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Invariance of essential spectra for generalized Schrödinger operators

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Abstract

We give a new sufficient condition for the invariance of the essential spectrum of $-\Delta + \mu$, where μ is a signed Radon measure. This condition is formulated in term of the behavior of the ratio of the $|\mu|$-measure of compact subsets by their 2-order capacity at infinity. Our method recovers a large class of measures.

Key words: Schrödinger operator, Spectrum, Measure, Capacity, Quadratic form.

1 Introduction

Spectral properties of Schrödinger operators $H_V := -\Delta + V$, where V is a real potential were extensively investigated. In the last years the case where V is replaced by a signed measure, gained much more interest. Such operators are called generalized Schrödinger operators. Many authors discussed spectral properties and dynamics of such operators. We here cite: [AGHKH88, BM90, Bra01, BEKŠ94, Maz85, Sto94]. This list does not exhaust the existing literature about the subject and much more relevant references can be found

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in the citations.
Generalized Schrödinger operators physically occur to describe interactions between a 'quantum mechanical particle' and potentials concentrated on surfaces (sphere, hyperplane), or more generally on subsets of positive capacity like d-sets (cf [JW84] for the notion of a d-set) or on curves. They also occur as generators of a Brownian motion on $\mathbb{R}^d$ superposed with a killing or jumping process on some subset of $\mathbb{R}^d$.
The rigorous construction of such operators, as operators related to quadratic forms or as self-adjoint realizations of some starting symmetric operator, was extensively discussed. For the reader who is interested to such problems we cite [AGHKH88, Bra95, Kat95, Maz85, Man02].
In this paper, we shall be concerned with essential spectra of operators of the type

$$H_\mu := -\Delta + \mu, \tag{1}$$

where μ is a suitable signed measure and H_μ is constructed via quadratic forms. We shall be mainly interested with giving sufficient conditions that guarantee the invariance of the essential spectrum of H_μ.

Our main result, Theorem 3.1 recovers the situation where the measure μ is absolutely continuous w.r.t. Lebesgue measure and also the known results for singular perturbations. The same question was already discussed by many authors [BEKŠ94, Bra01, Sto94, OS98, Sch86] and [Hem84] in an abstract setting.

For our purpose, we shall use a potential-theoretical method. With our method, we shall not demand from the measure $|\mu|$ to be simultaneously bounded w.r.t. the energy form (or $-\Delta$-bounded in the absolutely continuous case) and to vanish at infinity, (cf. [Sch86, p.84], [HS96, OS98] in the absolutely continuous case and [Bra01] in the singular case). Instead, we shall demand from the ratio

$$\frac{|\mu|(K)}{\mathrm{Cap}_2(K)}, \tag{2}$$

where Cap_2 is the 2-order capacity, to vanish at infinity, in a sense to be precised later. Proposition 3.1 shows that this condition is weaker than the first mentioned one.
Let us mention that a similar method was already adopted in [KS86, Maz85], getting partial informations about the essential spectrum of $-\Delta - \mu$ where μ is a positive measure.

We stress that our method and techniques still work if the Laplace operator is replaced by any second-order elliptic positive operator.

The paper is organized as follows: First we give the useful tools and preliminary results for solving the problem. Preparing the main result, we show the invariance of the essential spectrum if μ has compact support. Under some conditions, we approximate H_μ (in the norm resolvent sense) by generalized Schrödinger operators whose potentials (measures)

have compact support.
Then we give the main result about the invariance of the essential spectrum and discuss it. At the end, we give a criteria for the validity of the assumption adopted in Theorem 3.1 (assumption (38)) and show that, in some cases, it is equivalent to the fact that the measure μ vanishes at infinity.

2 Preliminaries

We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space. For a positive Radon measure μ, and a subset $\Omega \subset \mathbb{R}^d$, we denote by $L^2(\Omega, \mu)$ the space of measurable complex-valued (equivalence classes of) functions defined on Ω and which are square-integrable with respect to μ. If $\Omega = \mathbb{R}^d$, $L^2(\Omega, \mu)$ will be denoted simply by $L^2(\mu)$ and if μ is the Lebesgue measure on $\mathbb{R}^d$, which we denote by dx, the latter space will be denoted by L^2. We use the abbreviation a.e. to mean dx a.e. The spaces $W^{r,2}(\mathbb{R}^d), r > 0$ are the classical Sobolev spaces and $C_0^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable functions with compact support in $\mathbb{R}^d$.
For every $\alpha > 0$, we denote by H_α the operator $H_\alpha := -\Delta + \alpha$ defined in L^2, by g_α its Green function and by $\mathcal{E}_\alpha$ the form associated to H_α:

$$D(\mathcal{E}_\alpha) = W^{1,2}(\mathbb{R}^d),\ \mathcal{E}_\alpha[f] = \int_{\mathbb{R}^d} |\nabla f|^2 dx + \alpha \int_{\mathbb{R}^d} |f|^2 dx\,.$$

The energy form is

$$\mathcal{E},\ D(\mathcal{E}) = W^{1,2}(\mathbb{R}^d),\ \mathcal{E}[f] = \int_{\mathbb{R}^d} |\nabla f|^2\, dx.$$

For every $r > 0$, we denote by $V_r := (-\Delta + 1)^{-\frac{r}{2}}$ and by G_r its kernel. We also define the operator

$$V_r^\mu := L^2 \to L^2(\mu),\ V_r^\mu f = \int_{\mathbb{R}^d} G_r(\cdot, y) f(y)\, dy. \tag{3}$$

The r-capacity, which we denote by Cap_r, is defined as follows [AH96, p.20-25]: For a set $E \subset \mathbb{R}^d$, we define

$$\mathcal{L}_E := \{f : f \in L^2,\ V_r f \geq 1 \text{ on } E\} \tag{4}$$

and

$$\mathrm{Cap}_r(E) := \begin{cases} \inf_{f\in\mathcal{L}_E} \int_{\mathbb{R}^d} |f|^2\, dx & \text{if} \quad \mathcal{L}_E \neq \emptyset \\ +\infty & \text{if} \quad \mathcal{L}_E = \emptyset \end{cases}$$

The 1-capacity will be called simply the capacity. For $\alpha > 0$ and $r \geq 1$ we shall occasionally make use of the capacity $\mathrm{Cap}_r^{(\alpha)}$ obtained when changing V_r by $V_r^{(\alpha)} := (-\Delta + \alpha)^{-\frac{r}{2}}$.
We say that a property holds quasi-everywhere (q.e. for short) if it holds up to a set having zero capacity. A q.e. defined function f on $\mathbb{R}^d$ is said to be quasi continuous if for every $\epsilon > 0$, there is an open subset Ω such that $\mathrm{cap}(\Omega) < \epsilon$ and $f_{|\Omega^c}$ is continuous. According to [AH96, 156], every $f \in W^{1,2}(\mathbb{R}^d)$ can be modified so as to become quasi continuous. In

what follows we shall assume implicitly that elements from $W^{1,2}(\mathbb{R}^d)$ have been modified in this sense.

By $\mathcal{S}^+$, we designate the set of positive Radon measures charging no sets of zero capacity.
For a given $\mu \in \mathcal{S}^+$ and $\alpha > 0$, we define the operators

$$K_\alpha^\mu := L^2(\mu) \to L^2(\mu),\ f \mapsto K_\alpha^\mu f := \int_{\mathbb{R}^d} g_\alpha(.,y) f(y)\, d\mu(y), \tag{5}$$

and

$$I_{\mu,\alpha} := (W^{1,2}(\mathbb{R}^d), \mathcal{E}_\alpha^{\frac{1}{2}}) \to L^2(\mu),\ f \mapsto f\ \mu - \text{a.e.} \tag{6}$$

For $\alpha = 1$, $I_{\mu,\alpha}$ will be denoted by I_μ. Observe that since $\mu \in \mathcal{S}^+$, $I_{\mu,\alpha}$ is well defined and is closed.

By [BA04, Theorem 3.1], the operator K_α^μ is bounded if and only if the following inequality holds true:

$$\int_{\mathbb{R}^d} |f|^2\, d\mu \le C_\alpha^\mu \mathcal{E}_\alpha[f],\ \forall\ f \in D(\mathcal{E}), \tag{7}$$

where C_α^μ is a positive constant. Moreover the constant $\|K_\alpha^\mu\|_{L^2(\mu)}$ is the optimal one in the latter inequality and is equivalent to

$$\sup\{\frac{\mu(K)}{\mathrm{Cap}_1(K)}:\ K \text{ compact}\}. \tag{8}$$

Here the ratio is understood to be equal to zero if $\mathrm{Cap}_1(K) = 0$.
In other words the boundedness of K_α^μ is equivalent to the boundedness of the 'embedding' $I_{\mu,\alpha}$. For this reason, we shall call measures from $\mathcal{S}^+$ satisfying the latter assumption $\mathcal{E}$-bounded measures and shall denote them by $\mathcal{B}^+$. Those measures $\mu \in \mathcal{B}^+$ such that

$$C_\mu := \inf_{\alpha>0} \|K_\alpha^\mu\|_{L^2(\mu)} = \lim_{\alpha\to\infty} \|K_\alpha^\mu\|_{L^2(\mu)} < 1. \tag{9}$$

will be denoted by $\mathcal{B}_0^+$.
We also recall that extending the identity (cf. [BA04, (25)]) to complex-valued functions we get that, if $\mu \in \mathcal{B}^+$, then for every $f \in L^2(\mu)$, $K_\alpha^\mu f \in D(\mathcal{E})$ and

$$\mathcal{E}_\alpha\big(K_\alpha^\mu f, g\big) = \int_{\mathbb{R}^d} f\overline{g}\, d\mu,\ \forall g \in D(\mathcal{E}). \tag{10}$$

Let $\mu = \mu^+ - \mu^-$, where $\mu^+ \in \mathcal{S}^+$ and $\mu^- \in \mathcal{B}_0^+$. Define the form $\mathcal{E}^\mu$ by

$$D(\mathcal{E}^\mu) = \big\{f : f \in W^{1,2}(\mathbb{R}^d),\ \int_{\mathbb{R}^d} |f|^2\, d\mu^+ < \infty\big\},\ \mathcal{E}^\mu[f] = \mathcal{E}[f] + \int_{\mathbb{R}^d} |f|^2\, d\mu \tag{11}$$

By the KLMN theorem, the form $\mathcal{E}^\mu$ is closed and lower semi-bounded with lower bound equals to $-\alpha\|K_\alpha^\mu\|_{L^2(\mu)}$, for some $\alpha > 0$. We shall denote by H_μ the self-adjoint operator associated to $\mathcal{E}^\mu$ via the representation theorem [Kat95]:

$$D(H_\mu) \subset D(\mathcal{E}^\mu), \mathcal{E}^\mu(f,g) = (H_\mu f, g),\ \forall f \in D(H_\mu) \text{ and } \forall g \in D(\mathcal{E}^\mu). \tag{12}$$

When $\mu = 0$, H_μ will be simply denoted by $H := -\Delta$. We set $\sigma_{\rm ess}(H_\mu)$ the essential spectrum of H_μ.
For every $\rho > 0$ we denote by B_ρ the Euclidean ball of $\mathbb{R}^d$ with center zero and radius ρ, μ_ρ the restriction of the measure μ to B_ρ and $\mu^\rho := \mu - \mu_\rho$.

In this stage, we give some auxiliary results related to the operator $(H_\mu+\alpha)^{-1}$, namely we show that it is an integral operator and give an explicit formula for the resolvent difference $(H_\mu+\alpha)^{-1} - (H_{\mu^+}+\alpha)^{-1}$. We begin with the simple:

Lemma 2.1. *Let $\alpha > 0$. Then the operator $(H_{\mu^+}+\alpha)^{-1}$ is an integral operator: there is a symmetric Borel function*

$$g_\alpha^{\mu^+} := \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty],$$

such that

$$(H_{\mu^+}+\alpha)^{-1} f = \int_{\mathbb{R}^d} g_\alpha^{\mu^+}(\cdot,y) f(y)\, dy,\ \forall f \in L^2. \tag{13}$$

Moreover $g_\alpha^{\mu^+}$ satisfies

$$g_\alpha^{\mu^+}(x,y) = g_\alpha(x,y) - \int_{\mathbb{R}^d} g_\alpha(x,z) g_\alpha^{\mu^+}(y,z)\, d\mu^+(z). \tag{14}$$

Proof. By [BM90], the exponential operator $\exp(-tH_{\mu^+})$ has a continuous symmetric kernel p_t^+, for every $t > 0$ and it satisfies

$$p_t^+(x,y) \le c\mathrm{e}^{\beta t} p_{2t}(x,y),\ \forall\ x,y, \tag{15}$$

where c, β are positive constant and p_t is the kernel of $\exp(-tH)$.
Making use of the inversion formula

$$(H_{\mu^+}+\alpha)^{-1} = \int_0^\infty \mathrm{e}^{-\alpha t} \exp(-tH_{\mu^+})\, dt,\ \forall\ \alpha > 0, \tag{16}$$

together with the estimate (15) we conclude that $(H_{\mu^+}+\alpha)^{-1}$ has a kernel given by

$$g_\alpha^{\mu^+}(x,y) = \int_0^\infty \mathrm{e}^{-\alpha t} p_t^+(x,y)\, dt \le \frac{c}{2} g_{\frac{\alpha-\beta}{2}}(x,y),\ \forall\ x \neq y \text{ if } d \ge 2,$$

and every x, y if $d = 1$. The rest of the proof is easy to do. $\square$

For $\alpha > 0$, set $K_\alpha^+ := (H_{\mu^+} + \alpha)^{-1}$,

$$I_\alpha^+ := \big(D(\mathcal{E}^{\mu^+}), ((\mathcal{E}_\alpha^{\mu^+})^{1/2})\big) \to L^2(\mu^-),\ f \mapsto f\ \mu - a.e. \tag{17}$$

and

$$K_\alpha^- := L^2(\mu^-) \to L^2(\mu^-),\ f \mapsto \int_{\mathbb{R}^d} g_\alpha^{\mu^+}(.,y)f(y)\,d\mu^-(y). \tag{18}$$

The analysis of spectra of perturbed Schrödinger operators depends sometimes on an adequate formula for its resolvent. The following is inspired from [BEKŠ94, Lemma 2.3] and partially generalizes it.

Lemma 2.2. *Let α be such that $\|K_\alpha^{\mu^-}\|_{L^2(\mu^-)} < 1$. Then*

$$(H_\mu + \alpha)^{-1} - K_\alpha^+ = (I_\alpha^+ K_\alpha^+)^*(I - K_\alpha^-)^{-1}(I_\alpha^+ K_\alpha^+). \tag{19}$$

From the latter lemma together with Lemma 2.1 we derive that for big α, the operator $(H_\mu + \alpha)^{-1}$ is an integral operator. We shall denote by g_α^μ its kernel.
The identity (19) implies that

$$0 \le g_\alpha^{\mu^+}(.,y) \le g_\alpha^\mu(.,y) \text{ q.e.} \tag{20}$$

Proof. Let α be such that $\|K_\alpha^{\mu^-}\|_{L^2(\mu^-)} < 1$ be fixed. Then

$$\mathcal{E}_\alpha^\mu[f] \ge \big(1 - \|K_\alpha^{\mu^-}\|_{L^2(\mu^-)}\big)\mathcal{E}_\alpha[f],\ \forall\ f \in D(\mathcal{E}^\mu),$$

which implies that α lies in the resolvent set of H_μ. On the other hand since by (14), $g_\alpha^{\mu^+}(.,y) \le g_\alpha(.,y)$ q.e. we obtain $\|K_\alpha^-\|_{L^2(\mu^-)} \le \|K_\alpha^{\mu^-}\|_{L^2(\mu^-)} < 1$, so that $I - K_\alpha^-$ is invertible. Now since $\mu^- \in \mathcal{B}_0^+$, we conclude that I_α^+ is bounded. Hence the operator on the left hand side of (19) is bounded on L^2 with range $D(\mathcal{E}^{\mu^+}) = D(\mathcal{E}^\mu)$. Thereby to prove the lemma it suffices to prove

$$\mathcal{E}_\alpha^\mu(K_\alpha^+ f, g) + \mathcal{E}_\alpha^\mu((I_\alpha^+ K_\alpha^+)^*(I - K_\alpha^-)^{-1}(I_\alpha^+ K_\alpha^+)f, g) = \int_{\mathbb{R}^d} f\overline{g}\,dx\,\forall\ f \in L^2, g \in D(\mathcal{E}^\mu). \tag{21}$$

Let f and g be such functions. Without loss of generality, we assume that they are real-valued. A direct computation yields

$$\mathcal{E}_\alpha^\mu(K_\alpha^+ f, g) = \int_{\mathbb{R}^d} fg\,dx - \int_{\mathbb{R}^d} K_\alpha^+ fg\,d\mu^-. \tag{22}$$

and

$$(I_\alpha^+ K_\alpha^+)^* f = K_\alpha^- f \text{ a.e.} \tag{23}$$

Now since both $(I_\alpha^+ K_\alpha^+)^* f$ and $K_\alpha^- f$ lie in $W^{1,2}(\mathbb{R}^d)$, they are quasi continuous and whence they are equal q.e. [AH96, p.157]. Further, from $\mu \in \mathcal{S}^+$ we conclude that $(I_\alpha^+ K_\alpha^+)^* f = K_\alpha^- f$ μ a.e. Making use of this fact and of the identity (10), we get

$$\begin{aligned} \mathcal{E}_\alpha^\mu((I_\alpha^+ K_\alpha^+)^*(I - K_\alpha^-)^{-1}(I_\alpha^+ K_\alpha^+)f, g) \quad &= \mathcal{E}_\alpha^{\mu^+}((I_\alpha^+ K_\alpha^+)^*(I - K_\alpha^-)^{-1}(I_\alpha^+ K_\alpha^+)f, g) \\ &- \int_{\mathbb{R}^d} (I_\alpha^+ K_\alpha^+)^*(I - K_\alpha^-)^{-1}(I_\alpha^+ K_\alpha^+)fg\, d\mu^- \\ &= \int_{\mathbb{R}^d} (I - K_\alpha^-)^{-1}(I_\alpha^+ K_\alpha^+)fg\, d\mu^- \\ &- \int_{\mathbb{R}^d} K_\alpha^-(I - K_\alpha^-)^{-1}(I_\alpha^+ K_\alpha^+)fg\, d\mu^- \\ &= \int_{\mathbb{R}^d} I_\alpha^+ K_\alpha^+ fg\, d\mu^- = \int_{\mathbb{R}^d} K_\alpha^+ fg\, d\mu^-, \qquad (24) \end{aligned}$$

where the latter identity is justified by the fact that $I_\alpha^+ K_\alpha^+ f = K_\alpha^+ f$ everywhere since $K_\alpha^+ f$ is defined everywhere. Now putting (22) and (24) together we get what was to be proved. □

3 The essential spectrum of H_μ

We shall use known geometric characterization of the essential spectrum [HS96, Theorem 10.6 p.102] to prove the invariance of $\sigma_{\text{ess}}(H_\mu)$, when μ has compact support.

Lemma 3.1. *Let $\mu = \mu^+ - \mu^- \in \mathcal{S}^+ - \mathcal{B}_0^+$. Assume that μ has compact support. Then $\sigma_{\text{ess}}(H_\mu) = \sigma_{\text{ess}}(H) = [0, \infty)$.*

We here emphasize that unlike [BEKŠ94, Theorem 3.1], we do not suppose that the measure $|\mu| \in \mathcal{B}^+$.

Proof. For the proof we follow the idea of Hislop-Sigal [HS96, p.137]. The key is to apply [HS96, Theorem 10.6]. Next we shall prove that all assumptions required by this Theorem are fulfilled.
Let $\alpha > 0$. Since the operator $(H + \alpha)^{-1}$ is locally compact, i.e., for every open bounded subset $\Omega \subset \mathbb{R}^d$, the operator $\chi_\Omega(H + \alpha)^{-1}$ is compact, it follows from the variational formulation of the min-max principle and from the fact that $D(H_{\mu^+})$ is a core for $D(\mathcal{E}^{\mu^+})$, that $(H_{\mu^+} + \alpha)^{-1}$ is also locally compact. Hence using formula (19), we conclude that $(H_\mu + \alpha)^{-1}$ is locally compact as well.
In this step we prove that assumption (10.7) of [HS96, Theorem 10.6] (see identity (29)) is satisfied.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$ such that

$$0 \le \phi \le 1,\ \phi = 1 \text{ on } B_1,\ \text{supp}(\phi) \subset B_2.$$

Let (ϕ_n) be the sequence defined by $\phi_n(x) := \phi(x/n)$. Set

$$[H_\mu, \phi_n] := H_\mu \phi_n - \phi_n H_\mu. \tag{25}$$

Let $f \in D(H_\mu)$ be such that $\phi_n f \in D(H_\mu)$, $g \in D(\mathcal{E}^\mu)$ and α big. Then

$$\begin{aligned}\big([H_\mu, \phi_n]f, g\big) &= \big(H_\mu \phi_n f, g\big) - \big(H_\mu f, \phi_n g\big) \\ &= \mathcal{E}^\mu\big(\phi_n f, g\big) - \mathcal{E}^\mu\big(f, \phi_n g\big) &(26)\\ &= \int_{\mathbb{R}^d} (-f\Delta\phi_n - 2\nabla f \nabla \phi_n)\overline{g}\, dx. &(27)\end{aligned}$$

Hence

$$[H_\mu, \phi_n]f = -f\Delta\phi_n - 2\nabla f \nabla\phi_n = -\frac{1}{n^2} f\Delta\phi - \frac{2}{n}\nabla f \nabla \phi. \tag{28}$$

From the latter identity we infer that the commutator extends to $D(H_\mu)$ and that

$$\begin{aligned}\|[H_\mu, \phi_n](H_\mu + \alpha)^{-1} f\|_{L^2} &\le \frac{1}{n^2}\|\Delta\phi\|_\infty \|(H_\mu + \alpha)^{-1}\| \|f\|_{L^2} + \\ &\quad \frac{1}{n}\|\nabla\phi\|_\infty \| |\nabla (H_\mu + \alpha)^{-1} f| \| \\ &\le \frac{1}{n^2} C_1 \|f\|_{L^2} + \frac{C'_\alpha}{n}\big(\mathcal{E}^\mu_\alpha\big[(H_\mu + \alpha)^{-1} f\big]\big)^{\frac{1}{2}} \\ &= \frac{1}{n^2} C_1 \|f\|_{L^2} + \frac{C'_\alpha}{n}\Big(\int_{\mathbb{R}^d} f (H_\mu + \alpha)^{-1} f\, dx\Big)^{\frac{1}{2}} \le \frac{C}{n}\|f\|_{L^2}.\end{aligned}$$

Thereby

$$\lim_{n\to\infty} \|[H_\mu, \phi_n](H_\mu + \alpha)^{-1}\| = 0. \tag{29}$$

From [HS96, Theorem 10.6] we learn that $\lambda \in \sigma_{\text{ess}}(H_\mu)$ if and only if there is a sequence $(f_n) \subset D(H_\mu)$ such that

$$\|f_n\| = 1,\ \text{supp}(f_n) \subset B_n^c \text{ and } \lim_{n\to\infty} \|H_\mu f_n - \lambda f_n\| = 0. \tag{30}$$

Let $\lambda \in \sigma_{\text{ess}}(H_\mu)$. Take a sequence (f_n) as in (30). Observe that from the definition of H_μ, we have $H_\mu f = -\Delta f + f\mu$ in the sens of distributions for all $f \in D(H_\mu)$. Hence since μ has compact support, we conclude that $H_\mu f_n = -\Delta f_n \in L^2$ for large n yielding $f_n \in D(H)$ for large n and

$$\|H_\mu f_n - \lambda f_n\| = \|H f_n - \lambda f_n\| \to 0, \text{as } n \to \infty, \tag{31}$$

which implies $\sigma_{\text{ess}}(H_\mu) \subset \sigma_{\text{ess}}(H)$. The reversed inclusion can be proved exactly in the same manner, so we omit its proof. □

Our next purpose is to approximate the operator H_μ (in the norm resolvent sense) by a sequence (H_{ν_n}) such that the ν_n's have compact support. For this aim, we introduce the following operators:

Let $\mu \in \mathcal{S}^+ - \mathcal{B}_0^+$ and α big enough. For every $n \in \mathbb{N}$, set

$$K_\alpha^{\pm,n} := L^2((\mu^\pm)^n) \to L^2,\ f \mapsto \int_{\mathbb{R}^d} g_\alpha^\mu(\cdot, y) f(y)\, d(\mu^\pm)^n.$$

and

$$I_\alpha^{\pm,n} := \big(D(\mathcal{E}^{\mu_n}), (\mathcal{E}_\alpha^{\mu_n})^{1/2}\big) \to L^2((\mu^\pm)^n),\ f \mapsto f.$$

We claim that the operators $K_\alpha^{\pm,n}$ and $I_\alpha^{-,n}$ are bounded. Indeed: The boundedness of $I_\alpha^{-,n}$ follows from the fact that $\mu \in \mathcal{B}_0^+$.
Now set $\mathcal{G}_\alpha^\mu$ the kernel of $(H_\mu + \alpha)^{-\frac{1}{2}}$, and $\mathcal{K}_\alpha^{\pm,n}$ the operator

$$\mathcal{K}_\alpha^{\pm,n} := L^2 \to L^2((\mu^\pm)^n),\ f \mapsto \int_{\mathbb{R}^d} \mathcal{G}_\alpha^\mu(x, y) f(y)\, dy. \tag{32}$$

From $\mu^- \in \mathcal{B}_0^+$, we derive

$$\int_{\mathbb{R}^d} |f|^2\, d(\mu^\pm)^n \le (1 - \|K_\alpha^{\mu^-}\|_{L^2(\mu^-)})^{-1} \mathcal{E}_\alpha^\mu[f],\ \forall f \in D(\mathcal{E}^\mu), \tag{33}$$

which is equivalent to the boundedness of $\mathcal{K}_\alpha^{\pm,n}$. Hence their duals

$$\big(\mathcal{K}_\alpha^{\pm,n}\big)^* := L^2((\mu^\pm)^n) \to L^2,\ f \mapsto \int_{\mathbb{R}^d} \mathcal{G}_\alpha^\mu(x, y) f(y)\, d(\mu^\pm)^n(y). \tag{34}$$

are bounded and thereby

$$(H_\mu + \alpha)^{-\frac{1}{2}} \big(\mathcal{K}_\alpha^{\pm,n}\big)^* f = K_\alpha^{\pm,n} \tag{35}$$

are also bounded. Moreover

$$\sup_n \|K_\alpha^{\pm,n}\| \le \sup_n \|\mathcal{K}_\alpha^{\pm,n}\| < \infty, \tag{36}$$

by (33)-(35).

The following lemma has a central stage in the proof of the invariance of $\sigma_{ess}(H_\mu)$. It also has an independent interest since it gives a condition under which the norm resolvent convergence holds true.

Lemma 3.2. *Let n and α be as above. Set $\mu_n := 1_{B_n}\mu$, the restriction of the measure μ to the open Euclidean ball of radius n and center zero and $H_n := H_{\mu_n}$. Then*
i)

$$(H_\mu + \alpha)^{-1} - (H_n + \alpha)^{-1} = K_\alpha^{-,n} I_\alpha^{-,n} (H_n + \alpha)^{-1} - K_\alpha^{+,n} I_\alpha^{+,n} (H_n + \alpha)^{-1}. \tag{37}$$

ii) If moreover

$$\lim_{n\to\infty} \sup\Big\{\frac{|\mu|(K)}{\mathrm{Cap}_2(K)} : K \subset B_n^c,\ \text{compact}\Big\} = 0, \tag{38}$$

or equivalently

$$\lim_{n\to\infty} \|V_2^{|\mu|^n}\|_{L^2, L^2(|\mu|^n)} = 0, \tag{39}$$

then the norm resolvent convergence holds true:

$$\lim_{n\to\infty} \|(H_\mu + \alpha)^{-1} - (H_n + \alpha)^{-1}\| = 0. \tag{40}$$

The ratio appearing in (38) is understood to be equal to zero if $\mathrm{Cap}_2(K) = 0$. The equivalence between (38) and (39) follows from the equivalence between $\sup\big\{\frac{|\mu|(K)}{\mathrm{Cap}_2(K)} :\ K \subset B_n^c,\ \text{compact}\big\}$ and $\|V_2^{|\mu|^n}\|_{L^2, L^2(|\mu|^n)}$ (cf. [AH96, Theorem 7.2.1]).

Proof. i) We first prove that the right hand side of (37) is bounded on L^2. We have already observed that $K_\alpha^{\pm,n}$ and $I_\alpha^{-,n}$ are bounded. Thus the first operator in the right hand side of (37) is bounded. On the other hand $I_\alpha^{+,n}$ is closed, hence $I_\alpha^{+,n}(H_n+\alpha)^{-1}$ is closable with domain the whole space L^2, hence bounded and thereby the right hand side of (37) is bounded on L^2 as well. Now the rest of the proof is substantially similar to that of Lemma 2.2, so we omit it.

(ii) The key is to use formula (i). Fix $\alpha \geq 1$ such that

$$C_\alpha^{\mu^-} := \|K_\alpha^{\mu^-}\|_{L^2(\mu^-)} < 1.$$

Since by (36), $\sup_n \|K_\alpha^{\pm,n}\| < \infty$, proving (40) reduces to prove

$$\lim_{n\to\infty} \|I_\alpha^{\pm,n}(H_n + \alpha)^{-1}\| = 0. \tag{41}$$

Set

$$K_\alpha := (H+\alpha)^{-1}, \widetilde{K}_{\alpha,n} := (H + \mu_n^+ + \alpha)^{-1},$$

$\tilde{g}_{\alpha,n}$ the kernel of $\widetilde{K}_{\alpha,n}$ and

$$J_{\alpha,n} := \big(D(\mathcal{E}^{\mu_n^+}), (\mathcal{E}_\alpha^{\mu_n^+})^{1/2}\big) \to L^2(\mu_n^-),\ f \mapsto f.$$

Changing μ by μ_n in Lemma 2.2, we get

$$(H_n+\alpha)^{-1} = \widetilde{K}_{\alpha,n} + (J_{\alpha,n}\widetilde{K}_{\alpha,n})^*(I - \widetilde{K}_{\alpha,n}^-)^{-1}(J_{\alpha,n}\widetilde{K}_{\alpha,n}). \tag{42}$$

By the choice of α we have $\sup_n \|(I - \widetilde{K}_{\alpha,n}^{-})^{-1}\| \leq \big(1 - \|K_\alpha^{\mu^-}\|_{L^2(\mu^-)}\big)^{-1}$. On the other hand, from $\tilde{g}_{\alpha,n} \leq g_\alpha$ we obtain $\sup_n \|\widetilde{K}_{\alpha,n}\| \leq \|K_\alpha\|$, and a glance at the definition of $J_{\alpha,n}$ yields $\sup_n \|J_{\alpha,n}\| \leq \|K_\alpha^{\mu^-}\|_{L^2(\mu^-)}$.
Hence proving (41), reduces to prove

$$\lim_{n\to\infty} \|I_\alpha^{\pm,n}\widetilde{K}_{\alpha,n}\| = 0. \tag{43}$$

From the already established comparison of the relative kernels we have $\|I_\alpha^{\pm,n}\widetilde{K}_{\alpha,n}\| \leq \|I_\alpha^{\pm,n}K_{\alpha,n}\|$. Now the latter operator is given by

$$I_\alpha^{\pm,n}K_{\alpha,n} := L^2 \to L^2((\mu^\pm)^n),\ f \mapsto \int_{\mathbb{R}^d} g_\alpha(.,y)f(y)\,dy. \tag{44}$$

Whence by [AH96, Theorem 7.2.1], we achieve

$$\begin{aligned} \|I_\alpha^{\pm,n}K_{\alpha,n}\| \quad &\leq C\sup\{\frac{(\mu^\pm)^n(K)}{\mathrm{Cap}_2(K)} : K \text{ compact}\} \\ &\leq C\sup\{\frac{|\mu|(K)}{\mathrm{Cap}_2(K)} : K \text{ compact } K\subset B_n^c\} \\ \to 0 \text{ as } n\to\infty, & \end{aligned} \tag{45}$$

by assumption and the proof is finished. □

We are in a position now to state our main theorem:

Theorem 3.1. *Let $\mu = \mu^+ - \mu^- \in \mathcal{S}^+ - \mathcal{B}_0^+$. Suppose that assumption (38) is satisfied. Then $\sigma_{\mathrm{ess}}(H_\mu) = \sigma_{\mathrm{ess}}(H) = [0,\infty)$.*

Proof. Let $E_n(\lambda)$ (respectively $E(\lambda)$) be the spectral family related to H_n (respectively to H_μ). From the the norm resolvent convergence we derive (cf.[Kat95, p.362])

$$\lim_{n\to\infty} \|E(\lambda) - E_n(\lambda)\| = 0. \tag{46}$$

It follows that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \|E_n(\lambda+\epsilon) - E_n(\lambda-\epsilon) - (E(\lambda+\epsilon) - E(\lambda-\epsilon))\| = 0, \tag{47}$$

and thereby

$$\dim R\big(E_n(\lambda+\epsilon) - E_n(\lambda-\epsilon)\big) = \dim R\big(E(\lambda+\epsilon) - E(\lambda-\epsilon)\big),\ \text{for large } n. \tag{48}$$

From the characterization of the discrete spectrum [Wei80, p.202] and from Lemma 3.1 together with (48), we derive that $\sigma_{\mathrm{disc}}(H_\mu) \subset (-\infty, 0)$ so that $\sigma_{\mathrm{ess}}(H_\mu) \subset [0,\infty)$. An other time Lemma 3.1 together with (48) imply that $\sigma(H_\mu)$ can not have a gap at $\lambda > 0$. Thus $\sigma_{\mathrm{ess}}(H_\mu) = [0,\infty)$, which was to be proved. □

We would like to compare our criteria with those given in the literature. The main difference is that we do not assume that μ^+ is bounded nor the boundedness of

$$J^s_{\mu^+} := W^{s,2}(\mathbb{R}^d) \to L^2(\mu^+),\ f \mapsto f, \tag{49}$$

for any $1 \le s < 2$ (cf. [Bra01, Theorem 21]. Examples of measures violating this condition, but satisfying ours, are given on $\mathbb{R}^d$, $d > 4$ by μ such that $\mu^- \in \mathcal{B}_0^+$, μ^+ has compact support and

$$\mu^+(B_r(x)) \ge Cr^\beta,\ 0 < \beta < d - 2s,\ \forall\ 0 < r \le 1, \forall x. \tag{50}$$

For such measures we have

$$\frac{\mu^+(B_r(x))}{\mathrm{Cap}_s(B_r(x))} \ge Cr^{\beta - d + 2s},\ \forall\ 0 < r \le 1, \tag{51}$$

so that $\sup\{\frac{\mu^+(K)}{\mathrm{Cap}_s(K)},\ K \text{ compact}\} = \infty$ and by [AH96, p.191], $J^s_{\mu^+}$ is unbounded for every $1 \le s < 2$.
For $d = 2, 3$ the measure μ^+ can be chosen as follows : Set $\Sigma_{S_{d-1}(r)}$ the normalized surface measure of $S_{d-1}(r) := \{x \in \mathbb{R}^d : |x| = r\}$, $r > 0$. For $0 < \epsilon < 1$, define the measure

$$d\mu^+(x) := 1_{B_1(0)}(x)|x|^{-\epsilon} d\Sigma_{S_{d-1}(|x|)}.$$

Then μ^+ is a Radon measure on $\mathbb{R}^d$. Let $1 \le s < 2$. Then

$$\frac{\mu^+\big(S_{d-1}(r)\big)}{\mathrm{Cap}_s\big(S_{d-1}(r)\big)} \ge r^{-\epsilon}\big(\mathrm{Cap}_s\big(S_{d-1}(1)\big)\big)^{-1},\ \forall 0 < r < 1.$$

Recalling that $\mathrm{Cap}_s\big(S_{d-1}(1)\big) > 0$ for every $1 \le s < 2$ and $d = 2, 3$ (cf. [AH96, p.139]) we conclude that $J^s_{\mu^+}$ is unbounded, for the same considerations as before.

Many assumptions on the measure μ, in the literature, are made so as to imply the compactness of the resolvent difference. One of these assumptions (among others) is that $|\mu| \in \mathcal{B}^+$. However, in these circumstances, arguing as in the proof of Lemma 2.2, the resolvent difference can be written as

$$(H_\mu + \alpha)^{-1} - (H + \alpha)^{-1} = -(I_{\alpha,\mu}(H + \alpha)^{-1})^*\big(I + K^\mu_\alpha\big)^{-1} I_{\alpha,\mu}(H + \alpha)^{-1}. \tag{52}$$

for α large enough. Hence the compactness of the resolvent difference is equivalent to the compactness of the operator $I_{\alpha,\mu}(H + \alpha)^{-1}$, which by [AH96, Theorem 7.3.1 p.195] implies

$$\lim_{n\to\infty} \sup\big\{\frac{\mu(K)}{\mathrm{Cap}_2(K)} : K \subset B_n^c,\ \text{compact}\big\} = 0. \tag{53}$$

Thus the compactness of the resolvent difference, for $\mu^- = 0$, implies our criteria.
So, our assumptions on the measure μ are weaker, in the absolutely continuous and in the

singular case as well, than those existing in the literature cf. [Bra01, BEKŠ94, HS96, OS98] for the invariance of the essential spectrum, as long as Radon measures are considered. Moreover our conditions give supplement informations on the discrete spectrum. Namely every negative eigenvalue λ of H_μ is the limit of eigenvalues λ_n such that $\lambda_n \in \sigma_{\text{disc}}(H_{\mu_n})$ for large n.

Next we shall give a sufficient condition for the criteria (38) to be fulfilled. Then use it to show that our method recovers measures vanishing at infinity, in particular finite Kato measures and s-measures with compact support cf. [JW84].

From now on the letter K designate a nonempty compact subset of $\mathbb{R}^d$.

Theorem 3.2. *Let $\mu = \mu^+ - \mu^- \in \mathcal{S}^+ - \mathcal{B}_0^+$ be such that $(\mu^+)^\rho \in \mathcal{B}$ for some $\rho > 0$. Assume that*

$$\lim_{n\to\infty} \sup_{|x|\geq n} |\mu|(B_1(x)) = 0 \text{ if } d \leq 3, \tag{54}$$

$$\lim_{n\to\infty} \sup_{|x|\geq n} \int_{\{|x-y|<1\}} |\log(|x-y|)|\, d|\mu|(y) = 0 \text{ if } d = 4, \tag{55}$$

and

$$\lim_{n\to\infty} \sup_{|x|\geq n} \int_{\{|x-y|<1\}} |x-y|^{4-d}\, d|\mu|(y) = 0 \text{ if } d > 4. \tag{56}$$

Then assumption (38) is satisfied.

We observe that since for $d > 4$, $0 < s \leq 2$ and for every $x \in \mathbb{R}^d$, we have

$$|x-y|^{4-d} \leq |x-y|^{s-d} \text{ q.e. on } B_1(x),$$

then assumption (56) is still weaker than the one adopted by Schechter in [Sch86, Theorem 10.8, p.152].

Proof. We shall make the proof only for $d > 4$. For $d \leq 4$, the proof can be done exactly in the same way.
Let $d > 4$. For our purpose, we proceed to give an adequate estimate of the ratio appearing in (38).
For every K, set $\mathcal{P}_K$ the space of probability measures supported by K. Then making use of the dual definition of the capacity, [AH96, Theorem 2.5.5] and arguing as in [FŌT94, p.86-87], we conclude that

$$\big(\mathrm{Cap}_2(K)\big)^{-1} = \inf_{\nu\in\mathcal{P}_K} \sup_{x\in\mathbb{R}^d} \int_K G_4(x,y)\, d\nu(y). \tag{57}$$

Taking $\nu := |\mu|_{|K}\big(|\mu|(K)\big)^{-1}$ we achieve

$$\begin{aligned}\frac{|\mu|(K)}{\mathrm{Cap}_2(K)} \quad &\le \sup_{x\in\mathbb{R}^d}\int_K G_4(x,y)\,d|\mu|(y)\\ &\le A_n := \sup_{x\in\mathbb{R}^d}\int_{B_n^c} G_4(x,y)\,d|\mu|(y),\ \forall\ K\subset B_n^c. \end{aligned}\tag{58}$$

Now we are going to prove that $\lim_{n\to\infty} A_n = 0$. Since $G_4(\cdot,y)\le|\cdot-y|^{4-d}$ q.e., we have

$$A_n \le \sup_{x\in\mathbb{R}^d}\int_{B_x(1)\cap\{|y|\ge n\}}|x-y|^{4-d}\,d|\mu|(y) + \sup_{x\in\mathbb{R}^d}\int_{B_x^c(1)\cap\{|y|\ge n\}} G_4(x,y)\,d|\mu|(y). \tag{59}$$

Since the kernel G_4 decreases rapidly away from the diagonal, we derive

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\int_{B_x(1)\cap\{|y|\ge n\}} G_4(x,y)\,d|\mu|(y) \quad &= \sup_{|x|\ge n}\int_{B_x(1)\cap\{|y|\ge n\}} G_4(x,y)\,d|\mu|(y)\\ &\le \sup_{|x|\ge n}\int_{B_x(1)}|x-y|^{4-d}\,d|\mu(y)|\to 0 \text{ as } n\to\infty,\end{aligned}$$

by assumption.
By the same way we derive

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\int_{B_x^c(1)\cap\{|y|\ge n\}} G_4(x,y)\,d|\mu|(y) \quad &= \sup_{|x|\ge n}\int_{B_x^c(1)\cap\{|y|\ge n\}} G_4(x,y)\,d|\mu|(y)\\ &\le \sup_{|x|\ge n}\int_{B_x^c(1)} G_4(x,y)\,d|\mu|(y).\end{aligned}\tag{60}$$

We claim that

$$\sup_{|x|\ge n}\int_{B_x^c(1)} G_4(x,y)\,d|\mu|(y)\le C\sup_{|x|\ge n}\int_{B_x(1)}|x-y|^{4-d}\,d|\mu|(y), \tag{61}$$

where C is a constant depending only on d. Indeed:
For $k\ge 1$ let $B_1(z_j(k))_{1\le j\le N(k)}$ be a covering of $S_k := \{k\le|x-y|<k+1\}$ such that $z_j(k)\in S_k$. We choose $N(k)$ such that $N(k)\le k^d$ (which is possible). From the properties of the Bessel kernel we infer ([Ste70, p. 133]) that there is a constant $C(d)$ depending only on d such that

$$G_4(x,y)\le C(d)\mathrm{e}^{-\frac{|x-y|}{2}},\ \forall\ |x-y|\ge 1. \tag{62}$$

Now we have

$$
\begin{aligned}
\int_{B_x^c(1)} G_4(x,y)\,d|\mu|(y) \quad &\le \sum_{k=1}^{\infty} \int_{\{k\le|x-y|<k+1\}} G_4(x,y)\,d|\mu|(y) \\
&\le C(d) \sum_{k=1}^{\infty} (k+1)^{d-4} \mathrm{e}^{-\frac{k}{2}} \int_{\{k\le|x-y|<k+1\}} |x-y|^{4-d}\,d|\mu|(y) \\
&\le C(d) \sum_{k=1}^{\infty} (k+1)^{d-4} \mathrm{e}^{-\frac{k}{2}} \sum_{j=1}^{N(k)} \int_{\{|x+z_j(k)-y|<1\}\cap S_k} |x-y|^{4-d}\,d|\mu|(y) \\
&\le C(d) \sum_{k=1}^{\infty} (k+1)^{d-4} \mathrm{e}^{-\frac{k}{2}} \sum_{j=1}^{N(k)} k^{4-d} \\
&\cdot \int_{\{|x+z_j(k)-y|<1\}} |x+z_j(k)-y|^{4-d}\,d|\mu|(y). \qquad (63)
\end{aligned}
$$

Now taking the supremum on both side yields

$$
\sup_{|x|\ge n} \int_{B_1^c(x)} G_4(x,y)\,d|\mu|(y) \le 2^d C(d) \big(\sum_{k=1}^{\infty} k^d \mathrm{e}^{-\frac{k}{2}} \big) \sup_{|x|\ge n} \int_{B_1(x)} |x-y|^{4-d}\,d|\mu|(y)
$$

$$
\to 0 \text{ as } n\to\infty, \qquad (64)
$$

and the proof is finished. □

We note that condition (38) implies that μ vanishes at infinity, i.e.:

$$
\lim_{n\to\infty} \sup_{|x|\ge n} |\mu|\big(B_1(x)\big) = 0, \qquad (65)
$$

a condition which enters naturally for the invariance of the essential spectrum. To confirm this observation we will show that in many cases the two conditions are equivalent to each other.

Proposition 3.1. *Let $\mu = \mu^+ - \mu^- \in \mathcal{S}^+ - \mathcal{B}_0^+$ be such that $|\mu|^\rho \in \mathcal{B}^+$ for some $\rho > 0$. Then μ vanishes at infinity if and only if condition (38) is satisfied.*

Under these circumstances, it follows in particular that if μ vanishes at infinity then $\sigma_{\mathrm{ess}}(H_\mu) = [0,\infty)$, which generalizes a result due to Brasche [Bra01, Theorem 21].

Proof. The 'if' part was already proved above.
We prove the 'only if' part. Let μ be a measure vanishing at infinity. For $d < 4$, the result is a direct consequence of Theorem 3.2.
Now let $d > 4$ (for $d = 4$ the proof is essentially the same, so we omit it).
Let $0 < \delta < 1$, then

$$
\int_{\{|x-y|<1\}} |x-y|^{4-d}\,d|\mu|(y) \le \int_{\{|x-y|<\delta\}} |x-y|^{4-d}\,d|\mu|(y) + \int_{\{\delta\le|x-y|<1\}} |x-y|^{4-d}\,d|\mu|(y).
$$

The second integral satisfies

$$\int_{\{\delta\leq|x-y|<1\}} |x-y|^{4-d}\, d|\mu|(y) \leq \delta^{4-d}|\mu|(B_1(x)) \to 0 \text{ as } x\to\infty. \tag{66}$$

To estimate the first integral we proceed as follows: Let $n\in\mathbb{N}$ be large enough and x such that $|x|\geq n$. The boundedness assumption on $(\mu^+)^\rho$ and μ^-, together with [BA04] and [AH96, Prop.5.1.4] imply

$$(|\mu|)^\rho\big(B_\delta(x)\big) = |\mu|\big(B_\delta(x)\big) \leq C\mathrm{Cap}_1\big(B_\delta(x)\big) \leq C'\delta^{d-2},\ \forall\ 0<\delta<1. \tag{67}$$

Here the constant C' depends neither on x nor on δ.
So we get

$$\begin{aligned}\int_{|x-y|<\delta} |x-y|^{4-d}\, d|\mu|(y) &= \int_0^\delta t^{4-d}\, d|\mu|(B_x(t))\\ &= (d-4)\int_0^\delta t^{3-d}|\mu|(B_x(t))\, dt + \delta^{4-d}|\mu|(B_x(\delta))\\ &\leq (d-4)C'\int_0^\delta t\, dt + C'\delta^2, \forall\ 0<\delta\leq 1 \text{ and } \forall\ x,\ |x|\geq n,\end{aligned}$$

which together with (66) leads to

$$\lim_{n\to\infty}\sup_{|x|\geq n}\int_{B_1(x)} |x-y|^{4-d} \leq C\delta^2, \forall\ 0<\delta<1. \tag{68}$$

Now the result follows from Theorem 3.2-(56). □

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