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STABILITY AND BOUNDEDNESS PROPERTIES OF SOLUTIONS TO CERTAIN FIFTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider the nonlinear fifth order differential equation

$$x^{(v)} + ax^{(iv)} + b\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t; x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)})$$

and we used the Lyapunov's second method to give sufficient criteria for the zero solution to be globally asymptotically stable as well as the uniform boundedness of all solutions with their derivatives.

1. Introduction

We shall be concerned here with the differential equations of the form

$$x^{(v)} + ax^{(iv)} + b\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t; x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}) \tag{1.1}$$

with a and b being positive constants. The functions f, g, h and p are continuous in the respective arguments displayed explicitly. The dot means the derivative of the variable with respect to t. Furthermore, the functions are such that uniqueness and continuous dependence on initial condition is guaranteed.

The study of higher order nonlinear differential equations has received considerably much attention and still receiving such from various researchers. Boundedness and stability properties of solutions for various nonlinear third and fourth order differential equations have been considered by many authors (see [6–11], [15], [16], [18–22]). Some of the earlier results are summarized in [12].

Problems for various equations of the fifth order nonlinear differential equations have been examined to quite considerable extent (see [1–3], [5], [14], [17]) but not much as in the case of the third and fourth order equations.

In [2], the author employed frequency domain method to investigate the periodicity and stability for solution for nonlinear differential equation of the fifth

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order and gave conditions for the nonlinear functions under which the equation considered have bounded, globally exponentially stable and periodic solution using the frequency domain method.

In [1], [5], [14], [17], the authors employed the use of the Lyapunov second method to discuss these properties (boundedness and stability) of solutions of the classes of equations considered. In almost all these works, an incomplete Lyapunov (Yoshizawa) function have been prominent and the few complete Lyapunov functions constructed for the fifth order equations were made by the use of signum functions (see [5], [23]).

Since the Lyapunov second method has been established to be one of the most effective method to study the qualitative properties of solutions of differential equations, in this paper we shall give criteria for the existence of a unique solution to the equation (1.1) which is stable (globally asymptotically stable) and bounded (uniformly ultimately bounded) with its derivatives on the real line.

We shall achieve this by the use of a suitable single complete Lyapunov function without the use of any signum function and stringent condition on the functions other than the continuity condition. As in [10] and [11], we adapted Cartwright [4] for the construction of the Lyapunov function used in this work.

To be able to use the Lyapunov second method, the equation (1.1) is reduced to system of first order equations given as

$$\begin{gathered} \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = u, \quad \dot{u} = w \\ \dot{w} = -aw - bu - f(z) - g(y) - h(x) + p(t; x, y, z, u, w) \end{gathered} \tag{1.2}$$

In order to reach our main results, we will first give some important basic definitions for the general non-autonomous differential system. We consider the system

$$\dot{x} = f(t, x) \tag{1.3}$$

where $f \in C[I \times S_\rho]$, $I = [0, \infty)$, $t \geq 0$, and $S_\rho = \{x \in \Re^n : \|x\| < \rho\}$. Assume that f is smooth enough to ensure the existence and uniqueness of solutions of (1.3) through every point $(t_0, x_0) \in J \times S_\rho$. Also, let $f(t, 0) = 0$ so that (1.3) admits the zero solution $x \equiv 0$.

Definition 1.1. [23] The zero solution of (1.1) is said to be stable, if given $\epsilon > 0$ and $t_0 \in I_0$, there exists a $\delta(t_0, \epsilon) > 0$, such that $|x_0| < \delta(t_0, \epsilon)$, $|x(t; x_0), t_0| < \epsilon$ for all $t \geq t_0$.

Definition 1.2. [23] The solution $x(t) \equiv 0$ of (1.1) is asymptotically stable in the whole (globally asymptotically stable) if it is stable and every solution of (1.1) tends to zero as $t \to \infty$.

Definition 1.3. [23] The solution $x(t) \equiv 0$ of (1.1) is uniformly asymptotically stable if it is stable and there exists a $\delta(t_0) > 0$ such that $\|x(t; t_0, x_0)\| \to 0$ as $t \to \infty$ for all $x_0 \in S_{\delta_0}$.

DEFINITION 1.4. [23] The solution $x(t) \equiv 0$ of (1.1) is stable if for any $\epsilon > 0$ and any $t_0 \in I$ there exists a $\delta(t_0, \epsilon) < 0$ such that if $x_0 \in S_{\delta(t_0,\epsilon)}$ then $x(t; t_0, x_0) \in S_\epsilon$ for all $t \geq t_0$.

DEFINITION 1.5. [23] The solution $x(t)$ of (1.1) is bounded if there exists a $\beta > 0$, there exists a constant M such that $\|x(t, t_0, x_0)\| < M$ whenever $\|x_0\| < \beta$, $t \geq t_0$.

DEFINITION 1.6. [23] The solution $x(t)$ of (1.1) is ultimately bounded for bound **M**, if there exist $M > 0$ and $T > 0$, such that for every solution $x(t; t_0, x_0)$ of (1.1) $\|x(t, t_0, x_0)\| < M$ for all $t > t_0 + T$.

We shall also give the following definitions in our context.

DEFINITION 1.7. [11] A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be complete if for $X \in \Re^n$,

(i) $V(t, X) \geq 0$

(ii) $V(t, X) = 0$, if and only if $X = 0$ and

(iii) $\dot{V}|_{1.3}(t, X) \leq -c\,|X|$ where c is any positive constant and $|X|$ is given by $|X| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ such that $|X| \to \infty$ as $X \to \infty$.

DEFINITION 1.8. [11] A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be incomplete if for $X \in \Re^n$, conditions (i) and (ii) of Definition 1.5 are satisfied, and in addition

(iii) $\dot{V}(t, X)|_{1.3} \leq -c\,|X|_*$ where c is any positive constant and $|X|_*$ is given as $|X|_* = \left(\sum_{i=1}^{<n} x^2\right)^{1/2}$ such that $|X|_* \to \infty$ as $X \to \infty$.

To make our definition of complete and incomplete Lyapunov functions clearer we shall consider a simple case where $n = 2$.

Consider the simple 2nd order linear differential equation

$$\ddot{x} + a\dot{x} + bx = 0$$

(where a and b are all positive) with an equivalent system

$$\dot{x} = y, \quad y = -ay - bx \tag{1.4}$$

The following are some of the possible Lyapunov function for the system.

$$2V = \left(\frac{c+\delta}{a}\right) bx^2 + \left(\frac{c+\delta}{a}\right) y^2 \tag{1.5}$$

$$2V = \left(\frac{cb^2 + \delta a^2}{ab}\right) x^2 + \left(\frac{c}{a}\right) y^2 + 2\frac{\delta}{b} xy \tag{1.6}$$

$$2V = \left(\frac{b^2(b+c) + \delta a^2}{ab}\right) x^2 + \left(\frac{\delta + c}{a}\right) y^2 + 2\frac{\delta}{b} xy, \tag{1.7}$$

where $\delta > 0$.

Let $(x(t), y(t))$ be any solution of (1.4) then by a straightforward calculation form (1.5)–(1.7) and (1.4), we observe that

$$\dot{V} = -\delta y^2, \quad \dot{V} = -\delta x^2, \quad \text{and} \quad \dot{V} = -\delta(x^2 + y^2)$$

are the derivatives of V with respect to the system (1.4) respectively. Lyapunov functions defined as in (1.5) and (1.6) are referred to as incomplete while the one defined by (1.7) is complete.

2. Formulation of results

Let the functions f, g, h and p be continuous and the following conditions hold:

(i) $\theta = \frac{f(z)-f(0)}{z} \leq c \in I_0,\ z \neq 0$, with $I_0 = [\delta,\ \Delta] > 0$

(ii) $\beta = \frac{g(y)-g(0)}{y} \leq d \in I_0,\ y \neq 0$

(iii) $\gamma = \frac{h(x)-h(0)}{x} \leq e \in I_0,\ x \neq 0$

(iv) $f(0) = g(0) = h(0) = 0$.

Theorem 2.1. *Suppose the conditions (i)–(iv) are satisfied with* $p(t; x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}) \equiv 0$, *then the trivial solution of the equation (1.1) is globally asymptotically stable.*

Theorem 2.2 *In addition to the conditions (i)–(iv) suppose*

(v) $p(t; x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}) \equiv p(t)$ *and* $|p(t)| \leq M$ *for all* $t \leq 0$.

Then there exists a constant σ, $(0 < \sigma < \infty)$ *depending only on the constants* a, b, c, d, e *and* δ *such that every solution of (1.1) satisfies*

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) + {x^{(iv)}}^2(t) \leq e^{-\frac{1}{2}\sigma t} \left\{ A_1 + A_2 \int_{t_0}^{t} |p(\tau)|\, e^{\frac{1}{2}\sigma\tau}\, d\tau \right\}^2$$

for all $t \geq t_0$, *where the constant* $A_1 > 0$ *depends on* a, b, c, d, e, δ *as well as on* $t_0, x(t_0), \dot{x}(t_0), \ddot{x}(t_0), \dddot{x}(t_0), x^{(iv)}(t_0)$*; and the constant* $A_2 > 0$ *depends on* a, b, c, d, e *and* δ.

Theorem 2.3 *Suppose the conditions of Theorem 2.2 with condition (v) replaced with*

(vi) $|p(t; x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)})| \leq (|x| + |y| + |z| + |u| + |w|)\phi(t)$ *where* $\phi(t)$ *is a nonnegative and continuous function of* t, *and satisfies* $\int_0^t \phi(s)\, ds \leq M < \infty$ *and* M *a positive constant.*

Then there exists a constant K_0 *which depends on* M, K_1, K_2 *and* t_0 *such that every solution* $x(t)$ *of equation (1.1) satisfies*

$$|x(t)| \leq K_0, \quad |\dot{x}(t)| \leq K_0, \quad |\ddot{x}(t)| \leq K_0, \quad |\dddot{x}(t)| \leq K_0, \quad \left|x^{(iv)}(t)\right| \leq K_0$$

for all sufficiently large t.

Remark. We wish to remark here that while Theorem 2.1 is on the global asymptotic stability of the trivial solution, Theorems 2.2 and 2.3 are dealing with the boundedness and ultimate boundedness of the solutions respectively.

It is well known that all solution of corresponding linear equation to (1.1) given as

$$x^{(v)} + ax^{(iv)} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = p(t)$$

tend to trivial solution, as $t \to \infty$ (is asymptotically stable), provided the Routh-Hurwitz conditions $a > 0$, $(ab - c) > 0$, $(ab - c)c - (ad - e)a > 0$, $\Delta : (cd - be)(ab - c) - (ad - e)^2 > 0, e > 0$ hold.

Notations. Throughout this paper $K, K_0, K_1, \ldots, K_{14}$ will denote finite positive constants whose magnitudes depend only on the functions g, h and p as well as constants a, b, c, d and δ but are independent of solutions of the equation (1.1). $K_i's$ are not necessarily the same for each time they occur, but each $K_i, i = 1, 2, \ldots.$ retains its identity throughout.

3. Preliminary Results

We shall use as a tool to prove our main results—a function $V(x, y, z, u, w)$ defined by

$$\begin{aligned} 2V = {} & Ax^2 + By^2 + Cz^2 + Du^2 + Ew^2 \\ & + 2Fxy + 2Gxz + 2Hxu + 2Jxw + 2Lyu + 2Myw + 2Nzu + 2Ozw + 2Puw \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
A &= \frac{\delta}{ced\Delta}\left\{(e^2 + c^2 + ce)(ab - c) + cd^2\Delta)\right\} \\
B &= \frac{\delta}{ced\Delta}\left\{[(d(ed - bc) + (e^2 + c^2 + ce)c + e^2cd]\Delta\right. \\
&\quad \left. -ae^2[(e^2 + c^2 + ce)(ab - c) + cd[ce(b + 1) - a^2] + aced(1 - d)]\right\} \\
C &= \frac{\delta}{ced\Delta}\left\{[(e^2 + c^2 + ce)(ab - c) + cd[ce(b + 1) - a^2]\right. \\
&\quad \left. +aced(1 - d)](a - d)e + [c(1 - e) + edb]d\Delta\right\} \\
D &= \frac{\delta}{cd\Delta}\left\{(a^2 + b)[(e^2 + c^2 + ce)(ab - c) + cd[ce(b + 1) - a^2]\right. \\
&\quad \left. +aced(1 - d)] - d(1 + ac)\Delta\right\} \\
E &= \frac{\delta}{cd\Delta}\left\{[(e^2 + c^2 + ce)(ab - c) + cd[ce(b + 1) - a^2] + aced(1 - d)]\right\} \\
F &= \frac{\delta}{ce\Delta}\left\{\Delta(e^2 + c^2)\right\} \\
G &= \frac{\delta}{ced\Delta}\left\{ae^2[(e^2 + c^2 + ce)(ab - c) + cd[ce(b + 1) - a^2]\right. \\
&\quad \left. +aced(1 - d)] + (b - e^2)cd\Delta\right\}
\end{aligned}$$

$$H = \frac{\delta}{ced\Delta}\left\{e^2[(e^2+c^2+ce)(ab-c)+cd[ce(b+1)-a^2]+aced(1-d)]+acd\Delta\right\}$$
$$J = \frac{\delta}{e\Delta}\left\{\Delta\right\}$$
$$L = \frac{\delta}{ced\Delta}\left\{[(e^2+c^2)ae+c(ae^2-d)]\Delta+ed[(e^2+c^2+ce)(ab-c)\right.$$
$$\left.+cd[ce(b+c)-a^2]+aced(1-d)]\right\}$$
$$M = \frac{\delta}{ced\Delta}\left\{(e^2+c^2+ce)\Delta\right\}$$
$$N = \frac{\delta}{cd\Delta}\left\{b[(e^2+c^2+ce)(ab-c)+cd[ce(b+1)-a^2]+aced(1-d)]-cd\Delta\right\}$$
$$O = \frac{\delta}{c\Delta}\left\{\Delta\right\}$$
$$P = \frac{\delta}{cd\Delta}\left\{a[(e^2+c^2+ce)(ab-c)+cd[ce(b+1)-a^2]+aced(1-d)]-cd\Delta\right\}$$
$$\Delta = e[(ab-c)c-(ad-e)a].$$

LEMMA 3.1 *Subject to the assumptions of Theorem 2.1 there exist positive constants* $K_i = K_i(a,b,c,d,e,\delta)$, $i=1,2$ *such that*

$$K_1(x^2+y^2+z^2+u^2+w^2) \le V(x,y,z,u,w) \le K_2(x^2+y^2+z^2+u^2+w^2). \quad (3.2)$$

Proof. Clearly $V(0,0,0,0,0) \equiv 0$. Rearranging (3.1) we have

$$\begin{aligned}
2V = \frac{\delta}{ced\Delta}\Bigg\{ & \ell_1x^2+\ell_2y^2+\ell_3z^2+\ell_4u^2+\ell_5w^2+ce(ab-c)\left(x+\frac{Gz}{ce(ab-c)}\right)^2 \\
& +(e^2+c^2)c\Delta\left(y+\frac{x}{c}\right)^2+c(1-e)\Delta\left(z+\frac{Nu}{c(1-e)\Delta}\right)^2 \\
& +acde^2(1-d)(a-d)\left(z+\frac{Ow}{acde^2(1-d)(a-d)}\right)^2 \\
& +bce(ab-c)\left(u+\frac{Pw}{bce(ab-c)}\right)^2 \\
& +a^2cd[ce(b+1)-a^2]\left(u+\frac{Ly}{a^2cd[ce(b+1)-a^2]}\right)^2 \\
& +bcd[ce(b+1)-a^2]\left(u+\frac{Hx}{bcd[ce(b+1)-a^2]}\right)^2 \\
& +ce(ab-c)\left(w+\frac{Jx}{ce(ab-c)}\right)^2 \\
& +cd[ce(b+1)-a^2]\left(w+\frac{My}{cd[ce(b+1)-a^2]}\right)^2\Bigg\}
\end{aligned} \quad (3.3)$$

with

$$\ell_1 = \frac{bd[ce(b+1)-a^2]\left\{e(ab-c)[(d^2-1)c^2-e^2]\Delta - J^2\right\} - H^2 e(ab-c)}{bcde[ce(b+1)-a^2](ab-c)}$$

$$\ell_2 = \frac{(acde)^2\Delta - (L^2 + a^2M^2)}{a^2cd[ce(b+1)-a^2]}$$

$$\ell_3 = \frac{bc(de)^2 - G^2}{ce(ab-c)}$$

$$\ell_4 = \frac{c(1-e)(a^2+b)ace^2d(1-d)\Delta - N^2}{c(1-e)\Delta}$$

$$\ell_5 = \frac{b(acde^2)^2(ab-c)(1-d)^2(a-d) - O^2b(ab-c) - P^2aed(1-d)(a-d)}{abcde^2(1-d)(a-d)(ab-c)}$$

$$V \geq \frac{\delta}{ced\Delta}\left\{\ell_1x^2+\ell_2y^2+\ell_3z^2+\ell_4u^2+\ell_5w^2\right\} \geq K_1\left(x^2+y^2+z^2+u^2+w^2\right). \tag{3.4}$$

where

$$K_1 = \frac{\delta}{ced\Delta} \times \min\left\{\ell_1,\ell_2,\ell_3,\ell_4,\ell_5\right\}.$$

Applying the Cauchy-Schwartz inequality $|xy| \leq \frac{1}{2}|x^2+y^2|$ to the equation (3.1), we have

$$\begin{aligned}2V \leq \big\{&(A+F+G+H+J)x^2 + (B+F+L+M)y^2 + (C+G+N+O)z^2 \\ &+(D+H+L+N+P)u^2 + (E+J+M+O+P)w^2\big\}.\end{aligned} \tag{3.5}$$

From the equation (3.5), we have that

$$V \leq K_2\left(x^2+y^2+z^2+u^2+w^2\right), \tag{3.6}$$

with

$$\begin{aligned}K_2 = \frac{1}{2}\max\{&(A+F+G+H+J),(B+F+L+M),(C+G+N+O),\\ &(D+H+L+N+P),(E+J+M+O+P)\}.\end{aligned}$$

From the equations (3.4) and (3.6) we have that

$$K_1\left(x^2+y^2+z^2+u^2+w^2\right) \leq V \leq K_2\left(x^2+y^2+z^2+u^2+w^2\right) \tag{3.7}$$

This proves Lemma 3.1. ■

Lemma 3.2. *Suppose that the assumptions of Theorem 2.1 hold and in addition let the condition (ii) of the Theorem 2.2 be satisfied also. Then there are positive constants* $K_j = K_j(a,b,c,d,e,\delta)$ $(j=3,4)$ *such that for any solution* (x,y,z,u,w) *of system (1.2),*

$$\begin{aligned}\dot{V}|_{(1.2)} &\equiv \frac{d}{dt}V|_{(1.2)}(x,y,z,u,w)\\ &\leq -K_3(x^2+y^2+z^2++u^2+w^2) + K_4(|x|+|y|+|z|+|u|+|w|)\,|p(t)|.\end{aligned} \tag{3.8}$$

Proof. Differentiating (2.1) with respect to t we have

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} + \frac{\partial V}{\partial u}\dot{u} + \frac{\partial V}{\partial w}\dot{w} \tag{3.9}$$

using the system (1.2) in (3.9), we have

$$\begin{aligned}\dot{V} = \{&-Jh(x) - Mg(y)y - Of(z)z - Pbu^2 - Eaw^2\\ &+Jx(-aw - bu - f(z) - g(y) + p) + My(-aw - bu - f(z) - h(x) + p)\\ &Oz(-aw - bu - g(y) - h(x) + p) + Pu(-aw - f(z) - g(y) - h(x) + p)\\ &+Ew(-bu - f(z) - g(y) - h(x) + p) + w(Hx + Ly + Nz + Pw)\\ &+u(Gx + Cz + Nu + Dw) + (fx + by + Lu + Mw)\\ &+y(Ax + Fy + Gz + Hu + Jw)\}\end{aligned} \tag{3.10}$$

On simplifying using the conditions in the formulation of results, we have

$$\begin{aligned}\dot{V} = -\{&J\gamma x^2 + (M\beta - F)y^2 - O\theta z^2 - (Pb - N)u^2 + (Ea - P)w^2\\ &+(J\beta + M\gamma - A)xy - (J\theta + O\gamma - F)xz + (Jb + P\gamma - G)xu\\ &+(Ja + G\gamma - H)xw + (M\theta + O\beta - G - B)yz + (Mb + P\beta - H)yu\\ &+(Ma + E\beta - L - J)yw + (Ob + P\theta - L - C)zu\\ &+(E\theta + Oa - N - M)zw + (Pa + Eb - O - D)uw\\ &-(Jx + My + Oz + Pu + Ew)(h(0) + g(0) + f(0) + p)\}\end{aligned} \tag{3.11}$$

which reduces to

$$\begin{aligned}\dot{V} &\leq -\delta(x^2 + y^2 + z^2 + u^2 + w^2) + (Jx + My + Oz + Pu + Ew)p\\ &\leq -\delta(x^2 + y^2 + z^2 + u^2 + w^2) + K_5(|x| + |y| + |z| + |u| + |w|)p\end{aligned} \tag{3.12}$$

where $K_5 = \max\{J, M, O, P, E\}$. Since

$$(|x| + |y| + |z| + |u| + |w|) \leq \sqrt{5}(x^2 + y^2 + z^2 + u^2 + w^2)^{\frac{1}{2}},$$

inequality (3.12) becomes

$$\frac{dV}{dt} \leq -\delta(x^2 + y^2 + z^2 + u^2 + w^2) + K_6(x^2 + y^2 + z^2 + u^2 + w^2)^{\frac{1}{2}}\,|p(t)|\,, \tag{3.13}$$

where $K_6 = \sqrt{5}K_5$. Choosing $\delta = K_7$, we have

$$\begin{aligned}\dot{V}|_{(1.2)} &\equiv \frac{d}{dt}V|_{(1.2)}(x, y, z, u, w)\\ &\leq -K_7(x^2 + y^2 + z^2 + +u^2 + w^2) + K_6(x^2 + y^2 + z^2 + u^2 + w^2)^{\frac{1}{2}}\,|p(t)|\,.\end{aligned} \tag{3.14}$$

This completes the proof of Lemma 3.2. ■

4. Proofs of the main results

Proof of Theorem 2.1. From the proofs of Lemmas 3.1 and 3.2 it is established that the trivial solution of the equation (1.1) is globally asymptotically stable, i.e. every solution $(x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}(t), x^{(iv)}(t))$ of the system (1.2) satisfies $x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) + {x^{(iv)}}^2(t) \to 0$ as $t \to \infty$. ■

Proof of Theorem 2.2. Indeed, from the inequality (3.14),

$$\frac{dV}{dt} \leq -K_7(x^2 + y^2 + z^2 + u^2 + w^2) + K_6(x^2 + y^2 + z^2 + u^2 + w^2)^{\frac{1}{2}} |p(t)| ,$$

and also from the inequality (3.4), we have

$$(x^2 + y^2 + z^2 + u^2 + w^2)^{\frac{1}{2}} \leq \left(\frac{V}{K_1}\right)^{\frac{1}{2}} .$$

Thus the inequality (3.14) becomes

$$\frac{dV}{dt} \leq -K_8 V + K_9 V^{\frac{1}{2}} |p(t)| . \tag{4.1}$$

We note that $K_7(x^2 + y^2 + z^2 + u^2 + w^2) = K_7 \cdot \frac{V}{K_2}$ and

$$\frac{dV}{dt} \leq -K_8 V + K_9 V^{\frac{1}{2}} |p(t)| \tag{4.2}$$

where $K_8 = \frac{K_7}{K_2}$ and $K_9 = \frac{K_6}{K_2^{1/2}}$. These imply that $\dot{V} \leq -K_8 V + K_9 V^{\frac{1}{2}} |p(t)|$ and this can be written as

$$\dot{V} \leq -2K_{10} V + K_9 V^{\frac{1}{2}} |p(t)| , \tag{4.3}$$

where $K_{10} = \frac{1}{2} K_8$. Therefore

$$\dot{V} + K_{10} V \leq -K_{10} V + K_9 V^{\frac{1}{2}} |p(t)| \leq K_9 V^{\frac{1}{2}} \left\{ |p(t)| - K_{11} V^{\frac{1}{2}} \right\} , \tag{4.4}$$

where $K_{11} = \frac{K_{10}}{K_9}$. Thus the inequality (4.4) becomes

$$\dot{V} + K_{10} V \leq K_9 V^{\frac{1}{2}} V^* \tag{4.5}$$

where

$$V^* = |p(t)| - K_{11} V^{\frac{1}{2}} \leq V^{\frac{1}{2}} |p(t)| \leq |p(t)| . \tag{4.6}$$

When $|p(t)| \leq K_{11} V^{\frac{1}{2}}$,

$$V^* \leq 0 \tag{4.7}$$

and when $|p(t)| \geq K_{11} V^{\frac{1}{2}}$,

$$V^* \leq |p(t)| \cdot \frac{1}{K_{11}} . \tag{4.8}$$

On substituting the inequality (4.7) into the inequality (4.4), we have,

$$\dot{V} + K_{10} V \leq K_{12} V^{\frac{1}{2}} |p(t)|$$

where $K_{12} = \frac{K_9}{K_{11}}$. This implies that

$$V^{-\frac{1}{2}}\dot{V} + K_{10}V^{\frac{1}{2}} \leq K_{12}\left|p(t)\right|. \tag{4.9}$$

Multiplying both sides of the inequality (4.9) by $e^{\frac{1}{2}K_{10}t}$ we have

$$e^{\frac{1}{2}K_{10}t}\left\{V^{-\frac{1}{2}}\dot{V} + K_{10}V^{\frac{1}{2}}\right\} \leq e^{\frac{1}{2}K_{10}t}K_{12}\left|p(t)\right| \tag{4.10}$$

i.e.

$$2\frac{d}{dt}\left\{V^{\frac{1}{2}}e^{\frac{1}{2}K_{10}t}\right\} \leq e^{\frac{1}{2}K_{10}t}K_{12}\left|p(t)\right|. \tag{4.11}$$

Integrating both sides of (4.11) from t_0 to t gives

$$\left\{V^{\frac{1}{2}}e^{\frac{1}{2}K_{10}\gamma}\right\}_{t0}^{t} \leq \int_{t0}^{t}\frac{1}{2}e^{\frac{1}{2}K_9\tau}K_{12}\left|p(\tau)\right|d\tau \tag{4.12}$$

which implies that

$$\left\{V^{\frac{1}{2}}(t)\right\}e^{\frac{1}{2}K_{10}t} \leq V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_{10}t_0} + \frac{1}{2}K_{12}\int_{t0}^{t}\left|p(\tau)\right|e^{\frac{1}{2}K_{10}\tau}d\tau,$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-\frac{1}{2}K_{10}t}\left\{V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_{10}t_0} + \frac{1}{2}K_{12}\int_{t0}^{t}\left|p(\tau)\right|e^{\frac{1}{2}K_{10}\tau}d\tau\right\}.$$

Using (3.5) and (3.6) we have

$$\begin{aligned}
&K_1(x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) + x^{(iv)^2}(t)) \\
&\quad\leq e^{-\frac{1}{2}K_{10}t}\left\{K_2(x^2(t_0) + \dot{x}^2(t_0) + \ddot{x}^2(t_0) + \dddot{x}^2(t_0) + x^{(iv)^2}(t_0))\right. \\
&\qquad\qquad\left. e^{\frac{1}{2}K_{10}t_0} + \frac{1}{2}K_{12}\int_{t0}^{t}\left|p(\tau)\right|e^{\frac{1}{2}K_{10}\tau}d\tau\right\}^2
\end{aligned} \tag{4.13}$$

for all $t \geq t_0$. Thus,

$$\begin{aligned}
&x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) + x^{(iv)^2}(t_0) \\
&\leq \frac{1}{K_1}\left\{e^{-\frac{1}{2}K_{10}t}\left\{K_2(x^2(t_0) + \dot{x}^2(t_0) + \ddot{x}^2(t_0) + \dddot{x}^2(t_0) + x^{(iv)^2}(t_0))e^{\frac{1}{2}K_{10}t_0}\right.\right. \\
&+ \left.\left.\frac{1}{2}K_{12}\int_{t0}^{t}\left|p(\tau)\right|e^{\frac{1}{2}K_{10}\tau}d\tau\right\}^2\right\} \leq \left\{e^{-\frac{1}{2}K_{10}t}\left\{A_1 + A_2\int_{t0}^{t}\left|p(\tau)\right|e^{\frac{1}{2}K_{10}\tau}d\tau\right\}^2\right\}
\end{aligned} \tag{4.14}$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, \ldots K_{12}$ and $(x^2(t_0) + \dot{x}^2(t_0) + \ddot{x}^2(t_0)) + \dddot{x}^2(t_0) + x^{(iv)^2}(t_0)\}$. By substituting $K_{10} = \sigma$ in the inequality (4.14), we have

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) + x^{(iv)^2}(t) \leq \left\{e^{-\frac{1}{2}\sigma t}\left\{A_1 + A_2\int_{t0}^{t}\left|p(\tau)\right|e^{\frac{1}{2}\sigma\tau}d\tau\right\}^2\right\},$$

which completes the proof. ■

Proof of Theorem 2.3. From the function V defined above and the conditions of Theorem 2.3, the conclusion of Lemma 3.1 can be obtained, as

$$V \geq K_1\left(x^2+y^2+z^2+u^2+w^2\right), \tag{4.15}$$

and since $p \neq 0$ we can revise the conclusion of Lemma 3.2, i.e,

$$\dot{V} \leq -K_7(x^2+y^2+z^2+u^2+w^2)+K_6(|x|+|y|+|z|+|u|+|w|)\,|p(t)|\,,$$

and we obtain by using the condition on $p(t;x,y,z,u,w)$ as stated in the Theorem 2.3 that

$$\dot{V} \leq K_6(|x|+|y|+|z|+|u|+|w|)^2 r(t). \tag{4.16}$$

By applying the Schwartz inequality to (4.16), we have

$$\dot{V} \leq K_{13}(x^2+y^2+z^2+u^2+w^2)r(t), \tag{4.17}$$

where $K_{13}=4K_6$. From inequalities (4.15) and (4.17) we have

$$\dot{V} \leq K_{13}Vr(t). \tag{4.18}$$

Integrating inequality (4.18) from 0 to t, we obtain

$$V(t)-V(0) \leq K_{14}\int_0^t V(s)r(s)\,ds. \tag{4.19}$$

where $K_{14}=\frac{K_{13}}{K_1}=\frac{4K_6}{K_1}$

$$V(t) \leq V(0)+K_{11}\int_0^t V(s)r(s)ds. \tag{4.20}$$

Applying the Grownwall-Reid-Bellman theorem to the inequality (4.20) yields

$$V(t) \leq V(0)\exp\left(K_{14}\int_0^t r(s)ds\right). \tag{4.21}$$

This completes the proof of Theorem 2.3. ■

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