}$. Here we note that, for any subset L of N there exists no left N-subset X of N such that $X^nL=0$, for any $n\in\mathbb{Z}^+$. On the contrary,

2. in the near-ring $N(=\mathbb{Z}_8)$ [19 (22) p. 343] without unity, all the proper left N-subsets are $\{0,1\},\{0,2\},\{0,4\},\{0,4,5\},\{0,4,6\},\{0,2,4,6\}$, $\{0,2,4,6,7\}$ and $\{0,2,3,4,6\}$.

As $\{0,4,5\}\{0,5\}=\{0,4\}(\neq 0)$ and, in some sense, $\{0,5\}$ may be thought as nilpotent. We note that if $\{0,5\}$ is replaced by $\{0,4,5\}$, then we have $\{0,4,5\}^3=0$ and thereby $\{0,4,5\}$ is found as nilpotent.

3. In the near-ring $N(=\mathbb{Z}_8)$ [19 (84) p. 343], without unity the only proper left N-subset of it is $\{0,2\}$ and hence for any subset L of it different from $\{0,2\}$, we have $\{0,2\}L\neq 0$, but $\{0,2\}^2L=0$.

4. In the near-ring $N(=\mathbb{Z}_8)$ [19 (46) p. 343] with unity all the proper left N-subsets are $\{0,4\}$ and $\{0,2,4,6\}$. As $\{0,2,4,6\}\{0,2\}\neq 0$ and $\{0,2,4,6\}^2\{0,2\}=0$.

Near-rings with the above type of pseudo character in nilpotency of subsets are playing an important role in what we have attempted for and lead us to the following.

A subset L of E is ps(pseudo)-nilpotent w.r.t. a proper left N-subset B of N with nilpotency $n\in\mathbb{Z}^+$, if $BL\neq 0$ such that $B^nL=0$, for some $n(\geq 2)\in\mathbb{Z}^+$.

Subset L of E is ps-nilpotent if it is ps-nilpotent w.r.t. some proper left N-subset B of N with some nilpotency.

An element $e\in E$ is ps-nilpotent if $\{e\}$ is a ps-nilpotent subset of E.

In Ex. 1, N has no ps-nilpotent subset; on the other hand in Ex. 2, $\{0,5\}$ is a ps-nilpotent subset, in Ex. 3 any subset L of it different from $\{0,2\}$ is ps-nilpotent and Ex. 4 contains $\{0,2\}$ as ps-nilpotent.

5. In the near-ring $N(=\mathbb{Z}_8)$ [20 (22) p. 343] without unity, the singleton set $\{3\}$ is ps-nilpotent w.r.t. the proper left N-subset $\{0,4,5\}$. Hence 3 is a ps-nilpotent element.

6. In the near-ring N [19 (4) p. 340] of Klein 4-group with unity b is ps-nilpotent element.

In obvious sense, it is assumed that 0 (zero of E) is the ps-nilpotent element of E. Also if N has no proper left N-subset, then no element of E is ps-nilpotent.

Also we note that

7. In the near-ring $N(=\mathbb{Z}_6)$ [19 (27) p. 341] with unity there is no element a, which is ps-nilpotent w.r.t. the proper left N-subset Na.

8. In the near-ring $N(=\mathbb{Z}_8)$ [19 (84) p. 343] without unity, every element except 2 is ps-nilpotent w.r.t. the proper left N-subset $\{0,2\}$ but no element (say) $a(\in N)$ is ps-nilpotent w.r.t. the proper left N-subset Na.

9. In the near-ring $N(=\mathbb{Z}_8)$ [19 (46) p. 343] with unity, we have $(N2)2=\{0,4\}$ but $(N2)^2 2=0$.

10. In the near-ring $N(=\mathbb{Z}_8)$ [19 (22) p. 343] without unity, the element 3 is such that $N3=\{0,2,4,6\}$, $(N3)3=\{0,4\}$ and $(N3)^2 3= =0$.

We say, an element $a\in N$ is self-nilpotent if it is ps-nilpotent w.r.t. the proper left N-subset Na. Thus, in Examples 7 and 8, N has no self-nilpotent element; on the other hand in Ex. 9 and Ex. 10, 2 and 3 are respectively both self-nilpotent.

We see that a self-nilpotent element of N is nilpotent in N. Again we say, subset B of E is ps-nil if each element of B is ps-nilpotent.

Some sort of counterfeitness is observed with the semi-prime or strongly semi-prime character [7] with

11. The near-ring N [19 (7) p. 340] of Klein 4-group with unity, where it has non-zero proper left N-subsets $\{0,a\},\{0,b\}$ and $\{0,a,b\}$ such that $\{0,a\}^n L\neq 0$, for any subset $L(\neq 0)$ except $\{0,b\}$ and $\{0,b\}^n B\neq 0$, for any subset $B(\neq 0)$ except $\{0,a\}$ where $n\in\mathbb{Z}^+$. Hence N has no non-zero ps-nilpotent subset.

And in

12. the Klein 4-group, the near-ring N [19 (11) p. 340] without unity, the only non-zero proper left N-subsets are $\{0,a\},\{0,b\},\{0,a,c\}$ such that for any subset $L(\neq 0)$ of N $\{0,a\}^n L\neq 0$, $\{0,a,c\}^n L\neq 0$, for any $n\in\mathbb{Z}^+$ and $\{0,b\}L=0$. Thus here, N has no non-zero ps-nilpotent subset. In this sense, ${}_N N$ is a ps-strongly semi-prime. In other words, an N-group E is ps-strongly semi-prime if E has no non-zero ps-nilpotent subset. Another quasi character of nilpotency springs up from what we have cited below:

13. In the near-ring N ($= \mathbb{Z}_8$) [19 (46) p. 340] with unity, the units are 1, 3, 5, 7. Now for the set N_u of non-units of N, we have $N_u 2 \neq \{0, 4\}$, $N_u^2 2 = 0$ but $N_u^2 \neq 0$. In this sense, the element $2 \in N$ would be N_u-nilpotent. Thus, an element $x \in E$ is N_u-nilpotent, where N_u is the set of all non-unit elements of N, when there exists a least positive integer t (nilpotency) such that $N_u^t x = 0$, but $N_u^t \neq 0$.

Now it is noticeable that an element $\sum_{i=1}^{n} x_i \in \bigoplus_{i=1}^{n} E_i$, where each E_i is N-group, is N_u-nilpotent if each $x_i \in E_i$ is N_u-nilpotent, as for each i, $N_u^{t_i} x_i = 0$ but $N_u^{t_i} \neq 0$, for some t_i (least) $\in \mathbb{Z}^+$ and hence $\bigoplus_{i=1}^{n} N_u^m x_i = 0$, but $N_u^m \neq 0$, where $m = \max(t_1, t_2, \ldots, t_n)$.

And we make the following

Note. (i) It is clear that any proper left N-subset of N with nilpotency greater than 2 is ps-nilpotent.

(ii) If B is ps-nilpotent w.r.t. a proper left N-subset L of N, then LB is nilpotent.

(iii) It is obvious that an N_u-nilpotent element of E with nilpotency greater than one is ps-nilpotent.

Throughout our discussion Q' will denote the subset of E consisting of all N_u-nilpotent elements of E.

A non-empty proper N-subgroup of E is N_u-nil if it is contained in Q'.

The following example clarifies what we have proposed to carry out regarding the radical of E.

14. We consider a near-ring group ${}_N E$, where $E = \mathbb{Z}_3$ and N is obtained from a subset of the mappings of the group $(\mathbb{Z}_3, +)$, which elegantly expresses the N-group character with necessary requirements.

Consider $(E =)\mathbb{Z}_3 = \{0, 1, 2\}$ and $N = \{0, i, a, b, c, d, e, f, g\}$.

From the following table it follows that N is a right near-ring with unity i having proper left N-subgroups

$$A_1 = \{0, a, b\}, \quad A_2 = \{0, c, e\}, \quad A_3 = \{0, d, g\},$$

where A_1 and A_2 are left ideals. But $J(N) = A_1 \cap A_2 = 0$. Again $E = \{0, 1, 2\}$ is an N-group such that $J(E) = 0$. Hence $J(N)E = J(E)$.

Addition in N

$+$	0	i	a	b	c	d	e	f	g
0	0	i	a	b	c	d	e	f	g
i	i	f	c	d	g	e	b	0	a
a	a	c	b	0	d	i	f	g	e
b	b	d	0	a	i	c	g	e	f
c	c	g	d	i	e	f	0	a	b
d	d	e	i	c	f	g	a	b	0
e	e	b	f	g	0	a	c	d	i
f	f	0	g	e	a	b	d	i	c
g	g	a	e	f	b	0	i	c	d

Multiplication in N

$\times$	0	i	a	b	c	d	e	f	g
0	0	0	0	0	0	0	0	0	0
i	0	i	a	b	c	d	e	f	g
a	0	a	0	a	0	0	c	c	d
b	0	b	0	b	0	0	e	e	g
c	0	c	a	0	c	d	0	a	0
d	0	d	a	a	c	d	c	d	d
e	0	e	b	0	e	g	0	b	0
f	0	f	b	a	e	g	c	i	d
g	0	g	b	b	e	g	e	g	g

Product in E over N

$N \times E$	0	1	2
0	0	0	0
i	0	1	2
a	0	0	1
b	0	0	2
c	0	1	0
d	0	1	1
e	0	2	0
f	0	2	0
g	0	2	2

We shall consider near-ring groups in the above sense termed as fully radical character.

15. The near-ring $N(=\mathbb{Z}_8)$ [19 (46) p. 343] equipped with a topology

$$T(=\{\emptyset, N, \{0,4\}, \{1,5\}, \{2,6\}, \{3,7\}, \{0,1,4,5\}, \{0,3,4,7\}, \{0,2,4,6\}, \{1,2,5,6\}, \{2,3,6,7\}, \{1,3,5,7\}, \{0,2,3,4,6,7\}, \{0,1,3,4,5,7\}, \{0,1,2,4,5,6\}, \{1,2,3,5,6,7\}\})$$

is a topological near-ring. Here the left N-subsets are $\{0,4\}$ and $\{0,2,4,6\}$.

Now for the subset $L(=\{0,2\})$ of N we have $\{0,2,4,6\}L=\{0,4\}$ and $\{0,2,4,6\}^2L=0$, which belongs to every open subset of N (containing 0). In this sense we define the following:

A subset $\bigoplus_{i=1}^{n} D_i$ of $\bigoplus_{i=1}^{n} E_i$ is topologically nilpotent if for any open subset $\bigoplus_{i=1}^{n} U_i$ of $\bigoplus_{i=1}^{n} E_i$ containing 0, there exists a left N-subset $C_1 \cup \cup C_2 \cup \ldots \cup C_n$ of N such that $\bigoplus_{i=1}^{n} C_i^t D_i \subseteq \bigoplus_{i=1}^{n} U_i$, for some $t \in \mathbb{Z}^+$. It is clear that in the discrete topological N-group, a topologically nilpotent set, with nilpotency greater than or equal to 2, is ps-nilpotent.

16. The near-ring N of Klein's four group [19 (12) p. 340] with the topology $T(= \{\emptyset, N, \{0, a\}, \{b, c\}\})$ is a topological near-ring. Here we note that, for any subset $L(\neq 0)$ of N, $L\{b, c\} = \{o, a\}$, an open subset containing 0. In view of this we define that, a subset B of N is E-bounded, if for any open subset V of E containing 0, there exists an open subset U of E such that $BU \subseteq V$. If $B = N$, then N is itself E-bounded.

In $E = \bigoplus_{i=1}^{n} E_i$, where each E_i is an N-group, a subset B of N is E-bounded, if for any open subset $\bigoplus_{i=1}^{n} V_i$ of $\bigoplus_{i=1}^{n} E_i$ containing the zero, there exists an open subset $\bigoplus_{i=1}^{n} U_i$ of $\bigoplus_{i=1}^{n} E_i$ such that $\bigoplus_{i=1}^{n} BU_i \subseteq \bigoplus_{i=1}^{n} V_i$.

It is clear that if a subset B of N is E-bounded where $E = \bigoplus_{i=1}^{n} E_i$, then B is E_i-bounded for each i.

2. Preliminaries

In the following lemmas, we assume that N is a duo near-ring with the acc on annihilators of subsets of E in N and thus we call the N-group E as a $_{\text{duo}}$acc N-group E.

Lemma 2.1. *If E is a ps-strongly semi-prime $_{\text{duo}}$acc N-group, then E has no non-zero ps-nil N-subset.*

Proof. Let $B(\neq 0)$ be an N-subset of E and $L(\neq 0)$ be a left N-subset of N with $LB \neq 0$. So $aB \neq 0$, for some $a(\neq 0) \in L$. We have $aNaB \neq 0$, if not B is a ps-nilpotent subset of E, as $(Na)^2B = 0$ and $NaB \neq 0$ (as $1 \in N$), which is a contradiction. As E satisfies the acc on its annihilators, we can choose $ab(\neq 0) \in aB$, $(b \in B)$ with $\text{Ann}(ab)$ as large as possible. Now $aNab \neq 0$, if not, as above, the set $\{b\}$ is a ps-nilpotent subset of E which is a contradiction. So $axab \neq 0$, for some $x \in N$, and hence $xab \neq 0$, if not, $axab = 0$, giving thereby $x \notin \text{Ann}(ab)$. But $\text{Ann}(ab)$ being maximal and $\text{Ann}(ab) \subseteq \text{Ann}(axab)$, we get $\text{Ann}(ab) = \text{Ann}(axab)$. So $x \notin \text{Ann}(axab)$ or, $(xa)^2b \neq 0$ or, $(xax)ab \neq 0$ or, $xax \notin \text{Ann}(ab) = \text{Ann}(axab)$, or $(xa)^3b \neq 0$ and so on. Thus $(xa)^tb \neq 0$, for any $t \in \mathbb{Z}^+$. Therefore B possesses a non-zero non-ps-nilpotent element b. So B is not ps-nil. ◊

We note above: For $b(\neq 0) \in B$, we have $a \in L$ such that ab is a non-ps-nilpotent element of B. ◊

Lemma 2.2. *The set $Z_l(E)$ is a ps-nil N-subset of the* $_{\text{duo}}$acc *N-group E.*

Proof. By Lemma 2.10 of [6], it is clear that $Z_l(E)$ is an N-subset of E. Let $e \in Z_l(E)$ and if $Ne = 0$, then $e = 0$, which is ps-nilpotent. Again if $Ne \neq 0$, then $xe \neq 0$, for some $x(\neq 0) \in N$. So $B(= Nx)$ is a left N-subset of N such that $Be \neq 0$, as $1 \in N$. We have (N being a duo near-ring)

(1) $\quad \text{Ann}(e) \subseteq \text{Ann}(b_1e) \subseteq \text{Ann}(b_1b_2e) \subseteq \ldots$ (for any $b_i \in B$).

We claim that (1) is a strictly ascending chain. If possible let

(2) $\quad \text{Ann}(b_1b_2\ldots b_te) = \text{Ann}(b_1b_2\ldots b_{t+1}e)$, (for some $t \in \mathbb{Z}^+$).

At first, we show that

(3) $\quad \text{Ann}(b_1b_2\ldots b_te) = \text{Ann}(b_1b_2\ldots b_{t+n}e)$ (for any $n \in \mathbb{Z}^+$).

We have, (3) is true for $n = 1$, by (2). Suppose it is true for $n = r$. Now let $z \in \text{Ann}(b_1b_2\ldots b_te)$, then $zb_1b_2\ldots b_tb_{t+1}\ldots b_{t+r+1}e = xzb_1b_2\ldots b_te = x0 = 0$, for some $x \in N$, as N is a duo near ring, which gives $\text{Ann}(b_1b_2\ldots b_te) \subseteq \text{Ann}(b_1b_2\ldots b_{t+r+1}e)$. Now for $p \in \text{Ann}(b_1b_2\ldots b_{t+r+1}e)$, we get $pb_1b_2\ldots b_{t+r+1}e = 0$, which gives

$$pb_1(\text{Ann}(b_2b_3\ldots b_{t+r+1}e) = \text{Ann}(b_2b_3\ldots b_{t+1}e),$$

giving thereby $pb_1b_2\ldots b_{t+1}e = 0$ and hence $p \in \text{Ann}(b_1b_2\ldots b_{t+1}e)$ which implies $p \in \text{Ann}(b_1b_2\ldots b_te)$. So, by induction we get the result (3).

Now $Le = 0$, as $e \in Z_l(E)$, for some $L \subseteq_{eN} N$; this gives $L \subseteq \text{Ann}(e)$ and therefore $\text{Ann}(e) \subseteq_{eN} N$ as $L \subseteq_{eN} N$ giving thereby $\text{Ann}(b_1b_2\ldots b_{t+1}e) \subseteq_{eN} N$. Hence, if $nb_1b_2\ldots b_te \neq 0$, where $n \in N$, then

$$((Nb_1b_2\ldots b_t) \cap \text{Ann}(b_1b_2\ldots b_{t+1}e))e \neq 0.$$

Now $nb_1b_2\ldots b_tb_1b_2\ldots b_te = 0$, as $nb_1b_2\ldots b_t \in \text{Ann}(b_1b_2\ldots b_{t+1}e)$ $(= \text{Ann}(b_1b_2\ldots b_te))$ which gives

$$n \in \mathrm{Ann}((b_1b_2\dots b_t)^2e)\ (= \mathrm{Ann}(b_1b_2\dots b_te))$$

and so $nb_1b_2\dots b_te = 0$ which contradicts the choice of $nb_1b_2\dots b_te$. Hence (1) is a strictly ascending chain, which violates the hypothesis.

So we get $nb_1b_2\dots b_te = 0$, for all $b_i \in B$ and $n \in N$, which gives $B^te = 0$, for some $t(\geq 2) \in \mathbb{Z}^+$ (if $B = Nx = N$, then $B^t = N^t = N$ gives $Ne = 0$ and is not true).

Therefore B is proper left N-subset of N. Hence $e(\in Z_l(E))$ is a ps-nilpotent element and thus $Z_l(E)$ is a ps-nil subset of E. $\Diamond$

From above we get the following (as in case of a $_{\mathrm{duo}}$acc N-group E unless otherwise specified)

Lemma 2.3. *If E is a ps-strongly semi-prime N-group E, then $Z_l(E) =$ $= 0$.*

As in Lemma 2.11 [6] we have

Lemma 2.4. *If E is a ps-strongly semi-prime N-group such that N has no infinite direct sum of left ideals and an essential left ideal of N is strictly essential as N-subgroup of $_NN$, too, then the annihilators of subsets of E in N satisfy the* dcc.

Lemma 2.5. *Let I be a left N-subgroup of N and B be an N-subgroup of E with distributively generated annihilators of subsets of E in N. If i_1a_1 is a non-ps-nilpotent element of IB with $\mathrm{Ann}(i_1a_1)$ maximal and $i_2a_2 \in (\mathrm{Ann}(i_1a_1) \cap I)B$ with the same character as i_1a_1, then $\mathrm{Ann}(i_1a_1 + i_2a_2) = \mathrm{Ann}(i_1a_1) \cap \mathrm{Ann}(i_2a_2)$.*

Proof. Let $x \in \mathrm{Ann}(i_1a_1) \cap \mathrm{Ann}(i_2a_2)$, then $x = \sum_{fin} \pm s_j$ where $s_j \in S_1$ and $\mathrm{Ann}(i_1a_1) = \langle S_1 \rangle$, S_1 is a set of distributive elements.

Now, as each $s_j \in \mathrm{Ann}(i_1a_1)$, we get $\left(\sum_{fin} \pm s_j\right)(i_1a_1 + i_2a_2) =$ $= \left(\sum_{fin} \pm s_j\right)(i_2a_2) = xi_2a_2 = 0$ and hence $x \in \mathrm{Ann}(i_1a_1 + i_2a_2)$ giving thereby $\mathrm{Ann}(i_1a_1) \cap \mathrm{Ann}(i_2a_2) \subseteq \mathrm{Ann}(i_1a_1 + i_2a_2)$.

Conversely, let $y\left(= \sum_{fin} \pm t_j\right) \in \mathrm{Ann}(i_1a_1 + i_2a_2) = \langle S_2 \rangle$, where $t_j \in S_2$, a set of distributive elements, then $t_j \in \mathrm{Ann}(i_1a_1 + i_2a_2)$ which gives $t_j(i_1a_1 + i_2a_2) = 0$ or, $t_ji_1a_1 + t_ji_2a_2 = 0$.

Now $i_2(t_ji_1a_1 + t_ji_2a_2) = 0$ or, $\left(\sum_{fin} \pm s_k\right)(t_ji_1a_1 + t_ji_2a_2) =$ $= 0$, as $i_2\left(= \sum_{fin} \pm s_k\right) \in \mathrm{Ann}(i_1a_1) = \langle S_3 \rangle$, where $s_k \in S_3$ (a set of distributive elements), or, $\sum_{fin} \pm(s_kt_ji_1a_1 + s_kt_ji_2a_2) = 0$ or,

$\sum_{fin}\pm(ps_k i_1a_1+s_kt_ji_2a_2)=0$, for some $p\in N$, as N is a duo near-ring or, $\sum_{fin}\pm s_kt_ji_2a_2 = 0$ as $s_k\in \text{Ann}(i_1a_1)$ or, $\left(\sum_{fin}\pm s_k\right)t_ji_2a_2 = 0$ or, $i_2t_ji_2a_2 = 0$ or, $t_jqi_2a_2 = 0$, for some $q\in N$, being duo near-ring and so $t_j\in \text{Ann}(qi_2a_2)(\supseteq \text{Ann}(i_2a_2))$. Now $qi_2a_2(\in (\text{Ann}(i_1a_1)\cap I)B)$ is a non-ps-nilpotent element. If not, then $Sqi_2a_2\neq 0$ such that $S^nqi_2a_2 = = 0$, for some proper left N-subset S of N and some $n(\geq 2)\in\mathbb{Z}^+$. By taking $T=Sq$ we get $Ti_2a_2\neq 0$ such that $T^ni_2a_2=0$ giving thereby i_2a_2 is ps-nilpotent element which is not true. Thus $\text{Ann}(qi_2a_2) = = \text{Ann}(i_2a_2)$, as $\text{Ann}(i_2a_2)$ is maximal. So, $t_j\in\text{Ann}(i_2a_2)$ for each j, which gives $t_ji_2a_2=0$, for each j; this gives $t_ji_1a_1=0$ and hence $\left(\sum_{fin}\pm t_j\right)i_1a_1=0$ and $\left(\sum_{fin}\pm t_j\right)i_2a_2=0$, which implies $y\left(=\sum_{fin}\pm t_j\right)\in$ $\in (\text{Ann}(i_1a_1)\cap\text{Ann}(i_2a_2)$. Thus $\text{Ann}(i_1a_1+i_2a_2)\subseteq\text{Ann}(i_1a_1)\cap$ $\cap\,\text{Ann}(i_2a_2)$. ◊

In the near-ring $N(=\mathbb{Z}_8)$ [19 (22) p. 343], all the proper left N-subsets are $\{0,1\}$, $\{0,2\}$, $\{0,4\}$, $\{0,4,5\}$, $\{0,2,4,6\}$, $\{0,4,6\}$, $\{0,2,4,6,7\}$, $\{0,2,3,4,6\}$, where $\{0,4\}$ is distributively generated left annihilator and $\{0,3\}$ is a ps-nilpotent subset of N. As $\{0,2\}\{0,3\}\neq 0$ and $\{0,2\}^2\{0,3\}=0$. So N is not ps-strongly semi-prime. Moreover $\{0,4\}$ and $\{0,2,4,6\}$ are only two left N-subgroups as well as ideals. So each of them is essential left ideal as well as strictly essential as an N-subgroup of ${}_NN$. But ${}_NN$ contains no element e such that $\text{Ann}(e) = = 0$.

Thus we see how ps-strongly semi-prime character together with distributively generated annihilator and coincidence of essential left ideals and strictly essential N-subgroups of ${}_NN$ [17, Th. (7)] play a key role for the existence of an element e of ${}_NN$ such that $\text{Ann}(e)=0$. And we note the following

Lemma 2.6. *Let N-group E be as in* Lemma 2.4 *and the annihilators of subsets of E in N are distributively generated, then there exists $e\in E$ such that* $\text{Ann}(e)=0$.

Proof. Let B be a non-zero N-subgroup of E and let $I\subseteq_e {}_NN$. We have, by Note in Lemma 2.1, that B is not ps-nil having a non-ps-nilpotent element of the form ia, $(i\in I,\ a\in B)$. Now, by hypothesis, we consider $i_1a_1\in IB$ $(i_1\in I,\ a_1\in B)$, with i_1a_1 non-ps-nilpotent such that $\text{Ann}(i_1a_1)$ is as large as possible. If $\text{Ann}(i_1a_1)=0$, we stop. If not, then $\text{Ann}(i_1a_1)\cap I\neq 0$ as $I\subseteq_e {}_NN$. Again we choose, as above, a non-ps-nilpotent element $i_2a_2 \in (\mathrm{Ann}(i_1a_1) \cap I)B$ ($i_2 \in \mathrm{Ann}(i_1a_1) \cap I$, $a_2 \in B$), with $\mathrm{Ann}(i_2a_2)$ as large as possible. Now $i_1a_1 + i_2a_2 \in B$, if $\mathrm{Ann}(i_1a_1 + i_2a_2) = 0$, we stop, if not $\mathrm{Ann}(i_1a_1 + i_2a_2) \cap I \neq 0$ as $I \subseteq_{eN} N$. Now, by Lemma 2.5, $(\mathrm{Ann}(i_1a_1 + i_2a_2) \cap I)B = (\mathrm{Ann}(i_1a_1) \cap \mathrm{Ann}(i_2a_2) \cap I)B$. We choose as above a non-ps-nilpotent element $i_3a_3 \in (\mathrm{Ann}(i_1a_1) \cap \mathrm{Ann}(i_2a_2) \cap I)B$, ($i_3 \in \mathrm{Ann}(i_1a_1) \cap \mathrm{Ann}(i_2a_2) \cap I$, $a_3 \in B$), with $\mathrm{Ann}(i_3a_3)$ as large as possible. If $\mathrm{Ann}(i_1a_1 + i_2a_2 + i_3a_3) = 0$, we stop. If not we proceed as above and get a chain $\mathrm{Ann}(i_1a_1) \supseteq (\mathrm{Ann}(i_1a_1) \cap \mathrm{Ann}(i_2a_2)) \supseteq \ldots \supseteq (\mathrm{Ann}(i_1a_1) \cap \mathrm{Ann}(i_2a_2) \cap \ldots \cap \mathrm{Ann}(i_ta_t)) \ldots$. We get, by Lemma 2.4, some $n \in \mathbb{Z}^+$ such that $\mathrm{Ann}(i_1a_1 + i_2a_2 \cdots + i_na_n) = \mathrm{Ann}(i_1a_1 + i_2a_2 + \cdots + i_{n+1}a_{n+1}) = \ldots$. Now $\mathrm{Ann}(i_1a_1 + \cdots + i_na_n) = \mathrm{Ann}(i_1a_1 + \cdots + i_{n+1}a_{n+1}) = \mathrm{Ann}(i_1a_1 + \cdots + i_na_n) \cap \mathrm{Ann}(i_{n+1}a_{n+1})$, giving thereby $\mathrm{Ann}(i_1a_1 + \cdots + i_na_n) \subseteq \mathrm{Ann}(i_{n+1}a_{n+1})$. But, by our choice, $i_{n+1}a_{n+1} \in (\mathrm{Ann}(i_1a_1 + \cdots + i_na_n) \cap I)B$ ($\subseteq (\mathrm{Ann}(i_{n+1}a_{n+1}) \cap I)B$), where $i_{n+1} \in \mathrm{Ann}(i_{n+1}a_{n+1}) \cap I$ with $i_{n+1}a_{n+1}$ non-ps-nilpotent such that $\mathrm{Ann}(i_{n+1}a_{n+1})$ is as large as possible. Also $i_{n+1} \in \mathrm{Ann}(i_1a_1 + \cdots + i_na_n)$ ($\subseteq \mathrm{Ann}(i_{n+1}a_{n+1})$) which implies $i_{n+1}i_{n+1}a_{n+1} = 0$. If $i_{n+1}i_{n+1}a_{n+1}$ is a ps-nilpotent, then $Si_{n+1}i_{n+1}a_{n+1} \neq 0$ such that $S^t i_{n+1}i_{n+1}a_{n+1} = 0$, for some proper left N-subset S of N and some $t(\geq 2) \in \mathbb{Z}^+$. We get, by taking $L = Si_{n+1}$, a proper left N-subset of N, that $i_{n+1}a_{n+1}$ is a ps-nilpotent, which is not true. Hence i_{n+1} $(=0) \in \mathrm{Ann}(i_1a_1 + \cdots + i_na_n) \cap I \Rightarrow \mathrm{Ann}(i_1a_1 + \cdots + i_na_n) = 0$, as $I \subseteq_{eN} N$ or, $\mathrm{Ann}(e) = 0$ where $e = i_1a_1 + \cdots + i_na_n$. ◊

For $e \in E$, with $\mathrm{Ann}(e) = 0$, we get easily the following:

Lemma 2.7. (i) *An ideal B (N-subgroup) of ${}_NN$ is maximal if and only if Be is a maximal ideal (N-subgroup) of Ne.*

(ii) $J(N)e = J(Ne)$.

The following lemma is easy to see.

Lemma 2.8. *If S is an N-subgroup (ideal) of E, then $\overline{S}$ is also an N-subgroup (ideal) of E.*

We now consider an N-group of the type $E = \bigoplus_{i=1}^{n} Ne_i$, where each $\mathrm{Ann}(e_i) = 0$ (as observed in Lemma 2.6).

Lemma 2.9. *The radical $J(\bigoplus_{i=1}^{n} Ne_i)$ of the N-group $\bigoplus_{i=1}^{n} Ne_i$ contains all the N_u-nil N-subgroups of $\bigoplus_{i=1}^{n} Ne_i$.*

Proof. First we prove the result when $E = Ne$ with $\mathrm{Ann}(e) = 0$. Let Ce be an N_u-nil N-subgroup of Ne and suppose that Ce is not con-

N-subgroup of N with $l(b) = 0$ such that $Nb = N$.

Thus the above example is sufficient to explain the point that the converse of the above theorem is not true.

Moreover openness of Qe and E-boundedness of N lead us to the following expected results, mainly on openness of $J(E)$ together with discrete and finiteness of E.

Theorem 3.1.5. *If N is a E-bounded and E is with fully radical character, then $J(E)$ is open.*

Proof. As Qe is open and N is E-bounded, there exists an open subset V of E such that $NV \subseteq Qe$. Thus we can write $NV = \{qe \mid \text{for some } q \in Q\}$. Now, $V_1 = \{q \in Q \mid qe \in NV\}$ $(\subseteq Q)$. And $NV = V_1 e$, as $\text{Ann}(e) = 0$. Now $N(V_1 e) = N(NV) = N^2 V \subseteq NV = V_1 e$, which gives $NV_1 \subseteq V_1$, as $\text{Ann}(e) = 0$. So $NV_1 = V_1$. Moreover, for each $x \in V_1$, Nx is a quasi-regular left N subgroup of N, and as what have been stated above in case of A, we get $Nx \subseteq J(N)$ and hence $V_1 \subseteq J(N)$. Thus, we have $V = 1.\ V \subseteq NV = V_1 e \subseteq J(N)e \subseteq J(N)E = J(E)$. If $y \in J(E)$, $z \in V$, then $y + (-z) + V$ (an open subset of E containing $y) \subseteq J(E)$. Hence $J(E)$ is open. $\Diamond$

Corollary 3.1.6. *If N is E-bounded, E is with fully radical character and $J(E) = 0$, then E is discrete.*

The following example reveals that the vanishing of radical is essential for the discreteness of N-group E when N is E-bounded.

The topological near-ring $N(= \mathbb{Z}_8)$ [19 (127) p. 344] w.r.t. the topology $T(= \{\emptyset, N, \{0,4\}, \{1,5\}, \{2,6\}, \{3,7\}, \{0,1,4,5\}, \{0,3,4,7\}, \{0,2,4,6\}, \{1,2,5,6\}, \{2,3,6,7\}, \{1,3,5,7\}, \{0,2,3,4,6,7\}, \{0,1,3,4,5,7\}, \{0,1,2,4,5,6\}, \{1,2,3,5,6,7\}\})$ is N-bounded, $J(N) = \{0,4\}$, $l(e) = 0$, where $e = 1,2,3,5,6,7$ and N is not discrete.

Corollary 3.1.7. *If E is compact having fully radical character, N is E-bounded and $J(E) = 0$, then E is finite.*

We now note the following

Note. (i) Suppose $C^n D' \subseteq Q$, for some left N-subset C of N, $n \in \in \mathbb{Z}^+$ and $D' \subseteq N$. If $C^n D' \not\subseteq B$, for some left ideal B of N maximal as N-subgroup, then $Nc_1 c_2 \ldots c_n d' + B = N$ for some $c_1 c_2 \ldots c_n d'$ $(\notin B) \in C^n D'$, where $c_1, c_2, \ldots, c_n \in C$ and $d' \in D'$ giving thereby $nc_1 c_2 \ldots c_n d' + b = 1$, where some $n \in N$ and $b \in B$. And (C, being a left N-subset), $(nc_1)c_2 \ldots c_n d' \in C^n D'(\subseteq Q)$, we get $1 \in B$, a contradiction. Hence $C^n D' \subseteq J(N)$.

(ii) Suppose $n > 1$ and $C^{n-1} D' \not\subseteq J(N)$. Now as Note (i) $nc + + b = 1$, for some $n \in N$, $b \in B$ and $c(\notin B) \in C^{n-1} D'$. Again for any $c_1c_2\dots c_{n-1}d' \in C^{n-1}D'$, where $c_1, c_2, \dots c_{n-1} \in C$, $d' \in D$ we get $c_1c_2\dots c_{n-1}d' = c_1c_2\dots c_{n-1}d'(nc+b) - c_1c_2\dots c_{n-1}d'nc +$ $+ c_1c_2\dots c_{n-1}d'nc$ $(\in B)$, as C, a left N-subset giving thereby from Note (i) $c_1c_2\dots c_{n-1}d'nc(\in C^nD' \subseteq J(N)) \in B$ and B, a left ideal. Hence $C^{n-1}D' \subseteq B$, a contradiction. Therefore by induction, we get $CD' \subseteq J(N)$ and by Lemma 2.7 (ii) we get $CD \subseteq J(Ne)$.

(iii) When $C^nD \subseteq Qe$, we get $C^nD' \subseteq Q$ as $\text{Ann}(e) = 0$. By what we have got $CD \subseteq J(Ne)$.

Hence we get the following theorem establishing the link with the topologically nilpotent notion of a subset of $\bigoplus_{i=1}^{n} Ne_i$ and that of the radical of the N-group as follows:

Theorem 3.1.8. *If* $\bigoplus_{i=1}^{n} D_i\big(= \bigoplus_{i=1}^{n} D'_ie_i\big)$ *is a topologically nilpotent subset of* $\bigoplus_{i=1}^{n} Ne_i$*, then* $\bigoplus_{i=1}^{n} C_iD_i \subseteq J\big(\bigoplus_{i=1}^{n} Ne_i\big)$*, for some left N-subset* $C_1 \cup C_2 \cup \dots \cup C_n$ *of* N.

On the other hand, the squeezed character of each of the Qe_i in Ne_i with respect to the given topology, leads us to the closeness of what has been stated above regarding the direct sum of the group sum of ideals related to quasi-regular left ideal of N, when zero is the only element that kills the e_i's.

Theorem 3.1.9. *If* $\text{Ann}(e_i) = 0$ *and each* Qe_i *is closed in* Ne_i *for each* i*, then* $\bigoplus_{i=1}^{n} Se_i$ *is a closed ideal of* $\bigoplus_{i=1}^{n} Ne_i$.

Proof. As above the result for one component is sufficient. Now, by the proof of Th. 3.1.2. $Se \subseteq Qe$. So $\overline{Se} \subseteq Qe$, as Qe is closed. Again we have by the proof of Th. 3.1.2. $\overline{Se} \subseteq Se$ and hence Se is closed. ◇

Theorem 3.1.10. *As in case of* Th. 3.1.9 *if* $\bigoplus_{i=1}^{n} I_ie_i$ *is the unique maximal N-subgroup of* $\bigoplus_{i=1}^{n} Ne_i$*, then* $\bigoplus_{i=1}^{n} I_ie_i$ *is closed and each of* $\bigoplus_{i=1}^{n} Ae_i$ *and* $J\big(\bigoplus_{i=1}^{n} Ne_i\big)$ *is* $\bigoplus_{i=1}^{n} I_ie_i$.

Proof. Without loss of generality we see that, by hypothesis, Ie is the unique maximal N-subgroup of Ne and Qe is closed, so $Ae = Ie \subseteq$ $\subseteq \overline{Ae} \subseteq Qe$. Again Qe is a proper subset of Ne, if not $Q = N$, as $\text{Ann}(e) = 0$, which is not true as $1 \notin Q$ and hence by Lemma 2.8, we get Ie as closed. Again, for any left ideal B that is maximal as

N-subgroup of N, we have by Lemma 2.7 (i), Be, an ideal which is a maximal as N-subgroup. Thus, by uniqueness of Ie, we get $Be = Ie$. Hence $J(Ne) = Ie$. Thus $\bigoplus_{i=1}^{n} I_i e_i$ is closed and each of $\bigoplus_{i=1}^{n} Ae_i$ and $J(\bigoplus_{i=1}^{n} Ne_i)$ is $\bigoplus_{i=1}^{n} I_i e_i$. ◊

The notion of N_u-nilpotent element in N-group $\bigoplus_{i=1}^{n} Ne_i$ gives the following results, some of which are analogous to those obtained above. The results obtained here are on the assumption that each Q_i' is an open proper N-subset of Ne_i with $\mathrm{Ann}(e_i) = 0$.

Theorem 3.1.11. *If $\bigoplus_{i=1}^{n} B_i e_i$ is a maximal ideal of $\bigoplus_{i=1}^{n} Ne_i$, then $\bigoplus_{i=1}^{n} B_i e_i$ is closed.*

Proof. Suppose $\overline{Be} = Ne$. Since Q' is open, so is $-Q' + e$ and there is an element q' $(= qe) \in Q'$ and b' $(= be) \in Be$ such that $-q' + e = b'$ which gives $-q+1 = b$, as $\mathrm{Ann}(e) = 0$. Now, as $q' \in Q'$, we have $N_u^t q' = 0$ but $N_u^t \neq 0$, for some t(least) $\in \mathbb{Z}^+$ and hence $N_u^t q = 0$, as $\mathrm{Ann}(e) = 0$. Now for any $n_1 n_2 \dots n_t \in N_u^t$ we get $n_1 n_2 \dots n_t = n_1 n_2 \dots n_t(b + q) - n_1 n_2 \dots n_t q \in B$, as $N_u^t q = 0$ and B is a left ideal and thus $N_u^t \subseteq B$. Again $n_1 n_2 \dots n_{t-1} = n_1 n_2 \dots n_{t-1}(b + q) - n_1 n_2 \dots n_{t-1} q + n_1 n_2 \dots n_{t-1} q \in B$, as B is a left ideal and $n_1 n_2 \dots n_{t-1} q \in N_u^t$ $(\subseteq B)$ giving thereby $N_u^{t-1} \subseteq B$. So, by induction, $N_u \subseteq B$ which is a contradiction, as by Lemma 2.7 (i), B is maximal. But by Lemma 2.8, $\overline{Be}$ is closed. Thus it follows that $\bigoplus_{i=1}^{n} B_i e_i$ is closed. ◊

Corollary 3.1.12. *$J(\bigoplus_{i=1}^{n} Ne_i)$ is closed.*

Theorem 3.1.13. *If N is E-bounded where $E = \bigoplus_{i=1}^{n} Ne_i$, then $J(\bigoplus_{i=1}^{n} Ne_i)$ is open.*

Proof. As Q' is open subset of Ne containing 0 and N is E-bounded, so there exists an open subset V of Ne such that $NV \subseteq Q'$. Now, for each $x \in V$, Nx is an N_u-nil N-subgroup of Ne and hence by Lemma 2.9, $Nx \subseteq J(Ne)$, for each $x \in V$ and thus $V \subseteq J(Ne)$. Now for $y \in V$ and $x \in J(Ne)$, we get $x + (-y) + V$ as an open set containing x that is contained in $J(Ne)$. Hence $J(Ne)$ is open. ◊

Corollary 3.1.14. *If N is E-bounded where $E = \bigoplus_{i=1}^{n} Ne_i$, then $\bigoplus_{i=1}^{n} Ne_i$ is not connected.*

Corollary 3.1.15. *If N is E-bounded where $E = \bigoplus_{i=1}^{n} Ne_i$ and $J(\bigoplus_{i=1}^{n} Ne_i)$ is the zero ideal, then $\bigoplus_{i=1}^{n} Ne_i$ is discrete.*

3.2. Component of zero. For the remaining of the paper N will be a near-ring without unity. However we shall restrict our attention to the case where the topology on E is locally compact. By Th. (2) of [10], we see that this requirement forces $(E,+)$ to be topological group i.e. the function $f(x,y) = x - y$ is continuous on $E \times E$ to E.

Let Γ denote the component of 0 in E. We now note the following

Note. (i) The set Γ is a closed ideal in E and if it consists of 0 alone, then E is totally disconnected.

(ii) When E is totally disconnected and locally compact, it can be seen that it is o-dimensional (as Remark 1 on page 22 of [13]).

We attempt the following theorem, which is obvious from Section 4 of [14].

Theorem 3.2.1. *If E is locally compact, disconnected, contains no proper non-zero closed ideals and satisfies the* dcc *on closed subgroups of $(E,+)$, then E is discrete.*

Corollary 3.2.2. *If E is compact, disconnected, contains no proper non-zero closed ideals and satisfies the* dcc *on closed subgroups of $(E,+)$, then E is finite.* ◇

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