МАТЕМАТИЧКИ ВЕСНИК
53 (2001), 79–89

UDK 517.956.3
оригинални научни рад
research paper

ON A NONLOCAL SINGULAR MIXED EVOLUTION PROBLEM

Said Mesloub and Nadia Lekrine

Abstract. In the present paper, the existence and uniqueness of the strong solution of a mixed problem for a second order plurihyperbolic equation with an integral condition is proved. The proof is essentially based on an a priori bound and on the density of the range of the operator generated by the considered problem. In spite of the apparant simplicity of the problem, the solution requires a delicate set of techniques. It seems very difficult to extend these technics to the considered equation in more than one dimension without imposing complementary conditions.

1. Statement of the problem

In the region $Q = (0,a) \times (0,T_1) \times (0,T_2)$, with $a < \infty$, $T_1 < \infty$ and $T_2 < \infty$, we consider the one dimensional hyperbolic equation

$$\mathcal{L}v = v_{t_1 t_2} - \frac{1}{x}\left(xv_x\right)_x = F(x,t_1,t_2), \tag{1}$$

The equation (1) is supplemented by boundary and initial conditions

$$\ell_1 v = v(x,0,t_2) = \phi_1(x,t_2), \qquad (x,t_2) \in Q_2 = (0,a) \times (0,T_2), \tag{2}$$

$$\ell_2 v = v(x,t_1,0) = \phi_2(x,t_1), \qquad (x,t_1) \in Q_1 = (0,a) \times (0,T_1), \tag{3}$$

$$v_x(a,t_1,t_2) = \Phi(t_1,t_2), \qquad (t_1,t_2) \in (0,T_1) \times (0,T_2), \tag{4}$$

$$\int_0^a xv(x,t_1,t_2)\,dx = \Psi(t_1,t_2), \qquad (t_1,t_2) \in (0,T_1) \times (0,T_2). \tag{5}$$

where $\phi_1(x,t_2)$, $\phi_2(x,t_1)$, $\Phi(t_1,t_2)$, $\Psi(t_1,t_2)$ and $F(x,t_1,t_2)$ are given functions. The data functions have to satisfy the following compatibility conditions:

$$\frac{\partial\phi_1}{\partial x} = \Phi(0,t_2), \qquad \int_0^a x\phi_1(x,t_2)\,dx = \Psi(0,t_2),$$

$$\frac{\partial\phi_2}{\partial x} = \Phi(t_1,0), \qquad \int_0^a x\phi_2(x,t_1)\,dx = \Psi(t_1,0),$$

and $\phi_1(x,0) = \phi_2(x,0)$.

AMS Subject Classification: 35 L 20, 35 L 67
Key words and phrases: Strong solution, a priori bound, plurihyperbolic equation.

In the early sixties Cannon [6] has proved by means of an integral equation (Potential method), the existence and uniqueness of the solution for a mixed problem combining a classical condition (Dirichlet condition) and an integral one for the homogeneous equation. One year later Kamynin [10] has generalized the results of Cannon by using a system of integral equations (Potential method). The importance of mixed problems with integral conditions has been also pointed out by Samarskii [14]. Problem (1)–(5), can be viewed as a non-local problem for a plurihyperbolic equation (with the Bessel operator). A similar problem for which a homogeneous Dirichlet condition and the linear constraint $\int_0^a v(x,t)\,dx = 0$ are combined, has been investigated by Benouar and Yurchuk [1]. In their papers [2], [3], [4] and [5], the authors considered hyperbolic and parabolic equations having the operator $(\alpha(x,t)v_x)_x$ instead of the Bessel operator considered in equation (1). For some mixed problems for second order parabolic equations which combine classical and integral conditions the reader should refer to Cannon-van der Hoek [7], [8], Cannon-Esteva-van der Hoek [9], Kartynik [11], Shi [15], Yurchuk [16] and Mesloub-Bouziani [13]. In this paper, the existence and uniqueness of a strong solution of problem (1)–(5) is proved by means of an energy estimate and a density argument.

In point of view of the used method, it is preferable to transform the nonhomogeneous conditions to homogeneous ones. If we set:

$$u(x,t_1,t_2) = v(x,t_1,t_2) - w(x,t_1,t_2),$$

where

$$w(x,t_1,t_2) = (x - \frac{4(x-a)^2}{a}) \cdot \Phi(t_1,t_2) + \frac{12(x-a)^2}{a^4} \cdot \Psi(t_1,t_2),$$

then problem (1)–(5), becomes

$$\mathcal{L}u = F(x,t_1,t_2) - \mathcal{L}w = f(x,t_1,t_2), \tag{6}$$

$$\ell_1 u = u(x,0,t_2) = \phi_1(x,t_2) - \ell_1 w = \varphi_1(x,t_2), \tag{7}$$

$$\ell_2 u = u(x,t_1,0) = \phi_2(x,t_1) - \ell_2 w = \varphi_2(x,t_1), \tag{8}$$

$$u_x(a,t_1,t_2) = 0, \tag{9}$$

$$\int_0^a xu(x,t_1,t_2)\,dx = 0 \tag{10}$$

We now introduce the appropriate function spaces needed for the investigation of the posed problem. Let $L^2_\rho(Q)$ be the weighted L^2-space with finite norm

$$\|u\|^2_{L^2_\rho} = \int_Q xu^2\,dx\,dt,$$

$t = (t_1,t_2)$, $dt = dt_1\,dt_2$. The scalar product in $L^2_\rho(Q)$ is defined by $(u,v)_{L^2_\rho} = (xu,v)_{L^2}$. Let $V^{1,0}_\rho(Q_i)$, $V^{1,1}_\rho(Q_1)$, and $V^{1,1}_\rho(Q_2)$, $i = 1,2$ be the Hilbert spaces with scalar products respectively

$$
\begin{aligned}
(u,v)_{V_\rho^{1,0}(Q_i)} &= (u,v)_{L_\rho^2(Q_i)} + (u_x,v_x)_{L_\rho^2(Q_i)}, \qquad i=1,2,\\
(u,v)_{V_\rho^{1,1}(Q_1)} &= (u,v)_{L_\rho^2(Q_1)} + (u_x,v_x)_{L_\rho^2(Q_1)} + (u_{t_1},v_{t_1})_{L_\rho^2(Q_1)},\\
(u,v)_{V_\rho^{1,1}(Q_2)} &= (u,v)_{L_\rho^2(Q_2)} + (u_x,v_x)_{L_\rho^2(Q_2)} + (u_{t_2},v_{t_2})_{L_\rho^2(Q_2)},
\end{aligned}
$$

and with associated norms

$$
\begin{aligned}
\|u\|^2_{V_\rho^{1,0}(Q_i)} &= \|u\|^2_{L_\rho^2(Q_i)} + \|u_x\|^2_{L_\rho^2(Q_i)}\,, \quad i=1,2,\\
\|u\|^2_{V_\rho^{1,1}(Q_1)} &= \|u\|^2_{L_\rho^2(Q_1)} + \|u_x\|^2_{L_\rho^2(Q_1)} + \|u_{t_1}\|^2_{L_\rho^2(Q_1)}\,,\\
\|u\|^2_{V_\rho^{1,1}(Q_2)} &= \|u\|^2_{L_\rho^2(Q_2)} + \|u_x\|^2_{L_\rho^2(Q_2)} + \|u_{t_2}\|^2_{L_\rho^2(Q_2)}\,.
\end{aligned}
$$

The given problem (6)–(10) can be considered as the resolution of the operator equation

$$Lu = (\mathcal{L}u, \ell_1 u, \ell_2 u) = (f, \varphi_1, \varphi_2) = \mathcal{F},$$

where L is an operator defined on E into F, and E is the Banach space of functions $u \in L_\rho^2(Q)$, satisfying conditions (9) and (10), with the finite norm

$$
\begin{aligned}
\|u\|_E^2 = \sup_{0\le\tau_2\le T_2} &\left(\|u(\cdot,\cdot,\tau_2)\|^2_{V_\rho^{1,1}(Q_1)} + \|\Im_x(\xi u_{t_1}(\cdot,\cdot,\tau_2))\|^2_{L^2(Q_1)}\right)\\
&+ \sup_{0\le\tau_1\le T_1}\left(\|u(\cdot,\tau_1,\cdot)\|^2_{V_\rho^{1,1}(Q_2)} + \|\Im_x(\xi u_{t_2}(\cdot,\tau_1,\cdot))\|^2_{L^2(Q_2)}\right),
\end{aligned}
$$

where $\Im_x(\xi u) = \int_0^a \xi u(\xi,t_1,t_2)\,d\xi$, and F is the Hilbert space $L_\rho^2(Q)\times V_\rho^{1,1}(Q_2)\times V_\rho^{1,1}(Q_1)$, which consists of elements $\mathcal{F}=(f,\varphi_1,\varphi_2)$ with finite norm

$$\|\mathcal{F}\|_F^2 = \|\varphi_1\|^2_{V_\rho^{1,1}(Q_2)} + \|\varphi_2\|^2_{V_\rho^{1,1}(Q_1)} + \|\mathcal{L}f\|^2_{L_\rho^2(Q)}\,.$$

Let $D(L)$ be the set of all functions $u \in L^2(Q)$ for which u_{t_1}, u_{t_2}, $u_{t_1t_2}$, u_x, u_{xx}, u_{xt_1}, $u_{xt_2} \in L^2(Q)$ and satisfying conditions (9) and (10).

2. A priori bound and its consequences

THEOREM 2.1. *For any function $u \in D(L)$, there exists a positive constant c independent of the solution u such that*

$$\|u\|_E \le c\,\|Lu\|_F\,. \tag{11}$$

Proof. Taking the scalar product in $L^2(Q^\tau)$ of equation (6) and the integro-differential operator

$$\mathcal{M}u = x(u_{t_1}+u_{t_2}) - x\Im_x^2(\xi u_{t_1} + \xi u_{t_2}),$$

where $Q^\tau = (0,a)\times(0,\tau_1)\times(0,\tau_2)$ and $\Im_x^2 h = \int_0^x\int_0^\xi h(\zeta,t_1,t_2)\,d\zeta\,d\xi$, we obtain

$$
\begin{aligned}
(u_{t_1t_2}, u_{t_1}+u_{t_2})_{L_\rho^2(Q^\tau)} &- (u_{t_1t_2}, \Im_x^2(\xi u_{t_1} + \xi u_{t_2}))_{L_\rho^2(Q^\tau)}\\
&- (u_{t_1}+u_{t_2}, (xu_x)_x)_{L^2(Q^\tau)} + (\Im_x^2(\xi u_{t_1}+\xi u_{t_2}), (xu_x)_x)_{L^2(Q^\tau)}\\
&= (\mathcal{L}u, \mathcal{M}u)_{L^2(Q^\tau)}.
\end{aligned} \tag{12}
$$

The successive integration by parts of integrals on the left-hand side of (12) are straightforward but somewhat tedious. We only give their results

$$\begin{aligned}(u_{t_1t_2}, u_{t_1}+u_{t_2})_{L^2_\rho(Q^\tau)} &= \\ &= \frac{1}{2}\int_{Q_1^{\tau_1}} x(u_{t_1}(x,t_1,\tau_2))^2\,dx\,dt_1 - \frac{1}{2}\int_{Q_1^{\tau_1}} x(\frac{\partial\varphi_2}{\partial t_1})^2\,dx\,dt_1 \\ &\quad + \frac{1}{2}\int_{Q_2^{\tau_2}} x(u_{t_2}(x,\tau_1,t_2))^2\,dx\,dt_1 - \frac{1}{2}\int_{Q_2^{\tau_2}} x(\frac{\partial\varphi_1}{\partial t_2})^2\,dx\,dt_2, \quad (13)\end{aligned}$$

$$\begin{aligned}-(u_{t_1t_2}, \Im_x^2(\xi u_{t_1}+\xi u_{t_2}))_{L^2_\rho(Q^\tau)} &= \\ &= \frac{1}{2}\int_{Q_1^{\tau_1}} (\Im_x(\xi u_{t_1}(x,t_1,\tau_2))^2\,dx\,dt_1 - \frac{1}{2}\int_{Q_1^{\tau_1}} (\Im_x(\xi\frac{\partial\varphi_2}{\partial t_1}))^2\,dx\,dt_1 \\ &\quad + \frac{1}{2}\int_{Q_2^{\tau_2}} (\Im_x(\xi u_{t_2}(x,\tau_1,t_2))^2\,dx\,dt_2 - \frac{1}{2}\int_{Q_2^{\tau_2}} (\Im_x(\xi\frac{\partial\varphi_1}{\partial t_2}))^2\,dx\,dt_2, \quad (14)\end{aligned}$$

$$\begin{aligned}-(u_{t_1}+u_{t_2}, (xu_x)_x)_{L^2(Q^\tau)} &= \\ &= \frac{1}{2}\int_{Q_2^{\tau_2}} x(u_x(x,\tau_1,t_2))^2\,dx\,dt_2 - \frac{1}{2}\int_{Q_2^{\tau_2}} x(\frac{\partial\varphi_1}{\partial x})^2\,dx\,dt_2 \\ &\quad + \frac{1}{2}\int_{Q_1^{\tau_1}} x(u_x(x,t_1,\tau_2))^2\,dx\,dt_1 - \frac{1}{2}\int_{Q_1^{\tau_1}} x(\frac{\partial\varphi_2}{\partial x})^2\,dx\,dt_1, \quad (15)\end{aligned}$$

$$\begin{aligned}(\Im_x^2(\xi u_{t_1}+\xi u_{t_2}), (xu_x)_x)_{L^2(Q^\tau)} &= \\ &= -\int_{Q^\tau} xu_x(\Im_x(\xi u_{t_1})+\Im_x(\xi u_{t_2}))\,dx\,dt_1\,dt_2. \quad (16)\end{aligned}$$

First observe that

$$\|\Im_x u\|^2_{L^2(Q^\tau)} \le \frac{a^2}{2}\|u\|^2_{L^2(Q^\tau)}, \quad (17)$$

then by making use of (13)–(17), the Cauchy ε-inequality $\alpha\theta \le \varepsilon\alpha^2/2+\theta^2/2\varepsilon$, and the identity (12), we obtain

$$\begin{aligned}&\frac{1}{2}\|u_{t_1}(\cdot,t_1,\tau_2)\|^2_{L^2_\rho(Q_1^{\tau_1})} + \frac{1}{2}\|u_{t_2}(\cdot,\tau_1,t_2)\|^2_{L^2_\rho(Q_2^{\tau_2})} + \frac{1}{2}\|u_x(\cdot,t_1,\tau_2)\|^2_{L^2_\rho(Q_1^{\tau_1})} \\ &+\frac{1}{2}\|u_x(\cdot,\tau_1,t_2)\|^2_{L^2_\rho(Q_2^{\tau_2})} + \frac{1}{2}\|\Im_x(\xi u_{t_1}(\cdot,t_1,\tau_2))\|^2_{L^2(Q_1^{\tau_1})} + \frac{1}{2}\|\Im_x(\xi u_{t_2}(\cdot,\tau_1,t_2))\|^2_{L^2(Q_2^{\tau_2})} \\ &\le (\frac{a^4}{4}+\frac{1}{2})\left\|\frac{\partial\varphi_1}{\partial t_2}\right\|^2_{L^2_\rho(Q_2)} + (\frac{a^4}{4}+\frac{1}{2})\left\|\frac{\partial\varphi_2}{\partial t_1}\right\|^2_{L^2_\rho(Q_1)} + \frac{1}{2}\left\|\frac{\partial\varphi_2}{\partial x}\right\|^2_{L^2_\rho(Q_1)} + a\|u_x\|^2_{L^2_\rho(Q^\tau)} \\ &\quad + \frac{1}{2}\left\|\frac{\partial\varphi_1}{\partial x}\right\|^2_{L^2_\rho(Q_2)} + \frac{1}{2}\|u_{t_1}\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\|u_{t_2}\|^2_{L^2_\rho(Q^\tau)} + 2\|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)} \\ &\quad + (\frac{1}{2}+\frac{a^3}{4})\|\Im_x(\xi u_{t_1})\|^2_{L^2(Q^\tau)} + (\frac{1}{2}+\frac{a^3}{4})\|\Im_x(\xi u_{t_2})\|^2_{L^2(Q^\tau)}. \quad (18)\end{aligned}$$

Consider the elementary inequalities:

$$\|u(\cdot,\tau_1,t_2)\|^2_{L^2_\rho(Q_2^{\tau_2})} \le \|u\|^2_{L^2_\rho(Q^\tau)} + \|u_{t_1}\|^2_{L^2_\rho(Q^\tau)} + \|\varphi_1\|^2_{L^2_\rho(Q_2)}\,, \tag{19}$$

$$\|u(\cdot,t_1,\tau_2)\|^2_{L^2_\rho(Q_1^{\tau_1})} \le \|u\|^2_{L^2_\rho(Q^\tau)} + \|u_{t_2}\|^2_{L^2_\rho(Q^\tau)} + \|\varphi_2\|^2_{L^2_\rho(Q_1)}\,. \tag{20}$$

Adding side to side inequalities (18)–(20), we obtain

$$\begin{aligned}
&\|u_{t_1}(\cdot,t_1,\tau_2)\|^2_{L^2_\rho(Q_1^{\tau_1})} + \|u_{t_2}(\cdot,\tau_1,t_2)\|^2_{L^2_\rho(Q_2^{\tau_2})} + \|u_x(\cdot,t_1,\tau_2)\|^2_{L^2_\rho(Q_1^{\tau_1})}\\
&+ \|u_x(\cdot,\tau_1,t_2)\|^2_{L^2_\rho(Q_2^{\tau_2})} + \|\Im_x(\xi u_{t_1}(\cdot,t_1,\tau_2))\|^2_{L^2(Q_1^{\tau_1})} + \|\Im_x(\xi u_{t_2}(\cdot,\tau_1,t_2))\|^2_{L^2(Q_2^{\tau_2})}\\
&\qquad + \|u(\cdot,\tau_1,t_2)\|^2_{L^2_\rho(Q_2^{\tau_2})} + \|u(\cdot,t_1,\tau_2)\|^2_{L^2_\rho(Q_1^{\tau_1})}\\
&\le k\Big\{\|\varphi_1\|^2_{V_\rho^{1,1}(Q_2)} + \|\varphi_2\|^2_{V_\rho^{1,1}(Q_1)} + \|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)} + \|u\|^2_{L^2_\rho(Q^\tau)} + \|u_{t_1}\|^2_{L^2_\rho(Q^\tau)}\\
&\quad + \|u_{t_2}\|^2_{L^2_\rho(Q^\tau)} + \|u_x\|^2_{L^2_\rho(Q^\tau)} + \|\Im_x(\xi u_{t_1})\|^2_{L^2(Q^\tau)} + \|\Im_x(\xi u_{t_2})\|^2_{L^2(Q^\tau)}\Big\}\,,
\end{aligned} \tag{21}$$

where

$$k = \max\left\{2a,\ 4,\ 1+\frac{a^3}{2},\ 1+\frac{a^4}{2}\right\}.$$

Now, to eliminate the last six terms on the right-hand side of (21), we use the following lemma which can be proved in the same fashion as in lemma 7.1 from [12]. ■

LEMMA 2.2. *If $f_1(\tau_1,\tau_2)$, $f_2(\tau_1,\tau_2)$ and $f_3(\tau_1,\tau_2)$ are nonnegative functions on the rectangle $(0,T_1)\times(0,T_2)$, $f_1(\tau_1,\tau_2)$ and $f_2(\tau_1,\tau_2)$ are integrable, and $f_3(\tau_1,\tau_2)$ is nondecreasing in each of its variables separately, then it follows from*

$$\begin{aligned}
&\int_0^{\tau_1}\int_0^{\tau_2} f_1(\tau_1,\tau_2)\,dt_1\,dt_2 + f_2(\tau_1,\tau_2)\\
&\qquad\qquad \le c\int_0^{\tau_1} f_2(t_1,\tau_2)\,dt_1 + c\int_0^{\tau_2} f_2(\tau_1,t_2)\,dt_2 + f_3(\tau_1,\tau_2)
\end{aligned}$$

that

$$\int_0^{\tau_1}\int_0^{\tau_2} f_1(\tau_1,\tau_2)\,dt_1\,dt_2 + f_2(\tau_1,\tau_2) \le \exp(2c(\tau_1+\tau_2))\cdot f_3(\tau_1,\tau_2).$$

Then (21) takes the form

$$\begin{aligned}
&\|u(\cdot,t_1,\tau_2)\|^2_{V_\rho^{1,1}(Q_1^{\tau_1})} + \|\Im_x(\xi u_{t_1}(\cdot,t_1,\tau_2))\|^2_{L^2(Q_1^{\tau_1})}\\
&\quad + \|u(\cdot,\tau_1,t_2)\|^2_{V_\rho^{1,1}(Q_2^{\tau_2})} + \|\Im_x(\xi u_{t_2}(\cdot,\tau_1,t_2))\|^2_{L^2(Q_2^{\tau_2})}\\
&\qquad \le ke^{k(T_1+T_2)}\Big\{\|\varphi_1\|^2_{V_\rho^{1,1}(Q_2)} + \|\varphi_2\|^2_{V_\rho^{1,1}(Q_1)} + \|\mathcal{L}u\|^2_{L^2_\rho(Q^\tau)}\Big\}.
\end{aligned}$$

Since the right-hand side of the above inequality is independent of (τ_1,τ_2), we can take the least upper bound of the left side with respect to (τ_1,τ_2) from $[0,T_1)$ and $[0,T_2)$ respectively, we get the desired estimate (11) with $c=\sqrt{k}e^{k(T_1+T_2)/2}$.

We shall now prove that the operator L admits a closure. For this we must either show that it follows from a well known theorem in the theory of unbouded operators that the operator L^* adjoint to L is defined in a dense set, or else verify directly the following assertion: If $u_n \in D(L)$ is a sequence such that

$$u_n \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in the norm of } E, \tag{22}$$

and

$$Lu_n \underset{n\to\infty}{\longrightarrow} \mathcal{F} = (f, \varphi_1, \varphi_2) \qquad \text{in the norm of } F, \tag{23}$$

then $f = 0$, $\varphi_1 = 0$, $\varphi_2 = 0$.

Since (22) holds, then

$$u_n \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } \mathcal{D}'(Q), \tag{24}$$

where $\mathcal{D}'(Q)$ is the space of distributions on Q. By virtue of the continuity of derivation of $\mathcal{D}'(Q)$ in $\mathcal{D}'(Q)$, (24) implies that

$$\mathcal{L}u_n \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } \mathcal{D}'(Q). \tag{25}$$

But since

$$\mathcal{L}u_n \underset{n\to\infty}{\longrightarrow} f \qquad \text{in } L^2_\rho(Q), \tag{26}$$

then

$$\mathcal{L}u_n \underset{n\to\infty}{\longrightarrow} f \qquad \text{in } \mathcal{D}'(Q). \tag{27}$$

From the uniqueness of the limit in the space $\mathcal{D}'(Q)$, we conclude that $f = 0$.

According to (23), we have

$$\ell_1 u_n \underset{n\to\infty}{\longrightarrow} \varphi_1 \qquad \text{in } V^{1,1}_\rho(Q_2), \tag{28}$$

and by the fact that the canonical injection from $V^{1,1}_\rho(Q_2)$ into $\mathcal{D}'(Q_2)$ is continuous, (28) implies

$$\ell_1 u_n \underset{n\to\infty}{\longrightarrow} \varphi_1 \quad \text{in } \mathcal{D}'(Q_2). \tag{29}$$

Moreover, since (22) holds and

$$\|\ell_1 u_n\|_{V^{1,1}_\rho(Q_2)} \le \|u_n\|_E \qquad \forall n, \tag{30}$$

we have

$$\ell_1 u_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } V^{1,1}_\rho(Q_2). \tag{31}$$

Hence

$$\ell_1 u_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } \mathcal{D}'(Q_2). \tag{32}$$

By virtue of the uniqueness of the limit in $\mathcal{D}'(Q_2)$, we conclude from (29) and (32), that $\varphi_1 = 0$. In the same fashion, we can show that $\varphi_2 = 0$.

Let $\overline{L}$ be the closure of the operator L with domain of definition $D(\overline{L})$.

Definition 2.1. A solution of the operator equation

$$\overline{L}u = \mathcal{F},$$

is called the *strong solution* of the problem (6)–(10).

By passing to the limit, the estimate (11) can be extended to strong solutions, that is we have the inequality

$$\|u\|_E \le c \left\|\overline{L}u\right\|_F \qquad \forall u \in D(\overline{L}). \tag{33}$$

Hence

Corollary 2.3. *If a strong solution of* (6)–(10) *exists, it is unique and depends continuously on elements* $\mathcal{F} = (f, \varphi_1, \varphi_2) \in F$.

Corollary 2.4. *The range* $R(\overline{L})$ *of the operator* $\overline{L}$ *is closed in* F *and* $R(\overline{L}) = \overline{R(L)}$.

Hence, to prove that a strong solution of problem (6)–(10) exists for any element $(f, \varphi_1, \varphi_2) \in F$, it remains to prove that $\overline{R(L)} = F$.

3. Solvability of the posed problem

Theorem 3.1. *If, for some function* $\omega \in L^2(Q)$ *and for all* $u \in D(L)$ *verifying* $\ell_1 u = \ell_2 u = 0$, *we have*

$$\int_Q x\mathcal{L}u \cdot \omega \, dx\, dt = 0, \tag{34}$$

$dt = dt_1\, dt_2$, *then* ω *vanishes almost everywhere in the domain* Q.

Proof. Relation (34) holds for any function u in $D(L)$ such that $\ell_1 u = \ell_2 u = 0$, so it can be expressed in a particular form. Consider the function g_{ij} defined by

$$g_{ij}(t_1, t_2, x) = \int_{t_i}^{T_i} \omega_{ij}\, d\tau_i, \qquad i, j = 1, 2.$$

Let $\partial^2 u/\partial t_i \partial t_j$ be the solution of the equation

$$\partial^2 u/\partial t_i \partial t_j - \int_0^x \int_0^\xi \zeta \partial^2 u/\partial t_i \partial t_j \, d\zeta\, d\xi = g_{ij}(t_1, t_2, x) \tag{35}$$

and let

$$u = \begin{cases} 0, & 0 \le t_i \le s_i, \\ \int_{s_1}^{t_1} \int_{s_2}^{t_2} u_{\tau_1 \tau_2}\, d\tau_1\, d\tau_2, & s_i \le t_i \le T_i, \end{cases} \qquad i = 1, 2. \tag{36}$$

From the above relations, we have

$$\omega = \sum_{i=1}^{2} \sum_{j=1}^{2} \omega_{ij} = -\sum_{i=1}^{2} \sum_{j=1}^{2} \left(\partial^2 u/\partial t_i \partial t_j - \int_0^x \int_0^\xi \zeta \partial^2 u/\partial t_i \partial t_j \, d\zeta\, d\xi\right)_{t_i}. \tag{37}$$

LEMMA 3.2. *The function ω defined by* (37) *is in* $L^2(Q)$.

Proof. The proof can be derived as in [2]. ■

To continue the proof of Theorem 3.1, replacing ω in (34) by its representation (37), we have

$$\begin{aligned} &-(u_{t_1t_2},\sum_{j=1}^{2}u_{t_1t_1t_j})_{L^2_\rho(Q)}+(u_{t_1t_2},\sum_{j=1}^{2}\Im_x^2(\xi u_{t_1t_1t_j}))_{L^2_\rho(Q)}\\ &\quad+((xu_x)_x,\sum_{j=1}^{2}u_{t_1t_1t_j})_{L^2(Q)}-((xu_x)_x,\sum_{j=1}^{2}\Im_x^2(\xi u_{t_1t_1t_j}))_{L^2(Q)}\\ &\quad-(u_{t_1t_2},\sum_{j=1}^{2}u_{t_2t_2t_j})_{L^2_\rho(Q)}+(u_{t_1t_2},\sum_{j=1}^{2}\Im_x^2(\xi u_{t_2t_2t_j}))_{L^2_\rho(Q)}\\ &\quad+((xu_x)_x,\sum_{j=1}^{2}u_{t_2t_2t_j})_{L^2(Q)}-((xu_x)_x,\sum_{j=1}^{2}\Im_x^2(\xi u_{t_2t_2t_j}))_{L^2(Q)}=0. \end{aligned}\tag{38}$$

Using conditions (9), (10), the particular form of u given by the relations (35), (36) and then integrating by parts each term of (38), we get

$$-(u_{t_1t_2},\sum_{j=1}^{2}u_{t_1t_1t_j})_{L^2_\rho(Q)}=\frac{1}{2}\left\|u_{t_1t_1}(x,t_1,T_2)\right\|^2_{L^2_\rho(Q^1_{s_1})},\tag{39}$$

where $Q^1_{s_1}=(0,a)\times(s_1,T_1)$,

$$(u_{t_1t_2},\sum_{j=1}^{2}\Im_x^2(\xi u_{t_1t_1t_j}))_{L^2_\rho(Q)}=\frac{1}{2}\left\|\Im_x(\xi u_{t_1t_1}(x,t_1,T_2))\right\|^2_{L^2(Q^1_{s_1})},\tag{40}$$

$$((xu_x)_x,\sum_{j=1}^{2}u_{t_1t_1t_j})_{L^2(Q)}=\frac{1}{2}\left\|u_{xt_1}(x,t_1,T_2)\right\|^2_{L^2_\rho(Q^1_{s_1})},\tag{41}$$

$$\begin{aligned} -((xu_x)_x,\sum_{j=1}^{2}\Im_x(\xi u_{t_1t_1t_j}))_{L^2(Q)}&=-(u_{xt_1},\Im_x(\xi u_{t_1t_1}))_{L^2_\rho(Q_s)}-\\ -(u_{xt_2},\Im_x(\xi u_{t_1t_1}))_{L^2_\rho(Q_s)}&+(xu_x(x,t_1,T_2),\Im_x(\xi u_{t_1t_1}(x,t_1,T_2)))_{L^2_\rho(Q^1_{s_1})}, \end{aligned}\tag{42}$$

where $Q_s=(0,a)\times(s_1,T_1)\times(s_2,T_2)$,

$$-(u_{t_1t_2},\sum_{j=1}^{2}u_{t_2t_2t_j})_{L^2_\rho(Q)}=\frac{1}{2}\left\|u_{t_2t_2}(x,T_1,t_2)\right\|^2_{L^2_\rho(Q^2_{s_2})},\tag{43}$$

where $Q^2_{s_2}=(0,a)\times(s_2,T_2)$,

$$(u_{t_1t_2},\sum_{j=1}^{2}\Im_x^2(\xi u_{t_2t_2t_j}))_{L^2_\rho(Q)}=\frac{1}{2}\left\|\Im_x(\xi u_{t_2t_2}(x,T_1,t_2))\right\|^2_{L^2(Q^2_{s_2})},\tag{44}$$

$$((xu_x)_x,\sum_{j=1}^{2}u_{t_2t_2t_j})_{L^2(Q)}=\frac{1}{2}\left\|u_{xt_2}(x,T_1,t_2)\right\|^2_{L^2_\rho(Q^2_{s_2})},\tag{45}$$

$$\begin{aligned} -((xu_x)_x,\sum_{j=1}^{2}\Im_x^2(\xi u_{t_2t_2t_j}))_{L^2(Q)}&=-(u_{xt_2},\Im_x(\xi u_{t_2t_2}))_{L^2_\rho(Q_s)}-\\ -(u_{xt_1},\Im_x(\xi u_{t_2t_2}))_{L^2_\rho(Q_s)}&+(xu_x(x,T_1,t_2),\Im_x(\xi u_{t_2t_2}(x,T_1,t_2)))_{L^2_\rho(Q^2_{s_2})}. \end{aligned}\tag{46}$$

Combining equalities (38)–(46), we get

$$\begin{aligned}
&\frac{1}{2}\left\|u_{t_1t_1}(x,t_1,T_2)\right\|^2_{L^2_\rho(Q^1_{s_1})}+\frac{1}{2}\left\|\Im_x(\xi u_{t_1t_1}(x,t_1,T_2))\right\|^2_{L^2(Q^1_{s_1})}\\
&\qquad+\frac{1}{2}\left\|u_{xt_1}(x,t_1,T_2)\right\|^2_{L^2_\rho(Q^1_{s_1})}+\frac{1}{2}\left\|u_{t_2t_2}(x,T_1,t_2)\right\|^2_{L^2_\rho(Q^2_{s_2})}\\
&\qquad+\frac{1}{2}\left\|u_{xt_2}(x,T_1,t_2)\right\|^2_{L^2_\rho(Q^2_{s_2})}+\frac{1}{2}\left\|\Im_x(\xi u_{t_2t_2}(r,T_1,t_2))\right\|^2_{L^2(Q^2_{s_2})}\\
&=(u_{xt_1},\Im_x(\xi u_{t_1t_1}))_{L^2_\rho(Q_s)}+(u_{xt_2},\Im_x(\xi u_{t_1t_1}))_{L^2_\rho(Q_s)}+(u_{xt_2},\Im_x(\xi u_{t_2t_2}))_{L^2_\rho(Q_s)}\\
&\quad+(u_{xt_1},\Im_x(\xi u_{t_2t_2}))_{L^2_\rho(Q_s)}-(xu_x(x,T_1,t_2),\Im_x(\xi u_{t_2t_2}(x,T_1,t_2)))_{L^2_\rho(Q^2_{s_2})}\\
&\quad-(xu_x(x,t_1,T_2),\Im_x(\xi u_{t_1t_1}(x,t_1,T_2)))_{L^2_\rho(Q^1_{s_1})}.
\end{aligned}\tag{47}$$

We now estimate the terms on the right-hand side of (47). We have

$$(u_{xt_1},\Im_x(\xi u_{t_1t_1}))_{L^2_\rho(Q_s)}\le\frac{a}{2}\left\|u_{xt_1}\right\|^2_{L^2_\rho(Q_s)}+\frac{1}{2}\left\|\Im_x(\xi u_{t_1t_1})\right\|^2_{L^2(Q_s)},\tag{48}$$

$$(u_{xt_2},\Im_x(\xi u_{t_2t_2}))_{L^2_\rho(Q_s)}\le\frac{a}{2}\left\|u_{t_2x}\right\|^2_{L^2_\rho(Q_s)}+\frac{1}{2}\left\|\Im_x(\xi u_{t_2t_2})\right\|^2_{L^2(Q_s)},\tag{49}$$

$$(u_{xt_2},\Im_x(\xi u_{t_1t_1}))_{L^2_\rho(Q_s)}\le\frac{a}{2}\left\|u_{t_2x}\right\|^2_{L^2_\rho(Q_s)}+\frac{1}{2}\left\|\Im_x(\xi u_{t_1t_1})\right\|^2_{L^2(Q_s)},\tag{50}$$

$$(u_{xt_1},\Im_x(\xi u_{t_2t_2}))_{L^2_\rho(Q_s)}\le\frac{a}{2}\left\|u_{t_1x}\right\|^2_{L^2_\rho(Q_s)}+\frac{1}{2}\left\|\Im_x(\xi u_{t_2t_2})\right\|^2_{L^2(Q_s)},\tag{51}$$

$$\begin{aligned}
&-(xu_x(x,t_1,T_2),\Im_x(\xi u_{t_1t_1}(x,t_1,T_2)))_{L^2_\rho(Q^1_{s_1})}\\
&\qquad\le a\left\|u_x(x,t_1,T_2)\right\|^2_{L^2_\rho(Q^1_{s_1})}+\frac{1}{4}\left\|\Im_x(\xi u_{t_1t_1}(x,t_1,T_2))\right\|^2_{L^2(Q^1_{s_1})}.
\end{aligned}\tag{52}$$

Consider the elementary inequality

$$a\left\|u_x(x,t_1,T_2)\right\|^2_{L^2_\rho(Q^1_{s_1})}\le a\left\|u_x\right\|^2_{L^2_\rho(Q_s)}+a\left\|u_{xt_1}\right\|^2_{L^2_\rho(Q_s)}.\tag{53}$$

Applying the Poincare-Friedriks inequality to the first term on the right-hand side of (53), then (52) becomes

$$\begin{aligned}
&-(xu_x(x,t_1,T_2),\Im_x(\xi u_{t_1t_1}(x,t_1,T_2)))_{L^2_\rho(Q^1_{s_1})}\\
&\qquad\le(c_1a+a)\left\|u_{xt_1}\right\|^2_{L^2_\rho(Q_s)}+\frac{1}{4}\left\|\Im_x(\xi u_{t_1t_1}(x,t_1,T_2))\right\|^2_{L^2(Q^1_{s_1})}.
\end{aligned}\tag{54}$$

We also have

$$\begin{aligned}
&-(xu_x(x,T_1,t_2),\Im_x(\xi u_{t_2t_2}(x,T_1,t_2)))_{L^2_\rho(Q^2_{s_2})}\\
&\qquad\le(c_2a+a)\left\|u_{xt_2}\right\|^2_{L^2_\rho(Q_s)}+\frac{1}{4}\left\|\Im_x(\xi u_{t_2t_2}(x,T_1,t_2))\right\|^2_{L^2(Q^2_{s_2})}.
\end{aligned}\tag{55}$$

Combining the equality (47), the estimates (48)–(51), (54) and (55), we obtain

$$\begin{aligned}
&\left\|u_{t_1t_1}(x,t_1,T_2)\right\|^2_{L^2_\rho(Q^1_{s_1})}+\left\|\Im_x(\xi u_{t_1t_1}(x,t_1,T_2))\right\|^2_{L^2(Q^1_{s_1})}\\
&\qquad+\left\|u_{xt_1}(x,t_1,T_2)\right\|^2_{L^2_\rho(Q^1_{s_1})}+\left\|u_{t_2t_2}(x,T_1,t_2)\right\|^2_{L^2_\rho(Q^2_{s_2})}\\
&\qquad+\left\|u_{xt_2}(x,T_1,t_2)\right\|^2_{L^2_\rho(Q^2_{s_2})}+\left\|\Im_x(\xi u_{t_2t_2}(r,T_1,t_2))\right\|^2_{L^2(Q^2_{s_2})}\\
&\le c\left\{\left\|u_{xt_1}\right\|^2_{L^2_\rho(Q_s)}+\left\|\Im_x(\xi u_{t_1t_1})\right\|^2_{L^2(Q_s)}+\left\|u_{t_2x}\right\|^2_{L^2_\rho(Q_s)}+\left\|\Im_x(\xi u_{t_2t_2})\right\|^2_{L^2(Q_s)}\right\},
\end{aligned}\tag{56}$$

where

$$c = \max\{8a + 4c_1 a,\ 8a + 4c_2 a,\ 4\}.$$

It results from (56) that

$$\begin{aligned} \|u_{t_1t_1}(x,t_1,T_2)\|^2_{L^2_\rho(Q^1_{s_1})} &+ \|\Im_x(\xi u_{t_1t_1}(x,t_1,T_2))\|^2_{L^2(Q^1_{s_1})} \\ &+ \|u_{xt_1}(x,t_1,T_2)\|^2_{L^2_\rho(Q^1_{s_1})} + \|u_{t_2t_2}(x,T_1,t_2)\|^2_{L^2_\rho(Q^2_{s_2})} \\ &+ \|u_{xt_2}(x,T_1,t_2)\|^2_{L^2_\rho(Q^2_{s_2})} + \|\Im_x(\xi u_{t_2t_2}(r,T_1,t_2))\|^2_{L^2(Q^2_{s_2})} \leq 0, \quad (57) \end{aligned}$$

thanks to Gronwall's lemma 2.2. Hence (57) implies that $\omega = 0$ almost everywhere on Q. This achieves the proof of Theorem 3.1. ∎

THEOREM 3.3. *The range $R(L)$ of the operator L coincides with F.*

Proof. Suppose that, for some $W = (\omega, w_1, w_2) \in R(L)^\perp$,

$$(\mathcal{L}u,\omega)_{L^2_\rho(Q)} + (\ell_1 u, w_1)_{V^{1,0}_\rho(Q_2)} + (\ell_2 u, w_2)_{V^{1,0}_\rho(Q_1)} = 0. \quad (58)$$

We must prove that $W = 0$.

Let

$$D_0(L) = \{\, u \in D(L) : \ell_1 u = \ell_2 u = 0 \,\}$$

Putting $u \in D_0(L)$ in (58), we get $(\mathcal{L}u,\omega)_{L^2_\rho(Q)} = 0$, $u \in D_0(L)$. Hence, by virtue of Theorem 3.1 it follows that $\omega = 0$. Thus (58) becomes

$$(\ell_1 u, w_1)_{V^{1,1}_\rho(Q_2)} + (\ell_2 u, w_2)_{V^{1,1}_\rho(Q_1)} = 0. \quad (59)$$

$\ell_1 u$, and $\ell_2 u$ are independent, and the ranges of the operators ℓ_1 and ℓ_2 are everywhere dense in the spaces $V^{1,1}_\rho(Q_2)$, and $V^{1,1}_\rho(Q_1)$, respectively. Hence the equality (59) implies that $w_1 = w_2 = 0$. Consequently $W = 0$. This ends the proof of Theorem 3.3. ∎

REFERENCES

[**1**] N. E. Benouar and Yurchuk, *Mixed problem with an integral condition for parabolic equations with the Bessel operator*, Differ. Uravn. **27**, 12 (1991), 2094–2098.

[**2**] A. Bouziani, *Mixed problems with integral conditions for a certain parabolic equation*, J. Appl. Math. Stochastic Anal. **9**, 3, (1996), 323–330.

[**3**] A. Bouziani, *On a third order parabolic equation with a nonlocal boundary condition*, to appear in J. Appl. Math. Stochastic Anal.

[**4**] A. Bouziani and N. E. Benouar, *Problème mixte avec conditions intégrales pour une classe d'équations hyperboliques*, Bull. Belgian Math. Soc. Sim. Stev. **3** (1996), 137–145.

[**5**] A. Bouziani and N. E. Benouar, *Problème mixte avec conditions intégrales pour une classe d'équations paraboliques*, C. R. Acad. Sci. Paris **321**, Série I, (1995), 1177–1182.

[**6**] J. R. Cannon, *The solution of heat equation subject to the specification of energy*, Quart. Appl. Math. **21** (1963), 155–160.

[**7**] J. R. Cannon and J.van der Hoek, *The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy*, Boll. Univ. Math. Ital. Suppl. **1** (1981), 253–282.

[**8**] J. R. Cannon and J.van der Hoek, *An implicit finite difference schem for the diffusion of mass in a portion of the domain*, in: *Numerical solutions of partial differential equations*, (J. Noye, ed), North-Holland, Amsterdam (1982), 527–539.

[**9**] J. R. Cannon, S. P. Esteva and J. van der Hoek, *A Galerkin procedure for the diffusion equation subject to the specification of mass*, SIAM Numer. Anal. **24** (1987), 499–515.

[**10**] N. I. Kamynin, *A boundary value problem in the theory of heat conduction with non classical boundary condition*, Th., Vychisl., Mat., Fiz. **4**, 6 (1964), 1006–1024.

[**11**] A. V. Kartynnik, *Three point boundary value problem with an integral space variables conditions for second order parabolic equations*, Differ. Uravn. **26** (1990), 1568–1575.

[**12**] L. Garding, *Cauchy Problem for Hyperbolic Equations*, University of Chicago, Lecture notes, 1957.

[**13**] S. Mesloub and A. Bouziani, *Mixed problem with a weighted integral condition for a parabolic equation with Bessel operator*, to appear in J. Appl. Math. Stochastic Anal.

[**14**] A. A. Samarskii, *Some problems in differential equations theory*, Differents. Uravn. **16**, 11 (1980), 1925–1935.

[**15**] P. Shi, *Weak solution to an evolution problem with a nonlocal constraint*, SIAM J. Math. Anal. **24** (1993), 46–58.

[**16**] N. I. Yurchuk, *Mixed problem with an integral condition for certain parabolic equations*, Differ. Uravn. **22**, 12 (1986), 2117–2126.

(received 20.01.2001, in revised form 15.12.2001)

Department of Mathematics, University of Tebessa, 12002 Tebessa, Algeria

Department of Mathematics, University of Constantine, 25000 Constantine, Algeria

E-mail: mesloub@hotmail.com