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ON THE LARGE TRANSFINITE INDUCTIVE DIMENSION OF A SPACE BY A NORMAL BASE

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Abstract. The transfinite inductive dimensions of a space by a normal bases introduced by S. D. Iliadis are studied. These dimensions generalize both classical large transfinite inductive dimension and relative large transfinite inductive dimensions. The main theorems of dimension theory (sum theorem, subset theorem, product theorem) are proved.

1. Introduction

One of the approaches to define dimension of a topological space is inductive. Its origin is due to Poincaré and was developed in the works of Brouwer, Menger and Urysohn. It was Urysohn who hinted at the possibility of extending inductive dimensions by letting it assume ordinal values. The large transfinite inductive dimension, denoted here by trInd, was introduced by Yu. Smirnov. The desire to obtain the analogs of the main theorems of dimension theory in different classes of spaces which makes possibile the estimation of classical dimension caused the appearance of its different modifications.

The notion of a normal base for the closed subsets of a space was defined in [11]. The existence of such bases characterizes completely regular spaces. Thus by a space below we mean a completely regular space. Normal bases and compactifications correspondent to them were studied by many authors (see, for example, Bibliography from [1]. Natural normal bases are the following:

(a) the family of all closed subsets of a normal space;

(b) the family $Z(X)$ of all zero-sets of a completely regular space X; and

(c) the family $Z(X,Y) = X \cap Z(Y)$—traces on X of all zero-sets of a space Y.

These concrete normal bases are used in order to study finite inductuve dimensions Ind, Ind_0, I (see, for example, [2], [9], [14]. [5], [6], [7] and their Bibliographies)

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and their transfinite extensions (see, for example, [2], [9], [7], [8] and their Bibliographies).

In [15] normal bases are used for the definition of the so-called *base dimension-like functions* of the type Ind and of the type dim. These dimension-like functions are studied in [15] only with respect to the existence of universal elements and in [12] the dimensional properties of the finite variant of the dimension-like functions of the type Ind are studied. Below we investigate a transfinite base dimension-like function of the type Ind.

Let X be a space, Y a subset of X, and $\mathcal{F}$ a family of subsets of X. We set $\mathcal{F}^c = \{X \setminus F : F \in \mathcal{F}\}$ and $\mathcal{F}|_Y = \{Y \cap F : F \in \mathcal{F}\}$. By ω_α we denote the least ordinal of cardinality $\aleph_\alpha$.

We recall the definition of a normal base.

Definition 1. [11] A base $\mathcal{F}$ for the closed sets of a space X is said to be a *normal base* if the following conditions are satisfied:

(1) $\mathcal{F}$ is a *ring*: $\mathcal{F}$ is closed under finite unions and finite intersections.

(2) $\mathcal{F}$ is *disjunctive*: for every F in $\mathcal{F}$ and a point x of X not in F there is T in $\mathcal{F}$ such that $x \in T$ and $F \cap T = \emptyset$.

(3) $\mathcal{F}$ is *base-normal*: for every pair (F, T) of disjoint elements of $\mathcal{F}$ there exists a pair (R, S) of elements of $\mathcal{F}$ such that $F \cap S = \emptyset$, $T \cap R = \emptyset$, and $R \cup S = X$. (The pair (R, S) is called a *screening* of (F, T) in $\mathcal{F}$).

If, moreover, the normal base $\mathcal{F}$ is closed under countable intersections, then we say that $\mathcal{F}$ is *multiplicative* [12].

Below we shall always deal with normal bases for the closed sets.

Definition 2. [15] We denote by trI the unique dimensional-like function that has as domain the class of all pairs $(X, \mathcal{F})$ where $\mathcal{F}$ is a normal base on a space X, and as range the class $\mathcal{O} \cup \{-1, \infty\}$ where $\mathcal{O}$ is the set of all ordinals, such that:

(a) $\mathrm{trI}(X, \mathcal{F}) = -1$ if and only if $X = \emptyset$.

(b) $\mathrm{trI}(X, \mathcal{F}) \le \alpha$, where $\alpha \in \mathcal{O}$, if for every pair (F, T) of disjoint elements of $\mathcal{F}$ there exists a screening (R, S) of (F, T) such that $\mathrm{trI}(R \cap S, \mathcal{F}|_{R\cap S}) < \alpha$.

Therefore, $\mathrm{trI}(X, \mathcal{F}) = \infty$ if and only if the inequality $\mathrm{trI}(X, \mathcal{F}) \le \alpha$ is not true for every $\alpha \in \mathcal{O} \cup \{-1\}$.

Remark 3. (1) In [15] instead of "$\mathrm{trI}(X, \mathcal{F}) \le \alpha$" we say that "the normal base $\mathcal{F}$ is $(b^n - \mathrm{Ind} \le \alpha)$-dimensional". The restriction considered in [15] that the cardinality of C $(= \mathcal{F})$ is less than or equal to τ is not important since τ can be an arbitrary cardinal.

(2) In the case when X is a subspace of a space Y and $\mathcal{F} = Z(X, Y)$ (in particular, $X = Y$ and $\mathcal{F} = Z(X)$) $\mathrm{trI}(X, \mathcal{F})$ coincides with the *transfinite relative dimension* $\mathrm{I}(X, Y)$ $(\mathrm{Ind}_0(X))$ which is defined in [8] (see also [7]) and denoted below as $\mathrm{trI}(X, Y)$ $(\mathrm{trInd}_0(X))$.

(3) In [12] the finite variant of the dimension-like functions of the type Ind is studied. Let us recall its definition. Let $\mathcal{F}$ be a normal base on a space X. Put $\mathrm{I}(X, \mathcal{F}) = -1$ if and only if $X = \emptyset$. One says that $\mathrm{I}(X, \mathcal{F}) \leq n, n \geq 0, n \in \mathbb{N}$, if for every pair (F, T) of disjoint elements of $\mathcal{F}$ there exists its screening (R, S) such that $\mathrm{I}(R \cap S, \mathcal{F}|_{R\cap S}) \leq n - 1$.

Let us recall types of normal bases introduced in [12].

Let $\mathcal{F}$ be a normal base of a space X and Y a subspace of X. We say that Y is *$\mathcal{F}$-normal* [12, Definition 3] if the family

$$\mathcal{F}|_Y = \{Y \cap F : F \in \mathcal{F}\}$$

is a normal base on Y. A normal base $\mathcal{F}$ on X is said to be a *hereditarily normal base* [12, Definition 4] if any subspace Y of X is $\mathcal{F}$-normal.

2. Basic properties of transfinite inductive dimension of a space by a normal base

The following propositions are evident.

Proposition 4. *For any normal base $\mathcal{F}$ on X if $\mathrm{I}(X, \mathcal{F}) < \infty$, then $\mathrm{trI}(X, \mathcal{F}) = \mathrm{I}(X, \mathcal{F})$, and if $\mathrm{trI}(X, \mathcal{F}) < \omega_0$, then $\mathrm{I}(X, \mathcal{F}) = \mathrm{trI}(X, \mathcal{F})$.*

Proposition 5. [15] *For any normal base $\mathcal{F}$ on X and $Y \in \mathcal{F}$ we have $\mathrm{trI}(Y, \mathcal{F}|_Y) \leq \mathrm{trI}(X, \mathcal{F})$.*

Proposition 6. *If $\mathcal{F}$ is a normal base on X and $\mathrm{trI}(X, \mathcal{F}) = \alpha$ then for each ordinal number $\beta < \alpha$ there exists an $F \in \mathcal{F}$ such that $\mathrm{trI}(F, \mathcal{F}|_F) = \beta$.*

Proposition 7. [15] *For every normal base $\mathcal{F}$ on a space X we have*

$$\mathrm{trI}(X, \mathcal{F}) = \mathrm{trI}(w(X, \mathcal{F}), w\mathcal{F}),$$

where $w(X, \mathcal{F})$ is the Wallman type compactification by the normal base $\mathcal{F}$ and $w\mathcal{F} = \{\mathrm{Cl}_{w(X,\mathcal{F})} F = w(F, \mathcal{F}|_F) : F \in \mathcal{F}\}$ is the normal base on $w(X, \mathcal{F})$.

The proof of the following proposition is the same as in [12, Proposition 4].

Proposition 8. *The following propositions are true:*

(1) *If $\mathcal{F}'$ and $\mathcal{F}$ are normal bases on a compact space X such that $\mathcal{F}' \subset \mathcal{F}$, then $\mathrm{trI}(X, \mathcal{F}) \leq \mathrm{trI}(X, \mathcal{F}')$. In particular, for any normal base $\mathcal{F}$ on a compactum X, $\mathrm{trInd}X \leq \mathrm{trI}(X, \mathcal{F})$.*

(2) *If $\mathcal{F}'$ and $\mathcal{F}$ are normal bases on a Lindelöff space X such that $\mathcal{F}'$ is multiplicative and $\mathcal{F}' \subset \mathcal{F}$, then $\mathrm{trI}(X, \mathcal{F}) \leq \mathrm{trI}(X, \mathcal{F}')$. In particular, for any multiplicative normal base $\mathcal{F}$ on a Lindelöff space X, $\mathrm{trInd}X \leq \mathrm{trI}(X, \mathcal{F})$ ($\mathrm{trInd}_0 X \leq \mathrm{trI}(X, Y)$ for any $Y \supset X$).*

Proposition 9. *If $|\mathcal{F}| \leq \aleph_\alpha$ for a normal base $\mathcal{F}$ on X and $\mathrm{trI}(X, \mathcal{F}) \neq \infty$ then $\mathrm{trI}(X, \mathcal{F}) < \omega_{\alpha+1}$.*

Proof. Suppose that $\mathrm{trI}(X,\mathcal{F}) \geq \omega_{\alpha+1}$. Without loss of generality, by Proposition 6, we may assume that $\mathrm{trI}(X,\mathcal{F}) = \omega_{\alpha+1}$. For each pair (F,T) of disjoint sets from $\mathcal{F}$ there exists their screening (R,S) such that $\mathrm{trI}(R\cap S,\mathcal{F}|_{R\cap S}) < \omega_{\alpha+1}$. Since $|\mathcal{F}| \leq \aleph_\alpha$, there exists an ordinal number $\beta < \omega_{\alpha+1}$ such that for any pair (F,T) of disjoint sets from $\mathcal{F}$ there exists its screening (R,S) with $\mathrm{trI}(R\cap S,\mathcal{F}|_{R\cap S}) < \beta$. Hence, we have $\mathrm{trI}(X,\mathcal{F}) \leq \beta < \omega_{\alpha+1}$, a contradiction. ■

The following proposition shows that the condition in [9, Theorem 7.1.17] may be weakened.

Proposition 10. *If a compactum X with weight $w(X) \leq \aleph_\alpha$ has a normal base $\mathcal{F}$ such that* $\mathrm{trI}(X,\mathcal{F}) \neq \infty$*, then* $\mathrm{trInd}X < \omega_{\alpha+1}$.

Proof. By Proposition 8 (1) $\mathrm{trInd}X \leq \mathrm{trI}(X,\mathcal{F})$ for any normal base $\mathcal{F}$ on a compactum X. Thus, $\mathrm{trInd}X \neq \infty$. The rest follows from [9, Theorem 7.1.17]. ■

Corollary 11. *If a metrizable compactum X has a normal base $\mathcal{F}$ such that* $\mathrm{trI}(X,\mathcal{F}) \neq \infty$*, then* $\mathrm{trInd}X < \omega_1$.

Lemma 12. *Let $X = X_1 \cup X_2$ and $\mathcal{F}$ be either*

(a) *a hereditarily normal base on X or*

(b) *a normal base on X with $X_1 \in \mathcal{F}$ and X_2 be $\mathcal{F}$-normal,*

and $\mathrm{trI}(X_2,\mathcal{F}|_{X_2}) \leq \alpha \geq 0$. *Then, for any pair (F,T) of disjoint elements of $\mathcal{F}$ there exists their screening (R,S) such that $R\cap S = X_1' \cup X_2'$, $X_i' \in \mathcal{F}|_{X_i}$, $i = 1,2$, and* $\mathrm{trI}(X_2',\mathcal{F}|_{X_2'}) < \alpha$.

Proof. By [12, Remark 1(4)] for a pair (F,T) of disjoint elements of $\mathcal{F}$ there exist a disjoint pair (F',T') of elements of $\mathcal{F}$ and a pair (G,H) of elements of $\mathcal{F}^c$ such that $F \subset G \subset F'$ and $T \subset H \subset T'$. Since $\mathrm{trI}(X_2,\mathcal{F}|_{X_2}) \leq \alpha$ there exists a screening (R',S') of $(F'\cap X_2, T'\cap X_2)$ in X_2 such that

$$\mathrm{trI}(R'\cap S',\mathcal{F}|_{R'\cap S'}) < \alpha.$$

In the case (a) let $R'',S'' \in \mathcal{F}$ be such that $R''\cap X_2 = R'$, $S''\cap X_2 = S'$ and $R''\cap H = S''\cap G = \emptyset$. Put $F'' = F\cup R''$, $T'' = T\cup S''$, $X' = X\setminus(F''\cap T'')$. It is easy to check that $F,T \subset X'$ and $(F''\cap T'')\cap X_2 = R'\cap S'$. The sets $F''\cap X'$, $T''\cap X' \in \mathcal{F}|_{X'}$ and disjoint. Thus, there exists their screening (R^*,S^*) in $\mathcal{F}|_{X'}$. Besides $(R^*\cap S^*)\cap X_2 = \emptyset$. The pair $(R^*\cup(F''\cap T''), S^*\cup(F''\cap T''))$ is a screening of (F,T) which satisfies conditions of the lemma.

In the case (b) we set $U = G\cup(X\setminus(S'\cup X_1))$ and $V = H\cup(X\setminus(R'\cup X_1))$. It is easy to see that the sets U and V are disjoint, belong to $\mathcal{F}^c$ and $F\subset U$, $T\subset V$. If we put $R = X\setminus V$ and $S = X\setminus U$, then (R,S) is a screening of (F,T) and $R\cap S \subset X_1\cup(R'\cap S')$. The sets $X_1' = (R\cap S)\cap X_1$ and $X_2' = R'\cap S'$ satisfy conditions of the lemma. ■

If α is an ordinal number then it can be uniquely represented as $\alpha = \lambda + n$, where λ is a limit ordinal and n is a non-negative integer.

Proposition 13. [12, Proposition 7] *Let $\mathcal{F}$ be a hereditarily normal base on $X = X_1 \cup X_2$. Then*

$$\mathrm{I}(X, \mathcal{F}) \leq \mathrm{I}(X_1, \mathcal{F}|_{X_1}) + \mathrm{I}(X, \mathcal{F}|_{X_2}).$$

Proposition 13 may be examined as the base of induction in the proof of the next theorem.

Theorem 14. *Let $X = X_1 \cup X_2$ be a space and $\mathcal{F}$ a hereditarily normal base on X. If $\mathrm{trI}(X_i, \mathcal{F}|_{X_i}) = \alpha_i = \lambda_i + n_i$, $i = 1, 2$, then*

$$\mathrm{trI}(X, \mathcal{F}) \leq \begin{cases} \alpha_1, & \text{if } \lambda_1 > \lambda_2 \\ \alpha_2, & \text{if } \lambda_2 > \lambda_1 \\ \lambda_1 + n_1 + n_2 + 1, & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Proof. Suppose that the theorem does not hold. Evidently we may consider $\alpha_1 \leq \alpha_2$ such that the theorem is not valid for them but holds for $\mathrm{trI}(X_i, \mathcal{F}|_{X_i}) = \beta_i$, $i = 1, 2$, $\beta_1 \leq \beta_2$, whenever $\beta_2 < \alpha_2$ or $\beta_2 = \alpha_2$ and $\beta_1 < \alpha_1$. From Propositions 4 and 13 it follows that $\alpha_2 \geq \omega_0$.

Let (F, T) be disjoint pair elements of $\mathcal{F}$. Since $\mathrm{trI}(X_2, \mathcal{F}|_{X_2}) = \alpha_2$, by Lemma 12 (a) there exists its screening (R, S) such that $R \cap S = X_1' \cup X_2'$, $X_i' \in \mathcal{F}|_{X_i}$, $i = 1, 2$, and $\mathrm{trI}(X_2', \mathcal{F}|_{X_2'}) < \alpha_2$. Thus, by our supposition (remind that $\alpha_i = \lambda_i + n_i$, $i = 1, 2$),

$$\mathrm{trI}(R \cap S, \mathcal{F}|_{R \cap S}) \leq \begin{cases} \lambda_1 = \alpha_1, & \text{if } \lambda_1 = \lambda_2,\ n_2 = 0 \\ \lambda_1 + n_1 + n_2, & \text{if } \lambda_1 = \lambda_2,\ n_2 \neq 0 \\ \beta \text{ where } \beta < \alpha_2, & \text{in all other cases.} \end{cases}$$

From this it follows that

$$\mathrm{trI}(X, \mathcal{F}) \leq \begin{cases} \lambda_1 + 1, & \text{if } \lambda_1 = \lambda_2,\ n_2 = 0 \\ \lambda_1 + n_1 + n_2 + 1, & \text{if } \lambda_1 = \lambda_2,\ n_2 \neq 0 \\ \alpha_2, & \text{in all other cases.} \end{cases}$$

Thus, the statement of the theorem holds for the chosen α_1, α_2 and our supposition leads to a contradiction. ∎

The next theorem may be proved in the same way using Lemma 12 (b) and [12, Proposition 8] in the base of induction.

Theorem 15. *Let $X = X_1 \cup X_2$ be a space and $\mathcal{F}$ a normal base and $X_i \in \mathcal{F}$, $i = 1, 2$. If $\mathrm{trI}(X_i, \mathcal{F}|_{X_i}) = \alpha_i = \lambda_i + n_i$, $i = 1, 2$, then*

$$\mathrm{trI}(X, \mathcal{F}) \leq \begin{cases} \alpha_1, & \text{if } \lambda_1 > \lambda_2 \\ \alpha_2, & \text{if } \lambda_2 > \lambda_1 \\ \lambda_1 + n_1 + n_2 + 1, & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Remark 16. Theorem 15 generalizes results of Hattori [13], and Theorem 14 results of Levšenko [17] (see, for example, [9, Theorem 7.2.7, Problem 7.2.D] and results of Chigogidze [8].

Corollary 17. *Let* $X = \bigcup\{X_i : i = 1, \ldots, n\}$ *and* $\mathcal{F}$ *be either a hereditarily normal base on* X *or a normal base and* $X_i \in \mathcal{F}$, $i = 1, \ldots, n$. *If* $\mathrm{trI}(X_i, \mathcal{F}|_{X_i}) < \lambda$, $i = 1, \ldots, n$, *where* λ *is a limit ordinal, then* $\mathrm{trI}(X, \mathcal{F}) < \lambda$.

3. Product theorem

If $\mathcal{F}_j$ is a normal base on a space X_j, $j = 1, 2$, then, the family $\mathcal{F}$ of all finite unions of the elements of the form $F_1 \times F_2$ where $F_j \in \mathcal{F}_j$, $i = 1, 2$ is a normal base on $X_1 \times X_2$ which is called *product of normal bases* and it is denoted by the symbol $\mathcal{F}_1 \times \mathcal{F}_2$ [12, Proposition 11].

The following lemma is nearly the same as [12, Lemma 4]. Its proof is given for the completeness of presentation.

Lemma 18. *Let* $\mathcal{F}_j$ *be a normal base on a space* X_j, $j = 1, 2$, *and* $F, T \in \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ *be disjoint subsets of the product* $X = X_1 \times X_2$. *Then, there exist their screening* (R, S) *in* $\mathcal{F}$ *and the sets* $\Phi_j \in \mathcal{F}_j$, $j = 1, 2$, *such that*

$$R \cap S \subset \Phi_1 \times X_2 \cup X_1 \times \Phi_2.$$

Moreover, there exists $k \in \mathbb{N}$ *such that* $\Phi_j = \bigcup\{\varphi_j^{sl} \in \mathcal{F}_j : s, l = 1, \ldots, k\}$, *where* $\mathrm{trI}(\varphi_j^{sl}, \mathcal{F}_j|_{\varphi_j^{sl}}) < \mathrm{trI}(X_j, \mathcal{F}_j)$, $s, l = 1, \ldots, k$, $j = 1, 2$.

Proof. Let

$$F = \bigcup\{F_s = F_1^s \times F_2^s : s = 1, \ldots, k\}$$

and

$$T = \bigcup\{T_l = T_1^l \times T_2^l : l = 1, \ldots, k\},$$

where $F_j^s, T_j^l \in \mathcal{F}_j$, $s, l = 1, \ldots, k$ (without lose of generality we may assume that the sets F and T has equal number of rectangles), $j = 1, 2$.

Since any pair of rectangles $F_s = F_1^s \times F_2^s$ and $T_l = T_1^l \times T_2^l$ is disjoint it follows that either $F_1^s \cap T_1^l = \emptyset$ or $F_2^s \cap T_2^l = \emptyset$, $s, l = 1, \ldots, k$. Thus there exists a screening (R_{sl}, S_{sl}) of the pair (F_s, T_l) of the form either $R_{sl} = R_1^{sl} \times X_2$, $S_{sl} = S_1^{sl} \times X_2$, where (R_1^{sl}, S_1^{sl}) is a screening of the pair (F_1^s, T_1^l) if $F_1^s \cap T_1^l = \emptyset$, or of the form $R_{sl} = X_1 \times R_2^{sl}$, $S_{sl} = X_1 \times S_2^{sl}$, where (R_2^{sl}, S_2^{sl}) is a screening of the pair (F_2^s, T_2^l) if $F_2^s \cap T_2^l = \emptyset$. If both $F_1^s \cap T_1^l = \emptyset$ and $F_2^s \cap T_2^l = \emptyset$, then we take the screening of the first form. Besides, while putting $\varphi_1^{sl} = R_1^{sl} \cap S_1^{sl}$, $\varphi_2^{sl} = \emptyset$ if $F_1^s \cap T_1^l = \emptyset$ and $\varphi_1^{sl} = \emptyset$, $\varphi_2^{sl} = R_2^{sl} \cap S_2^{sl}$ otherwise, $s, l = 1, \ldots, k$, we may assume that $\mathrm{trI}(\varphi_j^{sl}, \mathcal{F}_j|_{\varphi_j^{sl}}) < \mathrm{trI}(X_j, \mathcal{F}_j)$, $j = 1, 2$.

The pair

$$(\bigcap\{R_{sl} : l = 1, \ldots, k\}, \bigcup\{S_{sl} : l = 1, \ldots, k\})$$

is a screening of the pair (F_s, T), $s = 1, \dots, k$. Besides

$$(\bigcap\{R_{sl} : l = 1, \dots, k\}) \cap (\bigcup\{S_{sl} : l = 1, \dots, k\}) \subset (\bigcup\{\varphi_1^{sl} : l = 1, \dots, k\}) \times X_2 \cup X_1 \times (\bigcup\{\varphi_2^{sl} : l = 1, \dots, k\}).$$

The pair $(\bigcup\{\bigcap\{R_{sl} : l = 1, \dots, k\} : s = 1, \dots, k\}, \bigcap\{\bigcup\{S_{sl} : l = 1, \dots, k\} : l = 1, \dots, k\})$ is a screening of the pair (F, T). Let us put $\Phi_j = \bigcup\{\varphi_j^{sl} : s, l = 1, \dots, k\}$, $j = 1, 2$. It follows that $(\bigcup\{\bigcap\{R_{sl} : l = 1, \dots, k\} : s = 1, \dots, k\}) \cap (\bigcap\{\bigcup\{S_{sl} : l = 1, \dots, k\} : l = 1, \dots, k\}) \subset \Phi_1 \times X_2 \cup X_1 \times \Phi_2$. ■

Definition 19. [12, Definition 6] For a normal base $\mathcal{F}$ its subsystem $\mathcal{T}$ is a *normal base in itself* if for any $T \in \mathcal{T}$ the family $\mathcal{F}|_T \subset \mathcal{T}$. (Note that $\mathcal{F}|_T = \mathcal{T}|_T \subset \mathcal{F}$ for any $T \in \mathcal{T}$).

Proposition 20. *For a normal base $\mathcal{F}$ on a space X the following conditions are equivalent:*

(1) $\operatorname{trI}(X, \mathcal{F}) \leq \alpha$*;*

(2) *there are normal in itself bases $\sigma_{-1}, \sigma_0, \dots, \sigma_\beta$, $\beta \leq \alpha$ such that*

(a) $\sigma_{-1} = \{\emptyset\}$*, $X \in \sigma_\beta$, $\sigma_\gamma \subset \sigma_\delta$ for any $\gamma \leq \delta \leq \beta$;*

(b) *for any $\gamma \leq \alpha$ any $T \in \sigma_\gamma$ and any two disjoint elements $A, B \in \mathcal{F}|_T$ there exists their screening (L, H) in T such that $L \cap H \in \sigma_\beta$ for some $\beta < \gamma$.*

Proof. In order to prove the implication (1) $\Longrightarrow$ (2) it is sufficient to set $\sigma_{-1} = \{\emptyset\}$ and $\sigma_\gamma = \{F : F \in \mathcal{F}, \operatorname{trI}(F, \mathcal{F}|_F) \leq \gamma\}$ for $\gamma \leq \alpha$. The normality of base σ_γ in itself, $\gamma \leq \alpha$, follows from Proposition 5. The fulfilment of conditions (a) and (b) is evident.

In order to prove the implication (2) $\Longrightarrow$ (1) it suffices to show that $\operatorname{trI}(F, \mathcal{F}|_F) \leq \gamma$ for any $F \in \sigma_\gamma$, $\gamma \leq \beta$. Indeed, if $\gamma = -1$, then $F = \emptyset$ and $\operatorname{trI}(F, \mathcal{F}|_F) = -1$. Let us assume that for all $\gamma < \delta \leq \beta$ the statement holds. Let $F \in \sigma_\delta$ and $A, B \in \mathcal{F}|_F$ be disjoint. From (b) it follows that there exists a screening (A', B') of (A, B) in F such that $A' \cap B' \in \sigma_{\delta'}$, where $\delta' < \delta$. By inductive assumption $\operatorname{trI}(A' \cap B', \mathcal{F}|_{A' \cap B'}) \leq \delta'$. Hence, $\operatorname{trI}(F, \mathcal{F}|_F) \leq \delta$. ■

Definition 21. A finite sum theorem is satisfied on X for the dimension by normal base $\mathcal{F}$ if for any $F_i \in \mathcal{F}$, $i = 1, \dots, k$,

$$\operatorname{trI}(\bigcup\{F_i : i = 1, \dots, k\}, \mathcal{F}|_{\bigcup\{F_i : i = 1, \dots, k\}}) \leq \max\{\operatorname{trI}(F_i, \mathcal{F}|_{F_i}) : i = 1, \dots, k\}.$$

Remark 22. (a) If $\operatorname{trI}(X, \mathcal{F}) \leq \alpha$ for a normal base $\mathcal{F}$ on a space X and the finite sum theorem is satisfied on X for the dimension by normal base $\mathcal{F}$ then each normal base in itself $\sigma_{-1}, \sigma_0, \dots, \sigma_\beta$, $\beta \leq \alpha$, from Proposition 20 may be assumed to be a ring.

(b) If in Lemma 18 the finite sum theorem is satisfied on X_j for the dimension by normal base $\mathcal{F}_j$, then we may additionally note that $\operatorname{trI}(\Phi_j, \mathcal{F}_j|_{\Phi_j}) \leq \max\{\operatorname{trI}(\varphi_j^{sl}, \mathcal{F}_j|_{\varphi_j^{sl}}) : s, l = 1, \dots, k\} < \operatorname{trI}(X_j, \mathcal{F}_j)$, $j = 1, 2$.

In the next theorem we shall use the natural sum of Hessenberg of two ordinals (see, for example, [16]). Let $\alpha = \sum \omega^\tau \cdot m_\tau$, $\beta = \sum \omega^\tau \cdot n_\tau$ where $m_\tau, n_\tau \in \omega$ be the unique representations of the ordinals. Then the natural sum of α and β is an ordinal $\alpha(+)\beta = \sum \omega^\tau \cdot (m_\tau + n_\tau)$. The operation of the natural sum has the following properties [16]:

(1) $\alpha(+)\beta = \beta(+)\alpha$;

(2) if $\alpha < \beta$ then $\alpha(+)\gamma < \beta(+)\gamma$; and

(3) $\infty(+)\alpha = \alpha(+)\infty = \infty$ and $(-1)(+)\alpha = \alpha(+)(-1) = \alpha$.

Theorem 23. [12, Theorem 4] *If the finite sum theorem is satisfied on X_j for the dimension by normal base $\mathcal{F}_j$, $j = 1, 2$, then*

$$\mathrm{I}(X = X_1 \times X_2, \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2) \leq \mathrm{I}(X_1, \mathcal{F}_1) + \mathrm{I}(X_2, \mathcal{F}_2).$$

Theorem 24. *If the finite sum theorem is satisfied on X_j for the dimension by normal base $\mathcal{F}_j$, $j = 1, 2$, then*

$$\mathrm{trI}(X = X_1 \times X_2, \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2) \leq \mathrm{trI}(X_1, \mathcal{F}_1)(+)\mathrm{trI}(X_2, \mathcal{F}_2).$$

Proof. Let $\mathrm{trI}(X_j, \mathcal{F}_j) = \alpha_j$, $j = 1, 2$. If either α_1 or $\alpha_2 = \infty$ then the statement of the theorem is evident. Otherwise, according to the Proposition 20 there are normal in itself bases σ^j_γ, $\gamma = -1, 0, \dots, \beta_j \leq \alpha_j$, of $\mathcal{F}_j$, $j = 1, 2$, satisfying conditions (a) and (b).

We set $\sigma_{-1} = \{\emptyset\}$ and

$$\sigma'_\delta = \{F_1 \times F_2 : F_j \in \sigma^j_{\gamma(j)}, \gamma(j) \geq 0, j = 1, 2, \gamma(1)(+)\gamma(2) \leq \delta\},$$

for $\delta = 0, \dots, \alpha_1(+)\alpha_2$. Let σ_δ be the set of all finite unions of elements of σ'_δ, $\delta = 0, \dots, \alpha_1(+)\alpha_2$. Since the finite intersection of finite unions of rectangles is a finite union of rectangles and the intersection of rectangle from $\mathcal{F}$ and rectangle from σ'_γ is a rectangle from σ'_γ it follows that σ_γ is a normal in itself base, $\gamma = -1, 0, \dots, \beta_j \leq \alpha_j$. Condition (a) is evidently satisfied. Condition (b) can be checked using induction. The base of induction follows from Proposition 4 and Theorem 23.

Since the finite sum theorem is satisfied on X_j for the dimension by normal bases $\mathcal{F}_j$, $j = 1, 2$, from Lemma 18 and Remark 22 (b) it follows that for any disjoint sets $A, B \in \mathcal{F}|_{F_1 \times F_2}$, where $F_1 \times F_2 \in \sigma'_\delta$ and $F_j \in \sigma^j_{\gamma(j)}, \gamma(j) \geq 0, j = 1, 2, \gamma(1)(+)\gamma(2) \leq \delta$, there exists their screening (G, H) in $F_1 \times F_2$ such that $G \cap H \in \sigma_\phi$, $\phi < \delta$, $\delta = 0, \dots, \alpha_1(+)\alpha_2$ (take into account condition (2) of the natural sum).

Now, let $F \in \sigma_\delta$ be of the form $F = F_1 \times F_2 \cup T_1 \times T_2$, where $F_1 \times F_2, T_1 \times T_2 \in \sigma'_\delta$ and $F_j \in \sigma^j_{i(j)}, T_j \in \sigma^j_{l(j)}, i(j), l(j) \geq 0, j = 1, 2, i(1)(+)i(2), l(1)(+)l(2) \leq \delta$,

$\delta = 0, \dots, \alpha_1(+)\alpha_2$ (the case of an arbitrary finite union can be examined in the same way).

Let the sets $A, B \in \mathcal{F}|_F$ be disjoint. Evidently $A, B \in \sigma_\delta \subset \mathcal{F}$. If either $i(1) \leq l(1)$ and $i(2) \leq l(2)$ or $i(1) \geq l(1)$ and $i(2) \geq l(2)$, then we can consider A and B as disjoint subsets of $F' = (F_1 \cup T_1) \times (F_2 \cup T_2) \in \sigma_\delta$ (see Remark 22 (a)), $\delta = 0, \dots, \alpha_1(+)\alpha_2$, and obviously $A, B \in \mathcal{F}|_{F'}$. Then, we can finish the proof in this case using the above observation.

Otherwise, by [12, Remark 1(4)] for a pair (A, B) of disjoint elements of $\mathcal{F}|_F$ there exist a disjoint pair (A', B') of elements of $\mathcal{F}|_F$ and a pair (G, H) of elements of $\mathcal{F}^c|_F$ such that $A \subset G \subset A'$ and $B \subset H \subset B'$.

With the use of Lemma 18 as in the proof of Lemma 12 there exist $\beta < \delta$; $H', G' \in \mathcal{F}^c|_{F_1 \times F_2}$ and $H'', G'' \in \mathcal{F}^c|_{T_1 \times T_2}$ such that:

$A' \cap (F_1 \times F_2) \subset H'$, $B' \cap (F_1 \times F_2) \subset G'$ and $(F_1 \times F_2) \setminus (H' \cup G') \in \sigma_\beta$;

$A' \cap (T_1 \times T_2) \subset H''$, $B' \cap (T_1 \times T_2) \subset G''$ and $(T_1 \times T_2) \setminus (H'' \cup G'') \in \sigma_\beta$;

$\min\{i(1), l(1)\}(+)\min\{i(2), l(2)\} < \delta$ by condition (2) of the natural sum.

We put

$$G^* = (G' \setminus (T_1 \times T_2)) \cup G \cup (G'' \setminus (F_1 \times F_2))$$

and

$$H^* = (H' \setminus (T_1 \times T_2)) \cup H \cup (H'' \setminus (F_1 \times F_2)).$$

Then, $(R = F \setminus H^*, S = F \setminus G^*)$ is a screening of (A, B) in F such that

$$R \cap S \subset \chi_1 \cup \chi_2 \cup ((F_1 \cap T_1) \times (F_2 \cap T_2)),$$

where $\chi_i \in \sigma_\beta$, $i = 1, 2$, and $(F_1 \cap T_1) \times (F_2 \cap T_2) \in \sigma_\beta$ since

$$\min\{i(1), l(1)\}(+)\min\{i(2), l(2)\} < \delta.$$

Thus, $R \cap S \in \sigma_\beta$, $\beta < \delta$, $\delta = 0, \dots, \alpha_1(+)\alpha_2$. ∎

Remark 25. The similar approach to the investigation of the transfinite inductive dimension of products was used in [3], [4], [20], [21]. Its origin is due to B. Pasynkov [18], [19].

Remark 26. There exists an example of compact spaces X_j, with $\mathrm{Ind} X_j = j$, $j = 1, 2$, such that $\mathrm{Ind} X_1 \times X_2 > 4$ [10]. Let $\mathcal{F}_j$ be the family of all closed sets of X_j, $j = 1, 2$. It is known, see for example [18], that the finite sum theorem for Ind is not satisfied in X_2. By Proposition 8(1), we have

$$\mathrm{I}(X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2) \geq \mathrm{Ind}(X_1 \times X_2) \geq 4.$$

Hence, if the finite sum theorem is not satisfied in factors for the dimension by normal bases then the inequality in Theorems 23 and 24 may not hold.

Remark 27. Let us note that the analog of the Theorem 24 may be given for finite and even infinite products after the product of normal bases is naturally defined.

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