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RIESZ SPACES OF MEASURES ON SEMIRINGS

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Abstract. It is shown that the spaces of finite valued signed measures (signed charges) on σ-semirings (semirings) are Dedekind complete Riesz spaces, which generalizes known results on σ-algebra and algebra cases.

In the literature, to the best of my knowledge, in measure theory there are two (slightly) different definitions of a "semiring" as a collection of subsets of a nonempty set with certain conditions. The notion of a "semiring" has first been defined in [2] as a nonempty collection $\mathcal{T}$ of a nonempty set X which satisfies, for each $A, B \in \mathcal{T}$, that $A \cap B \in \mathcal{T}$ and $A - B = \bigcup_n C_n$ for some pairwise disjoint sequence (C_n) in $\mathcal{T}$ (see also [5]). In [1], a nonempty set $\mathcal{T}$ of subsets of a nonempty set X is called a *semiring* on X if it is closed under finite intersections and for each $A, B \in \mathcal{T}$ there are pairwise disjoint sets $C_1, C_2, \dots, C_n$ such that $A - B = \bigcup_{i=1}^n C_i$. In this paper we use the notion of a semiring as in the later sense. A semiring $\mathcal{T}$ on X is called a *semi-algebra* if $X \in \mathcal{T}$ (see [4]). Of course algebras and σ-rings are semirings and there are plenty of examples of semirings which are not an algebra or a σ-ring.

A subset A of X is called a σ-*set* in a semiring $\mathcal{S}$ on X if $A = \bigcup_{n=1}^\infty A_n$ for some disjoint sequence (A_n) in $\mathcal{S}$. It is easy to see that if $A, A_1, A_2, \dots, A_n$ are in a semiring then $A - \bigcup_{i=1}^n A_i$ is a σ-set, but if $A \in \mathcal{S}$ and (A_n) is a sequence in $\mathcal{S}$ then $A - \bigcup_{n=1}^\infty A_n$ may not be a σ- set.

Example 1. i) Let $X = [0, 1)$ and $\mathcal{T} = \{[a, b) : 0 \le a \le b \le 1\}$ is a semiring on X, but $\{0\} = X - \bigcup_n [1/n, 1)$ is not a σ-set in $\mathcal{T}$.

ii) Let X be a countable infinite set, $\mathcal{T} = \{\{x\} : x \in X\} \cup \{\emptyset\}$. For each $A, A_1, A_2, \dots \in \mathcal{T}$, $A - \bigcup_n A_n$ is a σ-set, but $\mathcal{T}$ is neither an algebra nor a σ-ring.

This observation let us to introduce the following notion.

Definition 1. A semiring $\mathcal{S}$ on X is called a σ-*semiring* on a set X if for each $A \in \mathcal{S}$ and for each sequence (A_n) in $\mathcal{S}$ the set $A - \bigcup_n A_n$ is a σ- set.

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It should be noted that for sequences $(A_n), (B_n)$ in a σ-semiring $\mathcal{S}$ there exists a disjoint sequence (C_n) in $\mathcal{S}$ such that

$$\bigcup_n A_n - \bigcup_n B_n = \bigcup_n C_n.$$

If μ is a measure on $\mathcal{S}$ and $\bigcup_n A_n \subset \bigcup_n B_n$ then

$$\sum_n \mu(A_n) \leq \sum_n \mu(B_n).$$

For unknown definitions we refer to standard books [1] and [5].

1. The spaces of signed measures and charges as Riesz spaces

A map $\mu : \mathcal{S} \to \mathbf{R}$, where $\mathcal{S}$ is a semiring on a set X, is called a *signed measure* if it is the difference of two positive measures. Let Σ be a σ-algebra on a set X and let

$$M(\Sigma) = \{\mu : \mu : \Sigma \longrightarrow \mathbf{R} \quad \text{is a signed measure }\}.$$

It is well known that under the operations

$$(\mu_1 + \mu_2)(A) = \mu_1(A) + \mu_2(A), \quad (\alpha\mu)(A) = \alpha\mu(A)$$
$$\mu_1 \leq \mu_2 \Leftrightarrow \mu_1(A) \leq \mu_2(A) \quad \text{for all } A \in \Sigma$$

$M(\Sigma)$ is a Dedekind complete Riesz space and for any $\mu, \nu \in M(\Sigma)$ the supremum of μ and ν is determined by the formula

$$(\mu \vee \nu)(A) = \sup\{\mu(B) + \nu(A - B) : B \in \Sigma \quad \text{and} \quad B \subset A\}$$

(see [1] for a proof). The main result of this paper is to give a generalization of this as follows.

THEOREM 1. *Let $\mathcal{S}$ be a σ-semiring on a set X and let*

$$M(\mathcal{S}) = \{\mu : \mu : \mathcal{S} \longrightarrow \mathbf{R} \quad \textit{is a signed measure}\}.$$

Under the operations

$$(\mu_1 + \mu_2)(A) = \mu_1(A) + \mu_2(A), (\alpha\mu)(A) = \alpha\mu(A)$$
$$\mu_1 \leq \mu_2 \Leftrightarrow \mu_1(A) \leq \mu_2(A) \quad \textit{for all} \quad A \in \mathcal{S}$$

$M(\mathcal{S})$ is a Dedekind complete Riesz space and for any $0 \leq \mu, \nu \in M(\mathcal{S})$ the supremum of μ, ν is given by

$$w(A) = \sup\Big\{\sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \nu(B_n) : (A_n), (B_n) \quad \textit{are disjoint in} \quad \mathcal{S},$$
$$\bigcup_n A_n \subset A \quad \textit{and} \quad A - \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n\Big\}.$$

Proof. Firstly let $0 \le \mu, \nu \in M(\mathcal{S})$ be given and $w(A)$ be defined as above. It is easy to see that $w(A) \ge 0$ for each $A \in \mathcal{S}$ and $w(\emptyset) = 0$. Let $\{A_n\}$ be a disjoint sequence in $\mathcal{S}$ satisfying $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{S}$. Let (B_n) be a disjoint sequence in $\mathcal{S}$ with $\bigcup_{n=1}^{\infty} B_n \subset A$ and choose a disjoint sequence (C_n) in $\mathcal{S}$ such that $A - \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n$. For each m, let $({T_n}^m)$ be a disjoint sequence in $\mathcal{S}$ satisfying

$$\bigcup_{n=1}^{\infty}(C_n \cap A_m) \subset A_m - \bigcup_{n=1}^{\infty}(B_n \cap A_m) = \bigcup_{n=1}^{\infty} {T_n}^m$$

which implies

$$\sum_n \nu(C_n \cap A_m) \le \sum_n \nu({T_n}^m).$$

Also we have

$$\begin{aligned}\sum_n \mu(B_n) + \sum_n \nu(C_n) &= \sum_n \mu(\textstyle\bigcup_m (B_n \cap A_m)) + \sum_n \nu(\bigcup_m (C_n \cap A_m)) \\ &= \sum_n \sum_m \mu(B_n \cap A_m) + \sum_n \sum_m \nu(C_n \cap A_m) \\ &= \sum_m (\sum_n \mu(B_n \cap A_m) + \sum_n \nu(C_n \cap A_m)) \\ &\le \sum_m (\sum_n \mu(B_n \cap A_m) + \sum_n \nu({T_n}^m)) \\ &\le \sum_m w(A_m),\end{aligned}$$

so that

$$w(\bigcup_n A_n) \le \sum_m w(A_n).$$

For the converse direction let $\epsilon > 0$. For each n choose a disjoint sequences $({B_m}^n), ({T_m}^n)$ in $\mathcal{S}$ satisfying

$$\bigcup_m {B_m}^n \subset A_n \quad \text{and} \quad A_n - \bigcup_m {B_m}^n = \bigcup_m {T_m}^n$$

such that

$$w(A_n) - \epsilon/2^n \le \sum_m \mu({B_m}^n) + \sum_m \nu({T_m}^n).$$

So

$$\sum_n w(A_n) - \epsilon \le \sum_n \sum_m \mu({B_m}^n) + \sum_n \sum_m \nu({T_m}^n).$$

Note that for each i, j, m, n

$${B_m}^n \cap {T_i}^j = \emptyset \quad \text{and} \quad \bigcup_n \bigcup_m {B_m}^n \subset \bigcup_n A_n$$

and there exists a disjoint sequence (C_n) in $\mathcal{S}$ such that

$$\bigcup_n \bigcup_m {T_m}^n \subset \bigcup_n A_n - \bigcup_n \bigcup_m {B_m}^n = \bigcup_m C_m$$

and

$$\sum_n \sum_m \mu({T_m}^n) \leq \sum_m \mu(C_m).$$

Now it is clear that

$$\sum_n w(A_n) - \epsilon \leq w(\bigcup_n A_n).$$

Since $\epsilon > 0$ is arbitrary, we have

$$w(\bigcup_n A_n) = \sum_n w(A_n).$$

So far we have shown that w is an upper bound of μ, ν in $M(\mathcal{S})$. Now suppose that β is another upper bound of μ, ν. Let $A \in \mathcal{S}$ and $(A_n), (B_n)$ be arbitrary disjoint sequence in $\mathcal{S}$ with

$$\bigcup_n A_n \subset A \quad \text{and} \quad A - \bigcup_n A_n = \bigcup B_n$$

and

$$\sum_n \mu(A_n) + \sum_n \nu(B_n) \leq \sum_n \beta(A_n) + \sum \beta(B_n) = \beta(\bigcup_n A_n \cup \bigcup_n B_n) = \beta(A)$$

which implies that $w \leq \beta$ in $M(\mathcal{S})$. We have proved that $\mu \vee \nu$ exists for each $0 \leq \mu, \nu \in M(\sum)$. Let $\mu, \nu \in M(\mathcal{S})$ with $\alpha \leq \nu, \mu$ for some measure $-\alpha$. Now it is routine to check that

$$(\mu - \alpha) \vee (\nu - \alpha) + \alpha$$

is least upper bound on μ and ν, i.e. $\mu \vee \nu$ exists. Hence $M(\mathcal{S})$ is a Riesz space. If $\mu_\alpha \uparrow \leq \mu$ we define $\mu_\infty(A) = sup_\alpha \mu_\alpha(A)$ then it is easy to show that $\mu_\infty \in M(\mathcal{S})$ and $\mu_\alpha \uparrow \mu_\infty$, which proves that $M(\mathcal{S})$ is Dedekind complete. ∎

It is known that if $\mathcal{A}$ is an algebra on a set X, then the vector space

$$C(\mathcal{A}) = \{\mu : \mu : \mathcal{A} \longrightarrow \mathbf{R} \quad \text{is a signed charge}\}$$

is a vector lattice under pointwise order and supremum of $\mu, \nu \in C(\mathcal{A})$ is given by

$$\mu \vee \nu(A) = w(A) = \sup\{\mu(B) + \nu(A - B) : B \in \mathcal{A} \quad \text{and} \quad B \subset A\}$$

(see [2] for a proof). This result can be generalized as follows and its proof is similar to the proof of the above theorem.

Theorem 2. *Let $\mathcal{A}$ be a semiring (not necessarily a σ-semiring) on X. Then*

$$M(\Sigma) = \{\mu : \mu : \mathcal{A} \longrightarrow \mathbf{R} \quad \textit{is a signed charge}\}$$

is a Dedekind complete vector lattice under usual operations and supremum of any μ, ν is given by

$$(\mu \vee \nu)(A) = \sup\Big\{\sum_{i=1}^k \mu(A_i) + \sum_{i=1}^k \nu(B_i) : A_i, B_i \in \mathcal{A}, \quad \bigcup_{i=1}^k A_i \subset A \quad \textit{and}$$

$$A - \bigcup_{i=1}^k A_i = \bigcup_{i=1}^k B_i, \quad A_i \cap A_j = B_i \cap B_j = \phi \quad \textit{for all} \quad i \neq j\Big\}.$$

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