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THE INVARIANT SUBSPACE LATTICE OF AN ALGEBRAIC OPERATOR

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Abstract. The main objecr in this work is to analyze the invariant subspace lattice of an algebraic operator.

Let X be a Banach space. By $B(X)$ we mean the algebra of all bounded linear operators on X. A subspace $\mathcal{M}$ is *invariant* under an operator A if $Ax \in \mathcal{M}$ for every $x \in \mathcal{M}$. The collection of all subspaces of X invariant under A is denoted by $\operatorname{Lat} A$. The lattice $\operatorname{Lat} A$ is the direct sum of sublattices $\operatorname{Lat} A_1$ and $\operatorname{Lat} A_2$ if each $\mathcal{M} \in \operatorname{Lat} A$ is uniquely representable in the form $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_i \in \operatorname{Lat} A_i$, $i = 1, 2$. Notation: $\operatorname{Lat} A = \operatorname{Lat} A_1 \oplus \operatorname{Lat} A_2$.

An operator $A \in B(X)$ is *algebraic* if there exists a polynomial p other than 0 such that $p(A) = 0$. Let us consider the factorization $p(z) = \prod_{j=1}^{k}(z - \lambda_j)^{n_j}$ with the λ_j's mutually distinct. Then the spectrum of A is $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Every operator on a finite-dimensional space is algebraic. The algebraic operators on infinte-dimensional spaces can be characterized in terms of their invariant subspaces. An operator is algebraic if and only if the union of its finite-dimensional invariant subspaces is X.

The main motivation and the basis for the work in this paper are the results obtained by Brickman and Filmore in [1].

PROPOSITION 1. *Let A_1 and A_2 be algebraic operators with minimal polynomials p_1 and p_2 on the Banach spaces X_1 and X_2, respectively. Then*

$$\operatorname{Lat}(A_1 \oplus A_2) = \operatorname{Lat} A_1 \oplus \operatorname{Lat} A_2 \iff (p_1, p_2) = 1.$$

Proof. In general case, for every operator $A_i \in B(X_i)$, $i = 1, 2$, $\operatorname{Lat} A_1 \oplus \operatorname{Lat} A_2 \subset \operatorname{Lat}(A_1 \oplus A_2)$ holds. For the inverse inclusion, let $(p_1, p_2) = 1$. We must

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show that $\mathcal{M} \in \operatorname{Lat}(A_1 \oplus A_2)$ implies that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_i \in \operatorname{Lat} A_i$, $i = 1, 2$. Given $\mathcal{M} \in \operatorname{Lat}(A_1 \oplus A_2)$ let $\mathcal{M}_1 \oplus \{0\} = (1 \oplus 0)\mathcal{M}$ and $\{0\} \oplus \mathcal{M}_2 = (0 \oplus 1)\mathcal{M}$. Obviously $\mathcal{M} \subset \mathcal{M}_1 \oplus \mathcal{M}_2$. To prove that $\mathcal{M}_1 \oplus \mathcal{M}_2 \subset \mathcal{M}$, let r_1 and r_2 be polynomials such taht $r_1 p_1 + r_2 p_2 = 1$, and let $q_2 = r_2 p_2$. We have $q_2(A_1) = 1 - r_1(A_1)p_1(A_1) = 1$ so $q_2(A_1 \oplus A_2) = q_2(A_1) \oplus q_2(A_2) = 1 \oplus 0$. Then $\mathcal{M}_1 \oplus \{0\} = (1 \oplus 0)\mathcal{M} = q_2(A_1 \oplus A_2)\mathcal{M} \subset \mathcal{M}$. Similarly $\{0\} \oplus \mathcal{M}_2 \subset \mathcal{M}$. Thus $\mathcal{M}_1 \oplus \mathcal{M}_2 \subset \mathcal{M}$, and it follows that $\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{M}$. Clearly $\mathcal{M}_i \in \operatorname{Lat} A_i$ for $i = 1, 2$.

Conversely, suppose that p_1 and p_2 have a common prime factor q, i.e., $p_1 = qr_1$ and $p_2 = qr_2$. Deviding q by its largest coefficient, we can assume that the leading coefficient of q is 1. If $q(A_i)x_i \neq 0$ for each $x_1 \in X_1$ and $x_2 \in X_2$, then r_1 and r_2 are the minimal polynomials of A_1 and A_2. That means that there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that $q(A_i)x_i = 0$ for $i = 1, 2$.

Let $\mathcal{M} = \{\, r(A_1)x_1 \oplus r(A_2)x_2 : \deg r < \deg q$, the leading coefficient of r is $1 \,\}$. It is easy to verify that $\mathcal{M}$ is a linear manifold in $X_1 \oplus X_2$. If (y_n) is a sequence in $\mathcal{M}$ and $y_n \to y$, then $y \in \mathcal{M}$. For, suppose that, for each n, r_n is a non-zero polynomial of degree less than $\deg q$ such that $y_n = r_n(A_1)x_1 \oplus r_n(A_2)x_2$. Each coefficient of r_n has absolute value at most 1 and at least one coefficient has absolute value equal to 1. Then a subsequence of (r_n) converges coefficient-wise to a polynomial r of degree less than $\deg q$; r is not 0, since at least one of its coefficients has modulus 1. Re-label so that (r_n) is such a subsequence. Then $(r_n(A_1)x_1 \oplus r_n(A_2)x_2)$ converges to $r(A_1)x_1 \oplus r(A_2)x_2$. Thus $\mathcal{M}$ is closed, and so $\mathcal{M}$ is a subspace of $X_1 \oplus X_2$.

Now, let $x \in \mathcal{M}$, $x = r(A_1)x_1 \oplus r(A_2)x_2$, $\deg r < \deg q$. Then $(A_1 \oplus A_2)x = r_1(A_1)x_1 \oplus r_1(A_2)x_2$, where $r_1(z) = zr(z)$. The leading coefficient of r_1 is 1 and $\deg r_1 = \deg r + 1 \leqslant \deg q$. Since $\deg(r_1 - q) < \deg r_1 \leqslant \deg q$, we have $(A_1 \oplus A_2)x = r_1(A_1)x_1 \oplus r_1(A_2)x_2 = (r_1 - q)(A_1)x_1 \oplus (r_1 - q)(A_2)x_2 \in \mathcal{M}$. Thus $\mathcal{M} \in \operatorname{Lat}(A_1 \oplus A_2)$. If $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_i \in \operatorname{Lat} A_i$, $i = 1, 2$, we shall have $r(A_1)x_1 \oplus 0 \in \mathcal{M}$, $r(A_2)x_2 = 0$, and therefore $r = 0$ (because q is prime). Thus $\mathcal{M}_1 = \{0\}$ and similarly $\mathcal{M}_2 = \{0\}$. Hence $\mathcal{M} = \{0\}$, a contradiction. ■

PROPOSITION 2. *Let A be an algebraic operator on X with primary summands A_i. Then* $\operatorname{Lat} A = \bigoplus_i \operatorname{Lat} A_i$.

Proof. Using the induction and previous proposition gives the result. ■

PROPOSITION 3. *Let A be a prime algebraic operator, i.e., $(A - \lambda)^n = 0$. If $\mathcal{M} \in \operatorname{Lat} A$, then $(A - \lambda)\mathcal{M} \subset \mathcal{M}_-$, where $\mathcal{M}_- = \bigvee\{\, \mathcal{M}' : \mathcal{M}' \in \operatorname{Lat} A, \mathcal{M}' \subset \mathcal{M}, \mathcal{M}' \neq \mathcal{M} \,\}$.*

Proof. Suppose $\mathcal{M}_-$ is a proper subset of $\mathcal{M}$. Let $\hat{A}$ be the quotient operator on $X \setminus \mathcal{M}_-$. Then $\hat{\mathcal{M}} = \mathcal{M} \setminus \mathcal{M}_-$ is a minimal non-zero element of $\operatorname{Lat} \hat{A}$. But $(\hat{A} - \lambda)\hat{\mathcal{M}} \subset \hat{\mathcal{M}}$ and $(\hat{A} - \lambda)\hat{\mathcal{M}} \in \operatorname{Lat} \hat{A}$. Since $\hat{A} - \lambda$ is nilpotent, $(\hat{A} - \lambda)\hat{\mathcal{M}} \neq \hat{\mathcal{M}}$. Hence $\hat{A} - \lambda$ annihilates $\hat{\mathcal{M}}$, i.e., $(A - \lambda)\mathcal{M} \subset \mathcal{M}_-$. ■

PROPOSITION 4. *Let A be an algebraic operator and* $\operatorname{Lat} A \subset \operatorname{Lat} B$. *Then B is algebraic.*

Proof. The proof is similar to the proof of Theorem 4.8 [2]. Let A be an algebraic operator with the minimal polynomial p_A, $p_A(z) = \prod_{j=1}^{k}(z-\lambda_j)^{n_j}$, and $\operatorname{Lat} A \subset \operatorname{Lat} B$. Let $x \in X$ and $\mathcal{M}_x = \bigvee\{A^n x\}$. We have $\dim \mathcal{M}_x \leqslant \deg p_A$ and $\mathcal{M}_x \in \operatorname{Lat} A \subset \operatorname{Lat} B$. Thus there exists q_x, polynomial in B, such that $q_x(B)x = 0$. Let F_k denote the set of all vectors x such that $q_x(B)x = 0$, $\deg q_x \leqslant k$. Then $X = \bigcup_{k=1}^{\infty} F_k$. Let (x_n) is a sequence in F_k, $x_n \to x$. We shall prove that $x \in F_k$. For each n, $q_{x_n}(B)x_n = 0$, $\deg q_{x_n} \leqslant k$. We can assume that each coefficient of q_{x_n} has absolute value at most 1 and at least one coefficient has absolute value equal to 1. Then a subsequence of (q_{x_n}) converges coefficient-wise to a polynomial q; $\deg q \leqslant k$; q is not 0. Since

$$\begin{aligned}\|q(B)x\| &= \|q(B)x - q_{x_n}(B)x\| + \|q_{x_n}(B)x_n - q_{x_n}(B)x\| \\ &\leqslant \|q(B) - q_{x_n}(B)\|\,\|x\| + \|q_{x_n}(B)\|\,\|x_n - x\| \to 0,\end{aligned}$$

$q(B)x = 0$. Thus each F_k is closed. By the Baire category theorem, there exists k_0 such that the interior of F_{k_0} is not empty. For $x_0 \in \operatorname{int} F_{k_0}$, there exists $r > 0$ such that $B(x_0, r) = \{x \in X : \|x - x_0\| < r\} \subset F_{k_0}$. If $y \in B(0,r)$, then $y = x - x_0$ for some $x \in B(x_0, r)$. Then $q_x(B)q_{x_0}(B)y = 0$ and $\deg q_x q_{x_0} \leqslant 2k_0$ and so $B(0,r) \subset F_{2k_0}$. Since $q_x(B)q_{x_0}(B)(\alpha y) = \alpha q_x(B)q_{x_0}(B)(y)$, it follows that $X = F_{2k_0}$.

Let $x \in X$ and n_x is the degree of the lowest-degree non-zero polynomial of degree n such that $q(B)x = 0$, where $n = \max\{n_x\}$. We claim that $q(B)y = 0$ for all y. Given y, let $\mathcal{M} = \bigvee_{j=0}^{\infty}\{B^j x, B^j y\}$. Then $\mathcal{M} \in \operatorname{Lat} B$ and $\dim \mathcal{M} \leqslant 2n$. Let r be the minimal polynomial of $B|\mathcal{M}$, then q devides r. Moreover, $\deg q = \deg r$. Thus q is a multiple of r, and $q(B|\mathcal{M}) = 0$. Then $q(B)y = 0$. ∎

In the subsequent work we prove that every commutative set of algebraic operators is triangularizable. First we give some definitions.

DEFINITIONS. A collection of bounded linear operaors on a complex Banach space is *triangularizable* if there is a chain of subspaces which is maximal as a subspace chain and which consists of common invariant subspaces for the operators in the collection.

A collection of properties is said to be *inherited by quotients* if for every collection of quotients of a set satisfying the properties also satisfies the same properties.

THE TRIANGULARIZATION LEMMA. *Let P be a collection of properties inherited by quotients. If every set of operators on a space of dimension greater than one, which satisfies P, has a non-trivial invariant subspace, then every such set is trinagularizable.*

PROPOSITION 5. *Every commutative set of algebraic operators is triangularizable.*

Proof. Let $\mathcal{A}$ is a commutative set of algebraic operators. If every operator in $\mathcal{A}$ is a multiple of the identity, the result is trivial, so assume that $A \in \mathcal{A}$ is not a multiple of the identity. If p is a minimal polynomial of A, let $p(z)=\prod_{j=1}^{k}(z-\lambda_j)^{n_j}$. Then $\sigma(A)=\{\lambda_1,\lambda_2,\dots,\lambda_k\}$. Each $\lambda_j\in\Pi_0(A)$. For at least one $\lambda\in\sigma(A)$, the $\mathrm{cl}(A-\lambda)X\neq X$ (otherwise $\mathrm{cl}\big(\prod_{j=1}^{k}(A-\lambda_j)X\big)=X$). $\mathrm{Ker}(A-\lambda)$ is a nontrivial invariant subspace for A. Let $f\in\mathrm{Ker}(A-\lambda)$ and $B\in\mathcal{A}$. Then $(A-\lambda)Bf=B(A-\lambda)f=0$, i.e., $Bf\in\mathrm{Ker}(A-\lambda)$. It follows that the kernel of $A-\lambda$ is invariant under $\mathcal{A}$. The Triangularization Lemma completes the proof. ■

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