МАТЕМАТИЧКИ ВЕСНИК
56 (2004), 91–104

UDK 517.927
оригинални научни рад
research paper

ON UNIFORM CONVERGENCE ON CLOSED INTERVALS OF SPECTRAL EXPANSIONS AND THEIR DERIVATIVES, FOR FUNCTIONS FROM $W_p^{(1)}$

Nebojša L. Lažetić

Abstract. We consider the global uniform convergence of spectral expansions and their derivatives, $\sum_{n=1}^{\infty} f_n\, u_n^{(j)}(x)$, $(j = 0,1,2)$, arising by an arbitrary one-dimensional self-adjoint Schrödinger operator, defined on a bounded interval $G \subset \mathbb{R}$. We establish the absolute and uniform convergence on $\overline{G}$ of the series, supposing that f belongs to suitable defined subclasses of $W_p^{(1+j)}(G)$ $(1 < p \le 2)$. Also, some convergence rate estimates are obtained.

1. Introduction

Let L be an arbitrary self-adjoint differential operator of second order, defined by the differential expression

$$\mathcal{L}(u)(x) = -u''(x) + q(x)u(x), \qquad x \in G, \tag{1}$$

and the self-adjoint boundary conditions

$$\begin{aligned} \alpha_{10}u(a) + \alpha_{11}u'(a) + \beta_{10}u(b) + \beta_{11}u'(b) = 0,\\ \alpha_{20}u(a) + \alpha_{21}u'(a) + \beta_{20}u(b) + \beta_{21}u'(b) = 0, \end{aligned} \tag{2}$$

where $(\alpha_{i0}, \alpha_{i1}, \beta_{i0}, \beta_{i1}) \in \mathbb{R}^4$ $(i = 1,2)$ are linearly independent vectors, $G = (a,b)$ is a bounded interval of the real axis $\mathbb{R}$, and $q(x) \in L_1(G)$ is a real function. Denote by $D(L) \subset L_2(G)$ the domain of the operator $L : h(x) \in D(L)$ if functions $h(x), h'(x)$ are absolutely continuous on $\overline{G}$, $\mathcal{L}(h)(x) \in L_2(G)$, and $h(x)$ satisfies the conditions (2). If $h(x) \in D(L)$, then $L(h)(x) \stackrel{\text{def}}{=} \mathcal{L}(h)(x)$ (see [1, §18]). Spectrum of L is discrete. We can suppose, with no loss of generality, that L is a positive operator.

Let $\{u_n(x)\}_{n=1}^{\infty}$ be complete (in $L_2(G)$) and orthonormal system of eigenfunctions of L, and let $\{\lambda_n\}_{n=1}^{\infty}$ be the corresponding system of positive eigenvalues,

---

AMS Subject Classification: 34 L 10, 47 E 05

Keywords and phrases: Spectral expansion, uniform convergence, Schrödinger operator

This paper is supported by Ministry of Sciences, Technology and Development of Republic of Serbia (project #1856–*Structures of functional analysis and differential equations*).

enumerated in non-decreasing order. (By definition, $u_n(x)$ belongs to $D(L)$ and satisfies the differential equation

$$-u_n''(x) + q(x)u_n(x) = \lambda_n u_n(x) \tag{3}$$

almost everywhere in G.) If $f(x) \in L_1(x)$ and $\mu > 2$, we can form the partial sum of order μ:

$$\sigma_\mu(x, f) \overset{\text{def}}{=} \sum_{n=1}^{[\mu]} f_n u_n(x), \qquad f_n \overset{\text{def}}{=} \int_a^b f(x)u_n(x)\,dx.$$

($[\mu]$ denotes the entire part of μ.) The problem of behavior of functions $\sigma_\mu^{(j)}(x, f)$ on subsets $K \subseteq \overline{G}$, as $\mu \to +\infty$, is the classical one. (Note, $h^{(0)}(x) \overset{\text{def}}{=} h(x)$.) It is still actual and important, especially in the case of a non-self-adjoint operator L,by itself. The importance of the problem also follows from its relationship with the Fourier method for solving mixed boundary problems for one-dimensional hyperbolic (or parabolic) equations of second order, containing different classes of (non-) self-adjoint boundary conditions.

One of the most fruitful approaches to the problem is so-called "equiconvergence approach": one studies the behavior on K of the difference $\sigma_\mu^{(j)}(x, f) - S_\mu^{(j)}(x, f)$, as $\mu \to +\infty$, where $S_\mu(x, f)$ is the corresponding partial sum of the trigonometrical Fourier series of function $f(x)$. When $j = 0$ and $K \subset G$ is a compact set, the problem has been completely solved by V. A. Il'in (see [2–5]) for a large class of non-self-adjoint boundary conditions. The L_p-equiconvergence on $K \subseteq \overline{G}$ $(1 \le p \le +\infty, j = 0)$ was deeply studied by I. S. Lomov in papers [6–8]. A very extensive review of different results (case $j = 0$) can be found in [12]. As far as the convergence of derivatives of the spectral expansions concerned, it was less studied.

In this paper we consider the uniform convergence on the whole $\overline{G}$. A "naive" direct method is used:we point out some classes of functions $f(x)$ such that the equalities

$$f^{(j)}(x) = \sum_{n=1}^{\infty} f_n u_n^{(j)}(x), \qquad x \in \overline{G}, \quad j = 0, 1, 2,$$

are valid, where series $\sum_{n=1}^{\infty} |f_n u_n^{(j)}(x)|$ converges uniformly on $\overline{G}$. (We (will) say that the series converges "absolutely and uniformly" on $\overline{G}$.) Also, we establish some uniform (on $\overline{G}$) asymptotic estimates for the differences $f^{(j)}(x) - \sigma_\mu^{(j)}(x, f)$,as $\mu \to +\infty$.

2. Main results

Let $AC(\overline{G})$ be the class of (real-valued) absolutely continuous functions on the closed interval $\overline{G} = [a, b]$, and let $BV(\overline{G})$ be the class of functions having the bounded variation on that interval. By $W_p^{(k)}(G)$ we denote the set of functions

$h(x)$ such that $h(x), h'(x), \ldots, h^{(k-2)}(x)$ are continuously differentiable functions on $\overline{G}, h^{(k-1)}(x) \in AC(\overline{G})$, and $h^{(k)}(x) \in L_p(G)(1 \leq p < +\infty, k \in \mathbb{N})$. We say that function $h(x) \in L_p(G)$ belongs to the class $H_p^\alpha(G)(0 < \alpha \leq 1)$ if there exists a constant $C(h) > 0$ such that

$$\|h(x+t) - h(t)\|_{L_p(G_{|t|})} \leq C(h)|t|^\alpha$$

for every $t \in ((a-b)/2, (b-a)/2)$,where $G_{|t|} \stackrel{\text{def}}{=} (a+|t|, b-|t|)$.

We can now state our results.

Theorem 1. *Let $q(x) \in L_1(G)$, $f(x) \in W_p^{(1)}(G)$ $(1 < p \leq 2)$ and $f(a) = 0 = f(b)$. Then:*

(a) for $x \in \overline{G}$ the equality

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x) \tag{4}$$

is valid, and the series is absolutely and uniformly convergent on $\overline{G}$;

(b) the estimate

$$\max_{x \in \overline{G}} |f(x) - \sigma_\mu(x, f)| = o\left(\frac{1}{\mu^{1-1/p}}\right), \qquad \mu \to +\infty. \tag{5}$$

holds.

Theorem 2. *Let $q(x) \in L_1(G)$ and $f(x) \in D(L)$. Then for every $x \in \overline{G}$ we have the equalities*

$$f^{(j)}(x) = \sum_{n=1}^{\infty} f_n u_n^{(j)}(x), \qquad j = 0, 1, \tag{6}$$

the series (6) being absolutely and uniformly convergent on $\overline{G}$. Also, the following estimates are valid

$$\max_{x \in \overline{G}} |f^{(j)}(x) - \sigma_\mu^{(j)}(x, f)| = o\left(\frac{1}{\mu^{3/2-j}}\right), \qquad j = 0, 1. \tag{7}$$

Theorem 3. *Let $q(x) \in W_1^{(1)}(G)$ and $f(x) \in D(L) \cap W_1^{(3)}(G)$. If $\mathcal{L}(f)(x) \in W_p^{(1)}(G)$ $(1 < p \leq 2)$ and $\mathcal{L}(f)(a) = 0 = \mathcal{L}(f)(b)$, then:*

(a) the equalities

$$f^{(j)}(x) = \sum_{n=1}^{\infty} f_n u_n^{(j)}(x), \qquad j = 0, 1, 2, \tag{8}$$

hold on $\overline{G}$, and the series (8) converge absolutely and uniformly on $\overline{G}$;

(b) we have the estimates

$$\max_{x \in \overline{G}} |f^{(j)}(x) - \sigma_\mu^{(j)}(x, f)| = o\left(\frac{1}{\mu^{3-j-1/p}}\right), \qquad j = 0, 1, 2. \tag{9}$$

Remark 1. In some special cases estimate (5) can be improved. For example, if $f'(x) \in BV(\overline{G})$, then

$$\max_{x\in\overline{G}} |f(x) - \sigma_\mu(x,f)| = O\left(\frac{1}{\mu}\right).$$

Also, if $f'(x) \in L_\infty(G) \cap H_1^\alpha(G)$ $(0 < \alpha \le 1)$, then

$$\max_{x\in\overline{G}} |f(x) - \sigma_\mu(x,f)| = O\left(\frac{1}{\mu^\alpha}\right) + o\left(\frac{1}{\mu^{1/2}}\right).$$

These estimates were obtained in [13].

Remark 2. Propositions of Theorem 2 were actually proved in papers [14] and [13]. In order to keep the completeness of exposition of all our results obtained in the considered classes of smoothness for $q(x)$ and $f(x)$, we cite the propositions here. Note, the uniform convergence on $\overline{G}$ of the series $\sum_{n=1}^\infty f_n u_n(x)$, for every function $f(x) \in D(L)$, is a very well known fact (see [1, p.90]).

Related to Theorem 2, we can add the following. Let $q(x) \in L_1(G)$, $f(x) \in W_1^{(2)}(G)$, $\mathcal{L}(f)(x) \in L_p(G)$ $(1 < p \le 2)$, and $f^{(j)}(a) = 0 = f^{(j)}(b)$, $j = 0, 1$. Then the first proposition of the theorem holds, and the estimates

$$\max_{x\in\overline{G}} |f^{(j)}(x) - \sigma_\mu^{(j)}(x,f)| = o\left(\frac{1}{\mu^{2-j-1/p}}\right), \qquad j = 0, 1, \tag{10}$$

are valid, instead of estimates (7).

Remark 3. The estimates (9) can be also improved in some cases. If $\mathcal{L}(f)'(x) \in BV(G)$, supposing the other conditions of Theorem 3 are satisfied, then

$$\max_{x\in\overline{G}} |f^{(j)}(x) - \sigma_\mu^{(j)}(x,f)| = O\left(\frac{1}{\mu^{3-j}}\right), \qquad j = 0, 1, 2.$$

Further, if $\mathcal{L}(f)'(x) \in L_\infty(G) \cap H_1^\alpha(G)$, the following estimates hold:

$$\max_{x\in\overline{G}} |f^{(j)}(x) - \sigma_\mu^{(j)}(x,f)| = O\left(\frac{1}{\mu^{2-j+\alpha}}\right) + o\left(\frac{1}{\mu^{5/2-j}}\right), \quad j = 0, 1, 2,$$

(see paper [13]).

Remark 4. Suppose that the coefficients of boundary conditions (2) satisfy the relation

$$\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11} \ne 0, \tag{11}$$

as in the case of separated boundary conditions, for example. Then assumptions $f(a) = 0 = f(b)$ (in Theorem 1) and $\mathcal{L}(f)(a) = 0 = \mathcal{L}(f)(b)$ (in Theorem 3) can be dropped.

If one of the boundary conditions (2) has the form $\alpha_{j1}u'(a) + \beta_{j1}u'(b) = 0$, then $f(a) = 0 = f(b)$ (in Theorem 1) can be replaced by $\beta_{j1}f(a) + \alpha_{j1}f(b) = 0$, and in Theorem 3 we can suppose that $\beta_{j1}\mathcal{L}(f)(a) + \alpha_{j1}\mathcal{L}(f)(b) = 0$ instead of $\mathcal{L}(f)(a) = 0 = \mathcal{L}(f)(b)$.

Remark 5. In paper [9] I. S. Lomov has proved that the assertion of proposition (a) of Theorem 1 holds in the case of biorthogonal series $\sum_{n=1}^{\infty}(f, v_k)u_k(x)$, for $f(x) \in W_2^{(1)}(G)$, generated by a non-self-adjoint Schrödinger operator L, with a complex-valued potential $q(x) \in L_1(G)$. The operator L is defined on $G = (0, 1)$ by multi-point boundary conditions.

Proofs of Theorems 1 and 3 are based on suitable estimates for the Fourier coefficients of $f(x)$. These estimates are obtained by the known asymptotics for the eigenfunctions and eigenvalues of the operator L. On the other hand, the propositions concerned with Theorem 2 can be proved by using only some uniform estimates for the eigenfunctions and their derivatives, and an estimate for a number of the eigenvalues (see the next section).

3. Auxiliary results

In our proofs we will use the following results.

Proposition 1 [15]. *If $q(x) \in L_1(G)$, then there exists a constant $C_0 > 0$, independent of $n \in \mathbb{N}$, such that*

$$\max_{x\in\overline{G}} |u_n(x)| \le C_0, \qquad n \in \mathbb{N}. \tag{12}$$

Proposition 2 [15], [16]. *If $q(x) \in L_1(G)$, then there exists a constant $A > 0$ such that*

$$\sum_{t\le\sqrt{\lambda_n}\le t+1} 1 \le A, \tag{13}$$

for each $t \ge 0$, where A does not depend on t.

Proposition 3 [17]. (a) *Let $q(x) \in L_1(G)$. There is a constant $C_1 > 0$, not depending on $n \in \mathbb{N}$, such that the following estimates are valid*:

$$\max_{x\in\overline{G}} |u_n'(x)| \le C_1 \cdot (\sqrt{\lambda_n} + 1), \qquad n \in \mathbb{N}. \tag{14}$$

(b) *Let $q(x) \in C(\overline{G})$. Then the eigenfunctions $u_n(x)$ have continuous second derivative, satisfy the equation* (3) *everywhere on G, and there exists a constant $C_2 > 0$, independent on $n \in \mathbb{N}$, such that*

$$\max_{x\in\overline{G}} |u_n''(x)| \le C_2 \cdot (\lambda_n + 1), \qquad n \in \mathbb{N}. \tag{15}$$

We will also need the following known asymptotics for the eigenfunctions and eigenvalues of the (self-adjoint) operator L. Let $q(x) \in L_1(G)$, where $G = (-1, 1)$. There is a number $n_0 \in \mathbb{N}$ such that for every $n > n_0$ the equalities

$$u_n(x) = A_n(\cos\sqrt{\lambda_n}x)\left(1 + \frac{r_{1n}(x)}{\sqrt{\lambda_n}}\right) + B_n(\sin\sqrt{\lambda_n}x)\left(1 + \frac{r_{2n}(x)}{\sqrt{\lambda_n}}\right), \tag{16}$$

$$u_n'(x) = -A_n\sqrt{\lambda_n}(\sin\sqrt{\lambda_n}x)\left(1 + \frac{r_{3n}(x)}{\sqrt{\lambda_n}}\right) + B_n\sqrt{\lambda_n}(\cos\sqrt{\lambda_n}x)\left(1 + \frac{r_{4n}(x)}{\sqrt{\lambda_n}}\right) \tag{17}$$

are valid on $[-1,1]$. Here $|A_n|^2 + |B_n|^2 \in [c^{-1}, c]$ for some constant $c > 0$, and there is a constant $D > 0$ such that for every $n > n_0$ we have

$$\max_{x\in\overline{G}}\{|r_{in}(x)||i = 1, 2, 3, 4\} \le D.$$

Also, for every $n > n_0$ the following holds:

$$\sqrt{\lambda_n} = n\pi + \gamma + \frac{\rho_n}{n^{\beta-1}}, \tag{18}$$

where $\beta = 2$ if the boundary conditions (2) satisfy $\theta_0^2 - 4\theta_{-1}\theta_1 \neq 0$, and $\beta = 3/2$ if $\theta_0^2 - 4\theta_{-1}\theta_1 = 0$ (see [1, pp. 66–67, 74], [10], [11]). Here $\gamma > 0$ is a constant, and $\{\rho_n\}_{n=n_0}^{\infty}$ is a bounded sequence: $|\rho_n| \le \rho$.

4. Proof of Theorem 1

In order to avoid technicalities, we will prove the theorem supposing that $G = (-1, 1)$. Transition from the interval $(-1, 1)$ to an arbitrary interval (a, b) can be realized by the following change of variable:

$$x = \frac{b-a}{2}t + \frac{a+b}{2}, \quad -1 \le t \le 1.$$

Consequently, in formulae (16)–(17) one should put $(2x - a - b)/(b - a)$ instead of x, and π in equality (18) should be replaced by $\pi/(b - a)$.

An estimate for the Fourier coefficients. The first step in the proof is to establish an appropriate estimate for the Fourier coefficients $f_n (n > n_0)$.

Let λ_n be an arbitrary eigenvalue such that $n > n_0$. Using the equation (3), the integration by parts, and assumption $f(-1) = 0 = f(1)$, we obtain the equality

$$f_n = \frac{1}{\lambda_n} \cdot \left(\int_{-1}^{1} f'(x)u_n'(x)\,dx + \int_{-1}^{1} f(x)q(x)u_n(x)\,dx\right). \tag{19}$$

Consider the integral $\int_{-1}^{1} f'(x)u_n'(x)\,dx$. By virtue of equalities (17)–(18) it can be represented in the form

$$\begin{aligned}\int_{-1}^{1} f'(x)u_n'(x)\,dx &= -A_n\sqrt{\lambda_n}\cdot\big(I_1(n,\beta)+I_2(n,\beta)\big)+B_n\sqrt{\lambda_n}\cdot\big(I_3(n,\beta)+I_4(n,\beta)\big)\\ &- A_n\int_{-1}^{1} f'(x)r_{3n}(x)\sin\sqrt{\lambda_n}\,x\,dx + B_n\int_{-1}^{1} f'(x)r_{4n}(x)\cos\sqrt{\lambda_n}\,x\,dx,\end{aligned} \tag{20}$$

where $I_j(n, \beta)$ are defined as follows:

$$I_1(n,\beta) = \int_{-1}^{1} f'(x)\left[\sin n\pi x \cdot \cos\left(\gamma + \frac{\rho_n}{n^{\beta-1}}\right)x\right] dx,$$

$$I_2(n,\beta) = \int_{-1}^{1} f'(x)\left[\cos n\pi x \cdot \sin\left(\gamma + \frac{\rho_n}{n^{\beta-1}}\right)x\right] dx,$$

$$I_3(n,\beta) = \int_{-1}^{1} f'(x)\left[\cos n\pi x \cdot \cos\left(\gamma + \frac{\rho_n}{n^{\beta-1}}\right)x\right] dx,$$

$$I_4(n,\beta) = \int_{-1}^{1} f'(x)\left[\sin n\pi x \cdot \sin\left(\gamma + \frac{\rho_n}{n^{\beta-1}}\right)x\right] dx.$$

Let us introduce $\rho_{n,\beta} \overset{\text{def}}{=} \rho_n n^{1-\beta}$. In the further transformations of integrals $I_j(n,\beta)$ we will use the following fact: for any point $x \in [-1,1]$ there exist numbers $\theta_1(x), \theta_2(x) \in (0,1)$ such that

$$\cos \rho_{n,\beta} x = 1 - \frac{\rho_{n,\beta}^2 x^2}{2} \cos(\rho_{n,\beta}\theta_1(x)x),$$
$$\sin \rho_{n,\beta} x = \rho_{n,\beta} x - \frac{\rho_{n,\beta}^2 x^2}{2} \sin(\rho_{n,\beta}\theta_2(x)x).$$

Now, for the integral $I_1(n,\beta)$ we obtain

$$\begin{aligned} I_1(n,\beta) &= \int_{-1}^{1} \big[f'(x)\cos\gamma x\big] \sin n\pi x \cdot \cos(\rho_{n,\beta}x)\,dx \\ &\qquad - \int_{-1}^{1} \big[f'(x)\sin\gamma x\big] \sin n\pi x \cdot \sin(\rho_{n,\beta}x)\,dx \\ &= \int_{-1}^{1} \big[f'(x)\cos\gamma x\big] \sin n\pi x\,dx - \rho_{n,\beta} \cdot \int_{-1}^{1} \big[f'(x)x\sin\gamma x\big] \sin n\pi x\,dx \\ &\quad - \frac{\rho_{n,\beta}^2}{2} \cdot \int_{-1}^{1} \big[f'(x)x^2\cos\gamma x \cdot \cos(\rho_{n,\beta}\theta_1(x)x)\big] \sin n\pi x\,dx \\ &\qquad + \frac{\rho_{n,\beta}^2}{2} \cdot \int_{-1}^{1} \big[f'(x)x^2\sin\gamma x \cdot \sin(\rho_{n,\beta}\theta_2(x)x)\big] \sin n\pi x\,dx. \end{aligned}$$

If we define functions and Fourier coefficients

$$g_{jc}(x) \overset{\text{def}}{=} f'(x)x^j\cos\gamma x, \quad g_{js}(x) \overset{\text{def}}{=} f'(x)x^j \sin\gamma x, \qquad j = 0,1,2;$$
$$a_n(h) \overset{\text{def}}{=} \int_{-1}^{1} h(x)\cos n\pi x\,dx, \quad b_n(h) \overset{\text{def}}{=} \int_{-1}^{1} h(x)\sin n\pi x\,dx, \qquad n \in \mathbb{N},$$

then from the preceding equalities it follows that

$$I_1(n,\beta) = b_n(g_{0c}) - \frac{\rho_n}{n^{\beta-1}} \cdot b_n(g_{1s}) + O\left(\frac{1}{n^{2(\beta-1)}}\right).$$

Analogously, one can obtain the asymptotic relations

$$I_2(n,\beta) = a_n(g_{0s}) + \frac{\rho_n}{n^{\beta-1}} \cdot a_n(g_{1c}) + O\left(\frac{1}{n^{2(\beta-1)}}\right),$$
$$I_3(n,\beta) = a_n(g_{0c}) - \frac{\rho_n}{n^{\beta-1}} \cdot a_n(g_{1s}) + O\left(\frac{1}{n^{2(\beta-1)}}\right),$$
$$I_4(n,\beta) = b_n(g_{0s}) + \frac{\rho_n}{n^{\beta-1}} \cdot b_n(g_{1c}) + O\left(\frac{1}{n^{2(\beta-1)}}\right),$$

where $|O(n^{2(1-\beta)})| \le \rho^2 \|f'\|_{L_1(G)} n^{2(1-\beta)}$ in all the cases.

Let us return to the equality (20). Applying the preceding representations of the integrals $I_j(n,\beta)$,we can get the following estimate:

$$\left|\int_{-1}^{1} f'(x)u_n'(x)\,dx\right| \le \sqrt{c\lambda_n}\Big[|a_n(g_{0c}+g_{0s})|+|b_n(g_{0c}+g_{0s})| \\ + \frac{\rho}{n^{\beta-1}}(|a_n(g_{1c}+g_{1s})|+|b_n(g_{1c}+g_{1s})|)+\frac{4\rho^2\|f'\|_{L_1(G)}}{n^{2(\beta-1)}}\Big]+2\sqrt{c}\,D\|f'\|_{L_1(G)}. \tag{21}$$

That is why from (19) and (21) it follows, by virtue of estimate (12) and equality (18), that the final (desired) estimate

$$|f_n| \le \frac{D_1}{n}\cdot\big(|a_n(g_{0c}+g_{0s})|+|b_n(g_{0c}+g_{0s})|\big) \\ +\frac{D_2}{n^{\beta}}\cdot\big(|a_n(g_{1c}+g_{1s}))+|b_n(g_{1c}+g_{1s})|\big)+\frac{D_3}{n^2},\quad n>n_0, \tag{22}$$

holds, where the constants D_j have the following values:

$$D_1 \stackrel{\text{def}}{=} \frac{\sqrt{c}}{\pi},\qquad D_2 \stackrel{\text{def}}{=} \frac{\rho\sqrt{c}}{\pi}, \\ D_3 \stackrel{\text{def}}{=} \left(\frac{4\rho^2\sqrt{c}}{\pi n_0^{2\beta-3}}+\frac{2\sqrt{c}D}{\pi^2}\right)\cdot\|f'\|_{L_1(G)}+\frac{C_0}{\pi^2}\cdot\|f\|_{C(\overline{G})}\|q\|_{L_1(G)}. \tag{23}$$

Proof of equality (4). We first prove the absolute and uniform convergence on $\overline{G}=[-1,1]$ of the series (4), and the equality (4).

Note that the trigonometrical system $\{1/\sqrt{2},\cos n\pi x,\sin n\pi x \mid n\in\mathbb{N}\}$ is uniformly bounded and orthonormal in $L_2(G)$. Therefore, the corresponding Riesz inequality holds: for every function $h(x)\in L_p(G)(1<p\le 2)$ we have

$$\left(\sum_{n=0}^{\infty}\big(|a_n(h)|^r+|b_n(h)|^r\big)\right)^{1/r} \le \|h\|_{L_p(G)},$$

where $1/p+1/r=1$ and $a_0(h) \stackrel{\text{def}}{=} 2^{-1}\int_{-1}^{1}h(x)\,dx$, $b_0(h)\stackrel{\text{def}}{=}0$ (see [18]).

Now, the mentioned convergence of series (4) is a consequence of the following formal chain of equalities and inequalities:

$$\sum_{n=1}^{\infty}|f_nu_n(x)| = \sum_{n=1}^{n_0}|f_nu_n(x)|+\sum_{n=n_0+1}^{\infty}|f_nu_n(x)| \le n_0C_0^2\cdot\|f\|_{L_1(G)} \\ + C_0D_1\cdot\sum_{n=n_0+1}^{\infty}\left[\frac{1}{n}\big(|a_n(g_{0c}+g_{0s})|+|b_n(g_{0c}+g_{0s})|\big)\right] \\ + C_0D_2\cdot\sum_{n=n_0+1}^{\infty}\left[\frac{1}{n^{\beta}}\big(|a_n(g_{1c}+g_{1s})|+|b_n(g_{1c}+g_{1s})|\big)\right]+\sum_{n=n_0+1}^{\infty}\frac{C_0D_3}{n^2}$$

$$\leq E_1 + E_2\Big(\sum_{n=1}^{\infty}\frac{1}{n^p}\Big)^{1/p}\Big(\sum_{k=0}^{\infty}\big(|a_k(\cdot)|^r + |b_k(\cdot)|^r\big)\Big)^{1/r}$$

$$+ E_3\Big(\sum_{n=1}^{\infty}\frac{1}{n^{\beta p}}\Big)^{1/p}\Big(\sum_{k=0}^{\infty}\big(|a_k(\cdot\cdot)|^r + |b_k(\cdot\cdot)|^r\big)\Big)^{1/r} + E_4\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\leq E_1 + E_2\Big(\sum_{n=1}^{\infty}\frac{1}{n^p}\Big)^{1/p}\cdot\|g_{0c}+g_{0s}\|_{L_p(G)}$$

$$+ E_3\Big(\sum_{n=1}^{\infty}\frac{1}{n^{\beta p}}\Big)^{1/p}\cdot\|g_{1c}+g_{1s}\|_{L_p(G)} + E_4\cdot\sum_{n=1}^{\infty}\frac{1}{n^2},$$

the constants E_1–E_4 having an obvious meaning. Estimates (12),(22) and inequalities of Hölder and Riesz are used.

The equality (4) follows from the completeness of the system $\{u_n(x)\}_{n=1}^{\infty}$ in $L_2(G)$ and from the continuity of $f(x)$ on $\overline{G}$.

Proof of estimate (5). Let $\mu > n_0$ and $x \in [-1,1]$. Using equality (4), estimates (12) and (22), and the Hölder inequality, we obtain the relations

$$|f(x) - \sigma_\mu(x,f)| = \Big|\sum_{n=[\mu]+1}^{\infty} f_n u_n(x)\Big|$$

$$\leq E_2\Big(\sum_{n=[\mu]+1}^{\infty}\frac{1}{n^p}\Big)^{1/p}\Big(\sum_{n=[\mu]+1}^{\infty}\big(|a_n(g_{0c}+g_{1c})|^r + |b_n(g_{0c}+g_{1s})|^r\big)\Big)^{1/r}$$

$$+ E_3\Big(\sum_{n=[\mu]+1}^{\infty}\frac{1}{n^{\beta p}}\Big)^{1/p}\Big(\sum_{n=[\mu]+1}^{\infty}\big(|a_n(g_{1c}+g_{1s})|^r + |b_n(g_{1c}+g_{1s})|^r\big)\Big)^{1/r}$$

$$+ E_4\sum_{n=[\mu]+1}^{\infty}\frac{1}{n^2} \leq E_2\Big(\int_{[\mu]}^{+\infty}\frac{dt}{t^p}\Big)^{1/p}\cdot\alpha_{1,r}(\mu) + E_3\Big(\int_{[\mu]}^{+\infty}\frac{dt}{t^{\beta p}}\Big)^{1/p}\cdot\alpha_{2,r}(\mu)$$

$$+ E_4\int_{[\mu]}^{+\infty}\frac{dt}{t^2} \leq \frac{1}{\mu^{1-1/p}}\cdot\Big(E_5\cdot\alpha_{1,r}(\mu) + \frac{E_6}{\mu^{\beta-1}}\cdot\alpha_{2,r}(\mu) + \frac{2}{\mu^{1/p}}\Big),$$

where functions $\alpha_{1,r}(\mu)$ and $\alpha_{2,r}(\mu)$ are defined in the following way:

$$\alpha_{1,r}(\mu) = \Big(\sum_{n=[\mu]+1}^{\infty}\big(|a_n(g_{0c}+g_{0s})|^r + |b_n(g_{0c}+g_{0s})|^r\big)\Big)^{1/r},$$

$$\alpha_{2,r}(\mu) = \Big(\sum_{n=[\mu]+1}^{\infty}\big(|a_n(g_{1c}+g_{1s})|^r + |b_n(g_{1c}+g_{1s})|^r\big)\Big)^{1/r}.$$

The Riesz inequality gives $\lim_{\mu\to+\infty}\alpha_{1,r}(\mu) = 0 = \lim_{\mu\to+\infty}\alpha_{2,r}(\mu)$. Therefore, estimate (5) is a consequence of relations mentioned above.

Proof of Theorem 1 is completed. ∎

5. On Remark 2

In the second part of Remark 2 we have stated a proposition essentially related to Theorem 2. Proof of the proposition "contains" a proof of the theorem, and it is not based on asymptotic relations (16)–(18).

Let $f(x) \in W_1^{(2)}(G)$ and $\mathcal{L}(f)(x) \in L_p(G)$ $(1 < p \leq 2)$. Using differential equation (3) and the integration by parts twice, we can write

$$\begin{aligned} f_n &= \frac{1}{\lambda_n} \cdot \int_a^b f(x)\mathcal{L}(u_n)(x)\,dx \\ &= \frac{1}{\lambda_n}\Big(-f(x)u_n'(x)|_a^b + f'(x)u_n(x)|_a^b + \int_a^b \mathcal{L}(f)(x)u_n(x)\,dx \Big). \end{aligned}$$

By virtue of $f^{(j)}(a) = 0 = f^{(j)}(b)$ $(j = 0, 1)$, the double replacements vanish, and we get the basic relation

$$f_n = \frac{1}{\lambda_n} \cdot \int_a^b \mathcal{L}(f)(x)u_n(x)\,dx \equiv \frac{1}{\lambda_n} \cdot \mathcal{L}(f)_n, \tag{24}$$

playing in the following a role analogous to the one of estimate (22).

Let $x \in \overline{G}$. Using estimates (12)–(14), equality (24), and the Riesz inequality, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} |f_n u_n^{(j)}(x)| &= \sum_{0<\sqrt{\lambda_n}\leq 1} (\cdot) + \sum_{\sqrt{\lambda_n}>1} (\cdot) \leq AC_0C_j\|f\|_{L_1(G)} \\ &\quad + C_j\Big(\sum_{\sqrt{\lambda_n}>1} |\mathcal{L}(f)_n|^r\Big)^{1/r} \Big[\sum_{k=1}^{\infty}\Big(\sum_{k\leq\sqrt{\lambda_n}<k+1} \frac{1}{\lambda_n^{p(1-j/2)}}\Big)\Big]^{1/p} \\ &\leq E_7 + A^{1/p}C_0^{(2/p-1)}C_j\|\mathcal{L}(f)\|_{L_p(G)} \cdot \Big(\sum_{k=1}^{\infty} \frac{1}{k^{p(2-j)}}\Big)^{1/p}, \end{aligned}$$

where $j = 0, 1$ and $1/p + 1/r = 1$. Therefore, it follows that series (6) converge absolutely and uniformly on $\overline{G}$.

Proofs of equalities (6) are the standard ones. These equalities allow us to show that the estimates

$$|f^{(j)}(x) - \sigma_\mu^{(j)}(x,f)| \leq A^{1/p}C_j\Big(\sum_{n=[\mu]+1}^{\infty} |\mathcal{L}(f)_n|^r\Big)^{1/r}\Big(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^{p(2-j)}}\Big)^{1/p}$$

hold on $\overline{G}$, wherefrom the estimates (10) easily follow.

Let us suppose now that $f(x) \in D(L)$. Then $\mathcal{L}(f)(x) \in L_2(G)$, and equalities (24) are valid. The double replacements still vanish, as a consequence of the self-adjointness of the operator L. So, we see that the propositions of theorem 2 follow from the previous arguments (case $p = 2$).

6. Proof of Theorem 3

This proof is completely analogous to the one of Theorem 1; we suppose that $G = (-1, 1)$, and use relations (16)–(18).

Estimate for the Fourier coefficients. The starting point is equality (24), which is obviously valid in the case considered. So, using equation (3), the integration by parts, and assumption $\mathcal{L}(f)(-1) = 0 = \mathcal{L}(f)(1)$, we obtain the equalities

$$\begin{aligned} f_n &= \frac{1}{\lambda_n^2} \cdot \int_{-1}^{1} \mathcal{L}(f)(x)\mathcal{L}(u_n)(x)\,dx \\ &= \frac{1}{\lambda_n^2}\Big(\int_{-1}^{1} \mathcal{L}(f)'(x)u_n'(x)\,dx + \int_{-1}^{1} \mathcal{L}(f)(x)q(x)u_n(x)\,dx\Big). \end{aligned} \tag{25}$$

We need a suitable estimate for the integral $\int_{-1}^{1} \mathcal{L}(f)'(x)u_n'(x)dx$, $n > n_0$. Repeating, step by step, the arguments used in the first part of the proof of Theorem 1, we get the estimate

$$\begin{aligned} \Big|\int_{-1}^{1} \mathcal{L}(f)'(x)u_n'(x)\,dx\Big| &\le \sqrt{c\lambda_n}\Big[|a_n(h_{0c}+h_{0s})| + |b_n(h_{0c}+h_{0s})| \\ &+ \frac{\rho}{n^{\beta-1}}\big(|a_n(h_{1c}+h_{1s})|+|b_n(h_{1c}+h_{1s})|\big) + \frac{4\rho^2\|\mathcal{L}(f)'\|_{L_1(G)}}{n^{2(\beta-1)}}\Big] + 2\sqrt{c}D\|\mathcal{L}(f)'\|_{L_1(G)}, \end{aligned} \tag{26}$$

with functions $h_{jc}(x)$ and $h_{js}(x)$ defined as follows:

$$h_{jc}(x) = \mathcal{L}(f)'(x)x^j\cos\gamma x, \quad h_{js}(x) = \mathcal{L}(f)'(x)x^j\sin\gamma x, \qquad j = 0, 1.$$

From (25)–(26) we obtain, having in mind estimate (12) and equality (18), the necessary estimate for f_n:

$$\begin{aligned} |f_n| &\le \frac{D_4}{n^3} \cdot \big(|a_n(h_{0c}+h_{0s})| + |b_n(h_{0c}+h_{0s})|\big) \\ &\quad + \frac{D_5}{n^{\beta+2}} \cdot \big(|a_n(h_{1c}+h_{1s})| + |b_n(h_{1c}+h_{1s})|\big) + \frac{D_6}{n^4}, \qquad n > n_0, \end{aligned} \tag{27}$$

where the constants D_j have the values:

$$\begin{aligned} &D_4 \stackrel{\text{def}}{=} \frac{2\sqrt{c}}{\pi^3}, \qquad D_5 \stackrel{\text{def}}{=} \frac{\rho\sqrt{c}}{\pi^3}, \\ &\qquad D_6 \stackrel{\text{def}}{=} \Big(\frac{4\rho^2\sqrt{c}}{\pi^3 n_0^{2\beta-3}} + \frac{2\sqrt{c}D}{\pi^4}\Big) \cdot \|\mathcal{L}(f)'\|_{L_1(G)} + \frac{2C_0}{\pi^4} \cdot \|\mathcal{L}(f)q\|_{L_1(G)}. \end{aligned} \tag{28}$$

Proof of equalities (8). First we prove that series (8) converge absolutely and uniformly on $[-1, 1]$.

Having in mind estimates (12), (14), (15), (27), the equality (18), and inequalities of Hölder and Riesz, we obtain the following formal chain:

$$\sum_{n=1}^{\infty} |f_n u_n^{(j)}(x)| = \sum_{n=1}^{n_0}(\cdot) + \sum_{n=n_0+1}^{\infty}(\cdot) \le n_0 C_0 C_j \lambda_{n_0}^{j/2} \cdot \|f\|_{L_1(G)}$$

$$+ (2\pi)^j C_j D_4 \cdot \sum_{n=n_0+1}^{\infty} \frac{1}{n^{3-j}} \cdot \big(|a_n(h_{0c}+h_{0s})| + |b_n(h_{0c}+h_{0s})|\big)$$

$$+ (2\pi)^j C_j D_5 \cdot \sum_{n=n_0+1}^{\infty} \frac{1}{n^{\beta+2-j}} \cdot \big(|a_n(h_{1c}+h_{1s})| + |b_n(h_{1c}+h_{1s})|\big)$$

$$+ (2\pi)^j C_j D_6 \cdot \sum_{n=n_0+1}^{\infty} \frac{1}{n^{4-j}}$$

$$\le E_8 + E_9 \Bigg(\sum_{n=1}^{\infty} \frac{1}{n^{(3-j)p}}\Bigg)^{1/p} \cdot \|h_{0c}+h_{0s}\|_{L_p(G)}$$

$$+ E_{10} \Bigg(\sum_{n=1}^{\infty} \frac{1}{n^{(\beta+2-j)p}}\Bigg)^{1/p} \cdot \|h_{1c}+h_{1s}\|_{L_p(G)} + E_{11} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{4-j}}.$$

The numerical series on the right-hand side of the last inequality being convergent for $j = 0, 1, 2$, the convergence of series (8) holds as it has been claimed.

In the proof of equality (8) (case $j = 0$) one have to use, in a standard way, the continuity of $f(x)$ and the completeness of the system $\{u_n(x)\}_{n=1}^{\infty}$. The classical theorem on differentiability of the sum of a functional series helps in proving equality (8) in the cases $j = 1, 2$.

Proof of estimates (9). Let $\mu > n_0$ and $x \in [-1, 1]$. Starting from equalities (8), we can obtain, as in the preceding part of the proof, the inequalities

$$|f^{(j)}(x) - \sigma_\mu^{(j)}(x, f)| \le \sum_{n=[\mu]+1}^{\infty} |f_n u_n^{(j)}(x)|$$

$$\le E_9 \Bigg(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^{(3-j)p}}\Bigg)^{1/p} \cdot \alpha_{3,r}(\mu)$$

$$+ E_{10} \Bigg(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^{(\beta+2-j)p}}\Bigg)^{1/p} \cdot \alpha_{4,r}(\mu) + E_{11} \cdot \sum_{n=[\mu]+1}^{\infty} \frac{1}{n^{4-j}}$$

$$\le \frac{1}{\mu^{3-j-1/p}} \Bigg(E_{12} \cdot \alpha_{3,r}(\mu) + \frac{E_{13}}{\mu^{\beta-1}} \cdot \alpha_{4,r}(\mu) + \frac{E_{14}}{\mu^{1/p}}\Bigg), \quad (29)$$

where constants E_j have obvious meanings, and

$$\alpha_{3,r}(\mu) \stackrel{\text{def}}{=} \Bigg(\sum_{n=[\mu]+1}^{\infty} \big(|a_n(h_{0c}+h_{0s})|^r + |b_n(h_{0c}+h_{0s})|^r\big)\Bigg)^{1/r},$$

$$\alpha_{4,r}(\mu) \stackrel{\text{def}}{=} \Bigg(\sum_{n=[\mu]+1}^{\infty} \big(|a_n(h_{1c}+h_{1s})|^r + |b_n(h_{1c}+h_{1s})|^r\big)\Bigg)^{1/r}.$$

By virtue of $\lim_{\mu\to+\infty} \alpha_{3,r}(\mu) = 0 = \lim_{\mu\to+\infty} \alpha_{4,r}(\mu)$, estimates (9) follow from relations (29).

Theorem 3 is proved. ∎

7. On Remark 4

If the coefficients in boundary conditions (2) satisfy (11), then equations (2) can be solved with respect to the "variables" $u_n'(a)$ and $u_n'(b)$:

$$u_n'(a) = R_{1a}(\alpha_{ij}, \beta_{ij})u_n(a) + R_{1b}(\alpha_{ij}, \beta_{ij})u_n(b),$$
$$u_n'(b) = R_{2a}(\alpha_{ij}, \beta_{ij})u_n(a) + R_{2b}(\alpha_{ij}, \beta_{ij})u_n(b),$$

where constants R do not depend on n. Let us suppose that $f(a) \neq 0$ or/and $f(b) \neq 0$. Then we have, because of the above equalities and estimate (12), the inequality

$$|f(a)u_n'(b) - f(b)u_n'(b)| \leq 2C_0R_0(|f(a)| + |f(b)|), \tag{30}$$

where constant R_0 is independent of n.

Return now to the equality (19). In the general case considered, that equality must have the following form:

$$f_n = \frac{1}{\lambda_n} \cdot \int_a^b f(x)\mathcal{L}(u_n)(x)\,dx = \frac{1}{\lambda_n}\left[-f(a)u_n'(a) + f(b)u_n'(b)\right]$$
$$+ \frac{1}{\lambda_n}\left(\int_a^b f'(x)u_n'(x)\,dx + \int_a^b f(x)q(x)u_n(x)dx\right). \tag{31}$$

By virtue of estimate (30), the first member on the right-hand side of the second equality (31) has a "satisfactory" order with respect to λ_n. That is why the basic estimate (22) is still valid in this case, the constant D_3 being changed in the corresponding way.

Analogous arguments work in the cases $\mathcal{L}(f)(a) \neq 0$ or/and $\mathcal{L}(f)(b) \neq 0$.

If one of boundary conditions (2) has the form $\alpha_{j1}u'(a) + \beta_{j1}u'(b) = 0$ and $f(x)$ satisfies $\beta_{j1}f(a) + \alpha_{j1}f(b) = 0$, then the double replacements in (31) vanish, and we get estimate (22) again. The corresponding remark holds in the case of function $\mathcal{L}(f)(x)$ from Theorem 3.

Note that theorems 1 and 3 were reported at the Seminar for Functional Analysis and Operator Theory (Faculty of Mathematics, Belgrade) on March 21st, 2002.

Acknowledgment. We thank the referee for valuable suggestions.

Added in proof. Recently, V. M. Kurbanov and R. A. Safarov [19] have obtained, by a different approach, results stated in Theorem 1, with $O(\mu^{1/p-1})$ standing in the estimate (5) (instead of $o(\mu^{1/p-1})$).

REFERENCES

[1] M. A. Naímark, *Lineínye differentsial'nye operatory*, Nauka, Moskva, 1969.

[2] V. A. Il'in, *Neobkhodimye i dostatochnye usloviya bazisnosti i ravnoskhodimosti s trigonometricheskim ryadom spektral'nykh razlozhenií.* I, Diff. uravneniya **16**, 5 (1980), 771–794.

[3] V. A. Il'in, *Neobkhodimye i dostatochnye usloviya bazisnosti i ravnoskhodimosti s trigonometricheskim ryadom spektral'nykh razlozhenií.* II, Diff. uravneniya **16**, 6 (1980), 980–1009.

[4] V. A. Il'in, *Neobkhodimye i dostatochnye usloviya bazisnosti v L_p i ravnoskhodimosti s trigonometricheskim ryadom spektral'nykh razlozhenií i razlozhenií po sistemam eksponent*, Dokl.AN SSSR **273**, 4 (1983), 789–793.

[5] V. A. Il'in, *Ravnoskhodimost' s trigonometricheskim ryadom razlozhenií po kornevym funktsiyam odnomernogo operatora Shredingera s kompleksnym potentsialom iz klasa L_1*, Diff. uravneniya **27**, 4 (1991), 577–597.

[6] I. S. Lomov, *Conditions for convergence of biorthogonal expansions of functions on a closed interval*, Diff. equations **37**, 4 (2001), 562–565.

[7] I. S. Lomov, *The local convergence of biorthogonal series related to differential operators with nonsmooth coefficients*. I, Diff. equations **37**, 3 (2001), 328–342.

[8] I. S. Lomov, *The local convergence of biorthogonal series related to differential operators with nonsmooth coefficients*. II, Diff. equations **37**, 5 (2001), 648–660.

[9] I. S. Lomov, *Ravnomernaya skhodimost' biortogonal'nogo ryada dlya operatora Shredingera s mnogozhnachnymi kraevymi usloviyami*, Diff. uravneniya **38**, 7 (2002), 890–896.

[10] A. M. Minkin, *Teorema ravnoskhodomosti dlya razlozheniyi po oboshchennoyi spektral'noyi funktsii simmetricheskogo differentsial'nogo operatora i v integral Fur'e*, Funktsional'niyi analiz. Spektral'naya teoriya, t. **14**, 1980, 109–112.

[11] A. M. Minkin, *Almost orthogonality of Birkhoff's solutions*, Results in Math., **24**, 3/4 (1999), 280–287.

[12] A. M. Minkin, *Equiconvergence theorems for differential operators*, Journal of Math. Sci. **96**, 6 (1999), 3631–3715.

[13] N. L. Lažetić, *On uniform convergence of spectral expansions arising by self-adjoint extensions of an one-dimensional Schrödinger operator*, Publ. de l'Inst. Math. (Beograd) **69** (**83**) (2001), 59–77.

[14] N. L. Lazhetich, *O sushchestvovanii klassicheskogo resheniya smeshannoí zadachi dlya odnomernogo giperbolicheskogo uravneniya vtorogo poryadka*(II), Publ. de l'Inst. Math. (Beograd) **64**(**78**) (1998), 69–84.

[15] V. A. Il'in, I. Ío, *Ravnomernaya otsenka sobstvennykh funktsií i otsenka sverkhu chisla sobstvennykh znachenií odnomernogo operatora Shturma-Liuvillya s potentsialom iz klassa L_p*, Diff. uravneniya **15**, 7 (1979), 1165–1174.

[16] L. V. Krickov, *Ravnomernaya otsenka poryadka prisoedinennykh funktsií i raspredelenie sobstvennykh znachenií odnomernogo operatora Shredingera*, Diff. uravneniya **25**, 7 (1989), 1121–1129.

[17] N. Lazhetich, *Ravnomernye otsenki dlya proizvodnykh sobstvennykh funktsií samosopryazhennogo operatora Shturma-Liuvillya*, Diff. uravneniya **17**, 11 (1981), 1978–1983.

[18] A. Zygmund, *Trigonometrical Series*, Volume I and II, Cambridge, 1968.

[19] V. M. Kurbanov, R. A. Safarov, *On uniform convergence of orthogonal expansions in eigenfunctions of Sturm-Liouville operator*, Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **24**, 1 (2004), Math. Mech., 161–168.

(received 21.05.2003, in revised form 02.11.2004)

Faculty of Mathematics, Studentski trg 16, P.O. Box 550, 11000 Beograd, Serbia

E-mail: `lazetic@matf.bg.ac.yu`