FRACTIONAL DOUBLE NEWTON STEP PROPERTIES
FOR POLYNOMIALS WITH ALL REAL ZEROS

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Abstract. When doubling the Newton step for the computation of the largest zero of a real polynomial with all real zeros, a classical result shows that the iterates never overshoot the largest zero of the derivative of the polynomial. Here we show that when the Newton step is extended by a factor θ with $1 < \theta < 2$, the iterates cannot overshoot the zero of a different function. When $\theta = 2$, our result reduces to the one for the double-step case. An analogous property exists for the smallest zero.

1. Introduction

A polynomial with all real zeros has a minimum or maximum in between each pair of adjacent distinct zeros and is convex or concave to the left and right of the smallest and largest of these optima, respectively. This means that if one wanted to compute, for example, the largest zero with Newton's method, then the iterates would converge monotonically from an initial point to the right of that zero. A similar situation exists for the smallest zero. Newton's method for solving $f(z) = 0$ is defined by $z_{k+1} = z_k - f(z_k)/f'(z_k)$, starting from an appropriate z_0. Its geometric significance is that the next iterate is the zero of the tangent to $f(z)$ at the current iterate, which explains why starting, e.g., to the right of the largest zero of a convex or concave function causes the iterates to converge monotonically to that zero.

The convergence can be accelerated by increasing the Newton step $-f(z_k)/f'(z_k)$, although there is now a possibility of overshooting the zero. If an overshoot does occur, then a classical result for a doubling of the step for polynomials with all real zeros (see, e.g., [3], [4]) shows that the overshoot must lie to the right of the largest zero of the derivative of the polynomial. In addition, it describes what happens to a subsequent (regular) Newton step.

Although probably not of great practical importance, it is nevertheless an intriguing and natural question to ask how far an iterate can overshoot if the Newton

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step for a polynomial with all real zeros is increased by a factor θ that is less than two, i.e., $1 < \theta < 2$, which we will refer to as a fractional double Newton step. We will show that in this case too one can identify a function whose zero constitutes a lower bound on the overshoot. More specifically, consider a polynomial $p(x)$ with zeros $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$. Then, if the Newton step is multiplied by a factor $1 < \theta < 2$ and an overshoot occurs from a point to the right of the largest zero of the polynomial, we will show that the overshoot lies to the right of the largest zero of a function of the form

$$\frac{h(\theta)}{x - \xi_1} + \sum_{j=2}^{n} \frac{1}{x - \xi_j} \ , \tag{1}$$

where $0 < h(\theta) < 1$. For $\theta \to 2^+$, we recover the result for the double step method. As such, we provide a more complete treatment of what happens when a Newton step is extended continuously, rather than just doubled.

To conclude the introduction, we define the following functions for $1 < \theta \leq 2$ and $q \geq 0$:

$$\phi_{\theta,q}(x) = \frac{(\theta - 1)^q}{x - \xi_1} + \sum_{j=2}^{n} \frac{1}{x - \xi_j} \quad \text{and} \quad \phi(x) = \phi_{2,q}(x) = \phi_{\theta,0}(x) = \sum_{j=1}^{n} \frac{1}{x - \xi_j} \ .$$

For any value of $1 < \theta \leq 2$ and $q \geq 0$, the function $\phi_{\theta,q}$ is singular at each zero of $p(x)$ and strictly decreasing on intervals between distinct zeros. For $\xi_1 \neq \xi_2$, we denote its largest zero by $\eta_1^{(\theta,q)}$ and its graph on (ξ_2, ξ_1) looks as follows:
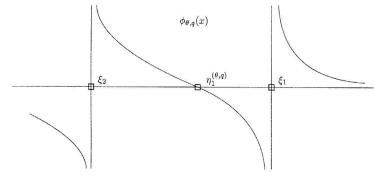

Fig. 1. The graph of $\phi_{\theta,q}(x)$ on (ξ_2, ξ_1)

Since $p(x) = a_n \prod_{j=1}^{n} (x - \xi_j)$, we note that $\phi(x) = \sum_{j=1}^{n} \frac{1}{x - \xi_j} = \frac{p'(x)}{p(x)}$ and that, when $\xi_1 \neq \xi_2$, the largest zeros of $p'(x)$ and $\phi(x)$ coincide, so that the classical result for the double-step Newton method can be rephrased in terms of the largest zero of the function in (1) with $h(\theta) \equiv 1$. Also, with our definitions, Newton's method can be written as $z_{k+1} = z_k - 1/\phi(z_k)$.

The main idea of the paper is to show that the function value of $\phi_{\theta,q}$ at an overshoot on the interval (ξ_2, ξ_1) is negative, which means that the overshoot must

lie to the right of the zero of that function. We then proceed to find the largest possible value of q for which this is true. The larger q, the larger the zero of the corresponding $\phi_{\theta,q}$ and the better the lower bound on the overshoot will be. This finally leads to the function $h(\theta)$.

First we prove a technical lemma, after which a string of lemmas following the above outline culminates in our main result.

2. Preliminary lemmas

To start, we show in the following lemma that if an overshoot occurs for an extended Newton step, i.e., a step multiplied by a factor θ with $0 < \theta \le 2$, for a polynomial with all real zeros of degree $n \ge 3$ and starting from a point to the right of its largest zero, then that largest zero must have multiplicity one. With our notation, this means that $\xi_2 > \xi_1$.

LEMMA 2.1. *Let $p(x)$ be a real polynomial of degree $n \ge 3$, all zeros $\xi_1 \ge \xi_2 \ge \cdots \ge \xi_n$ of which are real and let $1 < \theta \le 2$. Then for every $z > \xi_1$, $z - \theta\dfrac{p(z)}{p'(z)} < \xi_1$ implies that $\xi_1 \ne \xi_2$.*

Proof. First we note that $p(z)/p'(z) > 0$. Then, because $n \ge 3$, we have that

$$\frac{1}{2}\left(\frac{p'(z)}{p(z)}\right) = \frac{1}{2}\left(\sum_{j=1}^{n}\frac{1}{z - \xi_j}\right) > \frac{1}{2}\left(\frac{1}{z - \xi_1} + \frac{1}{z - \xi_2}\right)$$

and therefore $2\frac{p(z)}{p'(z)} < H(z - \xi_1, z - \xi_2)$, where $H(a,b)$ stands for the harmonic mean of a and b. Since $z - \xi_1$ and $z - \xi_2$ are positive numbers, this means that $H(z - \xi_1, z - \xi_2) \le z - \xi_2$ and therefore that $\xi_1 > z - \theta\dfrac{p(z)}{p'(z)} \ge z - 2\dfrac{p(z)}{p'(z)} > z - H(z - \xi_1, z - \xi_2) \ge \xi_2$, so that $\xi_1 > \xi_2$. ∎

Before we continue, we define $\gamma_j = \phi(z)(z - \xi_j)$, i.e., γ_j is the distance from z to ξ_j in terms of Newton steps from z. The next lemma bounds $\phi_{\theta,q}\left(z - \frac{\theta}{\phi(z)}\right)$ in terms of $\phi(z)$, θ, and γ_1.

LEMMA 2.2. *Let $\phi_{\theta,q}(x)$ and $\phi(x)$ be as defined before and let $p(x)$ be a real polynomial of degree $n \ge 2$, all zeros $\xi_1 \ge \xi_2 \ge \cdots \ge \xi_n$ of which are real. Let $1 < \theta \le 2$, $q \ge 0$, $z > \xi_1$, and $y = z - \frac{\theta}{\phi(z)}$. Then $y < \xi_1$ implies that*

$$\phi_{\theta,q}(y) \le \phi(z)\left(\frac{(\theta - 1)^q}{\gamma_1 - \theta} + \frac{\gamma_1 - 1}{\theta - (\theta - 1)\gamma_1}\right). \tag{2}$$

When $n = 2$, we require that $\xi_1 \ne \xi_2$.

Proof. First, we have either by assumption or by Lemma 2.1 that $\xi_1 > \xi_2$, so that the function $\phi_{\theta,q}$ looks exactly as in Figure 1, i.e., its largest zero $\eta_1^{(\theta,q)}$

satisfies $\xi_2 < \eta_1^{(\theta,q)} < \xi_1$. We have

$$
\begin{aligned}
\phi_{\theta,q}(y) &= \frac{(\theta-1)^q}{y-\xi_1} + \sum_{j=2}^{n} \frac{1}{y-\xi_j} = \frac{(\theta-1)^q}{z-\theta/\phi(z)-\xi_1} + \sum_{j=2}^{n} \frac{1}{z-\theta/\phi(z)-\xi_j} \\
&= \frac{(\theta-1)^q}{z-\xi_1-\theta/\phi(z)} + \sum_{j=2}^{n} \frac{1}{z-\xi_j-\theta/\phi(z)} \\
&= \phi(z) \left(\frac{(\theta-1)^q}{\phi(z)(z-\xi_1)-\theta} + \sum_{j=2}^{n} \frac{1}{\phi(z)(z-\xi_j)-\theta} \right).
\end{aligned}
$$

Recalling that $\gamma_j = \phi(z)(z-\xi_j)$, we obtain

$$
\sum_{j=1}^{n} \frac{1}{\gamma_j} = \sum_{j=1}^{n} \frac{1}{(\phi(z))(z-\xi_j)} = \frac{1}{\phi(z)} \sum_{j=1}^{n} \frac{1}{z-\xi_j} = 1 .
$$

In terms of the variables γ_j, we have

$$
\phi_{\theta,q}(y) = \phi(z) \left(\frac{(\theta-1)^q}{\gamma_1-\theta} + \sum_{j=2}^{n} \frac{1}{\gamma_j-\theta} \right). \tag{3}
$$

Since $\phi(z) > 0$, we can derive an upper bound on $\phi_{\theta,q}(y)$ for given γ_1 by computing an upper bound on the function $\sum_{j=2}^{n} 1/(\gamma_j - \theta)$ over all values $\gamma_j > 0$ $(j = 2,3,\ldots,n)$, subject to the restriction that $\sum_{j=2}^{n} 1/\gamma_j = 1 - 1/\gamma_1$.

We note that, because $z - \theta/\phi(z) < \xi_1$ and $z - 1/\phi(z) > \xi_1$, it follows that $1 < \gamma_1 < \theta$, which implies that $0 < 1 - \frac{1}{\gamma_1} < \frac{\theta-1}{\theta}$ so that, for $j = 2,\ldots,n$, $\frac{1}{\gamma_j} < \frac{\theta-1}{\theta}$. Therefore, $\gamma_j > \theta$ for $j = 2,\ldots,n$ and the function $\sum_{j=2}^{n} 1/(\gamma_j - \theta)$ is well-defined.

The problem of finding such an upper bound is solved in [3] for $\theta = 2$ but the proof is entirely analogous for $1 < \theta < 2$. The upper bound is obtained by setting $\gamma_2 = \frac{\gamma_1}{\gamma_1 - 1}$ and $\gamma_j = +\infty$ for $j \geq 3$ (when $n \geq 3$), which yields $\frac{\gamma_1 - 1}{\theta - (\theta-1)\gamma_1}$. Substituting this value into (3) gives

$$
\phi_{\theta,q}(y) \leq \phi(z) \left(\frac{(\theta-1)^q}{\gamma_1-\theta} + \frac{\gamma_1-1}{\theta-(\theta-1)\gamma_1} \right).
$$

This completes the proof. ∎

For $1 \leq x < \theta$, $1 < \theta \leq 2$, and $q \geq 0$, we define

$$
\psi_{\theta,q}(x) = \frac{(\theta-1)^q}{x-\theta} + \frac{x-1}{\theta-(\theta-1)x}. \tag{4}
$$

We can then rewrite the result in Lemma 2.2 as $\phi_{\theta,q}(y) \leq \phi(z)\psi_{\theta,q}(\gamma_1)$.

It can easily be verified that

$$
\frac{d}{dx}\psi_{\theta,q}(x) = \frac{-(\theta-1)^q}{(x-\theta)^2} + \frac{1}{(\theta-(\theta-1)x)^2}. \tag{5}
$$

In the following lemma, we prove a few basic properties of the function $\psi_{\theta,q}(x)$ on the interval $1 \leq x < \theta$, namely that for $1 < \theta < 2$ it is a decreasing function

when $0 \leq q < 2$ and that when $q \geq 2$ it has a unique maximum at some point $x = \bar{\gamma}_{\theta,q}$ on $[1, \theta)$. If $\psi_{\theta,q}(\bar{\gamma}_{\theta,q}) \leq 0$, we obtain with Lemma 2.2 that $\phi_{\theta,q}(y) \leq \phi(z)\psi_{\theta,q}(\bar{\gamma}_{\theta,q}) \leq 0$. Figure 1 then shows that the zero of $\phi_{\theta,q}(x)$ on (ξ_2, ξ_1) must be a lower bound on y. As was mentioned earlier, our goal is to find the largest possible value of q for which $\psi_{\theta,q}(\bar{\gamma}_{\theta,q})$ is still negative because the larger the value of q, the larger the zero of $\phi_{\theta,q}(x)$ will be, and, consequently, the better (larger) the lower bound on $y = z - \theta/\phi(z)$. This is illustrated in Figure 2, which, as an example, shows the graphs of $\phi_{1.8,1}$ and $\phi_{1.8,6}$ for $p(x) = (x+1)(x-2)(x-4)(x-8)$.
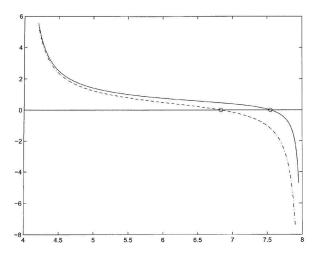

Fig. 2. The graphs of the functions $\phi_{1.8,1}$ (dashed line) and $\phi_{1.8,6}$ (solid line).

LEMMA 2.3. *For $1 < \theta < 2$ the function $\psi_{\theta,q}(x)$ has the following properties on $[1, \theta)$:*

(1) *For $q \geq 0$, $\psi_{\theta,q}(1) < 0$.*

(2) *For $0 \leq q < 2$, $\psi_{\theta,q}(x)$ is strictly decreasing in x.*

(3) *For $q \geq 2$, $\psi_{\theta,q}(x)$ has a unique maximum at $x = \bar{\gamma}_{\theta,q}$, with*

$$\bar{\gamma}_{\theta,q} = 1 + \frac{(\theta - 1) - (\theta - 1)^{q/2}}{1 - (\theta - 1)^{1+q/2}}. \tag{6}$$

When $\theta = 2$, $\psi_{2,q}(x) = -1$ for any $x \in [1, 2)$ and any value of q.

Proof. Part (1) in the statement of the proof is immediate from (4). Let us now consider

$$\psi'_{\theta,q}(x) < 0 \iff \frac{(\theta - 1)^{q/2}}{\theta - x} > \frac{1}{\theta - (\theta - 1)x}$$

$$\iff x > \frac{1 - (\theta - 1)^{q/2}}{1 - (\theta - 1)^{1+q/2}} \cdot \theta \tag{7}$$

$$\iff x > \frac{1 - (\theta - 1)^{q/2}}{1 - (\theta - 1)^{1+q/2}} \cdot (1 + (\theta - 1))$$

$$\Longleftrightarrow \; x > \frac{1 - (\theta - 1)^{1+q/2} + (\theta - 1) - (\theta - 1)^{q/2}}{1 - (\theta - 1)^{1+q/2}}$$

$$\Longleftrightarrow \; x > 1 + \frac{(\theta - 1) - (\theta - 1)^{q/2}}{1 - (\theta - 1)^{1+q/2}}. \tag{8}$$

If $q < 2$, then $\theta - 1 < (\theta - 1)^{q/2}$ so that statement (8) is true for $x \in [1, \theta)$. Therefore, $\psi'_{\theta,q}(x) < 0$ on $[1, \theta)$ when $q < 2$.

Proceeding exactly as before, we find

$$\psi'_{\theta,q}(\bar{\gamma}_{\theta,q}) = 0 \; \Longleftrightarrow \; \bar{\gamma}_{\theta,q} = 1 + \frac{(\theta - 1) - (\theta - 1)^{q/2}}{1 - (\theta - 1)^{1+q/2}}.$$

If $q \geq 2$, then $\theta - 1 \geq (\theta - 1)^{q/2}$, so that $\bar{\gamma}_{\theta,q} \geq 1$. In view of (7), we have that

$$\psi'_{\theta,q}(\bar{\gamma}_{\theta,q}) = 0 \; \Longleftrightarrow \; \bar{\gamma}_{\theta,q} = \frac{1 - (\theta - 1)^{q/2}}{1 - (\theta - 1)^{1+q/2}} \cdot \theta, \tag{9}$$

and since $(\theta - 1)^{q/2} > (\theta - 1)^{1+q/2}$, equation (9) implies that $\bar{\gamma}_{\theta,q} < \theta$. We have therefore obtained a single critical point on $[1, \theta)$. Furthermore, because $q \geq 2$, we have that $\psi'_{\theta,q}(1) = 1 - (\theta - 1)^{q-2} \geq 0$ and $\lim_{x \to \theta} \psi'_{\theta,q}(x) < 0$. The critical point must therefore represent a maximum.

It is also easily verified that $\psi_{2,q}(x) = -1$ for any $x \in [1, 2)$. This concludes the proof. ∎

Next, we compute the largest value of q for which $\psi_{\theta,q}(\bar{\gamma}_{\theta,q})$ remains negative. Figure 3 shows a few curves, corresponding to $\theta = 1.4$ and various values of q, ranging from 1.5 to 5.5. Higher curves correspond to higher values of q. As can clearly be seen, the maximum value of $\psi_{\theta,q}$ remains negative until a certain value of q is reached. That value is obtained in the following lemma.
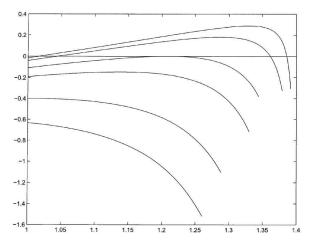

Fig. 3. The graphs of the functions $\psi_{1.4,q}$ for $q = 1.5, 2, 2.8, 3.4199, 4.5, 5.5$.

LEMMA 2.4. *With $\psi_{\theta,q}(x)$ and $\bar{\gamma}_{\theta,q}$ defined as before, $1 < \theta < 2$, and $q > 2$, we have that $\psi_{\theta,q}(\bar{\gamma}_{\theta,q}) \leq 0$ as long as*

$$q \leq 2 - 2\,\frac{\ln\left(1 + \sqrt{\theta(2-\theta)}\right)}{\ln(\theta-1)}.$$

Proof. The proof begins with a straightforward substitution. We recall that

$$\bar{\gamma}_{\theta,q} = 1 + \frac{(\theta-1) - (\theta-1)^{q/2}}{1 - (\theta-1)^{1+q/2}}\;,$$

which implies

$$
\begin{aligned}
\psi_{\theta,q}(\bar{\gamma}_{\theta,q}) \leq 0 &\iff \frac{(\theta-1)^q}{\bar{\gamma}_{\theta,q} - \theta} \leq \frac{1 - \bar{\gamma}_{\theta,q}}{\theta - (\theta-1)\bar{\gamma}_{\theta,q}} \\
&\iff (\theta-1)^q(\theta - (\theta-1)\bar{\gamma}_{\theta,q}) \geq (1 - \bar{\gamma}_{\theta,q})(\bar{\gamma}_{\theta,q} - \theta) \\
&\iff (\theta-1)^q(1 + (\theta-1) - (\theta-1)\bar{\gamma}_{\theta,q}) \\
&\qquad \geq (1 - \bar{\gamma}_{\theta,q})(\bar{\gamma}_{\theta,q} - (\theta-1) - 1) \\
&\iff (\theta-1)^q\left(1 - \frac{(\theta-1)^2 - (\theta-1)^{1+q/2}}{1 - (\theta-1)^{1+q/2}}\right) \\
&\qquad \geq \left(-\frac{(\theta-1) - (\theta-1)^{q/2}}{1 - (\theta-1)^{1+q/2}}\right)\left(\frac{(\theta-1) - (\theta-1)^{q/2}}{1 - (\theta-1)^{1+q/2}} - (\theta-1)\right) \\
&\iff (\theta-1)^q\left(1 - (\theta-1)^2\right) \\
&\qquad \geq \left((\theta-1)^{q/2} - (\theta-1)\right)\left(\frac{(\theta-1)^{2+q/2} - (\theta-1)^{q/2}}{1 - (\theta-1)^{1+q/2}}\right) \\
&\iff (\theta-1)^{q/2} \geq \frac{(\theta-1) - (\theta-1)^{q/2}}{1 - (\theta-1)^{1+q/2}}.
\end{aligned}
$$

From the last inequality, one readily obtains

$$(\theta-1)^q - \frac{2}{\theta-1}(\theta-1)^{q/2} + 1 \leq 0. \tag{10}$$

As $q \longrightarrow 2^+$ for a given θ, the function of q in the left-hand side of inequality (10) approaches $(\theta-1)^2 - 1$, which is a negative number. Its derivative with respect to q is given by

$$(\theta-1)^q \ln(\theta-1) - (\theta-1)^{q/2}\frac{\ln(\theta-1)}{\theta-1} = \left((\theta-1)^{q/2} - \frac{1}{\theta-1}\right)(\theta-1)^{q/2}\ln(\theta-1)\,.$$

This is a strictly positive number which means that the function in the left-hand side of inequality (10) is strictly increasing as a function of q, from which we conclude that the maximum function value of $\psi_{\theta,q}(x)$ on $[1, \theta)$ remains negative as q increases from $q = 2^+$, until it vanishes for the value of q that satisfies

$$(\theta-1)^q - \frac{2}{\theta-1}(\theta-1)^{q/2} + 1 = 0\,. \tag{11}$$

We compute this value by noting that (11) is a quadratic equation in $(\theta - 1)^{q/2}$. Its solutions are given by

$$(\theta - 1)^{q/2} = \frac{1}{2}\left(\frac{2}{\theta - 1} \pm \sqrt{\frac{4}{(\theta - 1)^2} - 4}\right) = \frac{1 \pm \sqrt{1 - (\theta - 1)^2}}{\theta - 1}.$$

Since $\theta - 1 < 1$, the only feasible solution is the one with the minus sign, which can be rewritten as

$$(\theta - 1)^{q/2} = \frac{\theta - 1}{1 + \sqrt{1 - (\theta - 1)^2}}. \tag{12}$$

Taking the natural logarithm on both sides then easily yields

$$q = 2 - 2\,\frac{\ln\left(1 + \sqrt{\theta(2 - \theta)}\right)}{\ln(\theta - 1)}.$$

This completes the proof. ∎

We are interested in the largest possible value for q. That means that the case $q > 2$ is the only one of interest as Lemma 2.3 shows that $\psi_{\theta,q}$ is always negative on $[1, \theta)$ for any $0 \le q \le 2$.

In Figure 3, the curve touching the x-axis from below corresponds to $q \approx 3.4199$, the value given by Lemma 2.4.

3. Main result: the overshooting theorem

Our main result is stated in the following theorem.

THEOREM 3.1. *Let $\phi_{\theta,q}(x)$ and $\phi(x)$ be as defined before and let $p(x)$ be a real polynomial of degree $n \ge 2$, all zeros $\xi_1 \ge \xi_2 \ge \cdots \ge \xi_n$ of which are real. Let $1 < \theta \le 2$, $z > \xi_1$, and $y = z - \dfrac{\theta}{\phi(z)}$. When $n = 2$, we also require that $\xi_1 \ne \xi_2$. Then, if y overshoots ξ_1, i.e., if $y < \xi_1$, it will not overshoot the largest zero of the function*

$$\mu(x) = \frac{h(\theta)}{x - \xi_1} + \sum_{j=2}^{n} \frac{1}{x - \xi_j} \quad with \quad h(\theta) = \left(\frac{\theta - 1}{1 + \sqrt{\theta(2 - \theta)}}\right)^2.$$

Proof. We start by noticing that $\theta = 2$ simply reduces to the double step case which was already dealt with in [3] and [4]. Now let $1 < \theta < 2$ and let γ_j, $\phi_{\theta,q}(x)$, $\phi(x)$, $\psi_{\theta,q}(x)$, and $\bar{\gamma}_{\theta,q}$ be as defined before. Then for any $q > 2$ we have with Lemma 2.2 that

$$\phi_{\theta,q}(y) \le \phi(z)\psi_{\theta,q}(\gamma_1) \le \phi(z)\psi_{\theta,q}(\bar{\gamma}_{\theta,q}). \tag{13}$$

Lemma 2.4 shows that $\psi_{\theta,q}(\bar{\gamma}_{\theta,q}) \le 0$, which means that y lies to the right of the zero of $\phi_{\theta,q}$ on (ξ_2, ξ_1), as long as

$$q \le 2 - 2\,\frac{\ln\left(1 + \sqrt{\theta(2 - \theta)}\right)}{\ln(\theta - 1)},$$

or, with (12), as long as

$$(\theta - 1)^q \geq \left(\frac{\theta - 1}{1 + \sqrt{\theta(2 - \theta)}} \right)^2 .$$

We are interested in the smallest possible value for $(\theta - 1)^q$ as it yields the largest zero for $\phi_{\theta,q}$ on (ξ_2, ξ_1). Substituting this smallest value in the expression for $\phi_{\theta,q}$ completes the proof. ∎

REMARKS.

(1) The function

$$h(\theta) = \left(\frac{\theta - 1}{1 + \sqrt{\theta(2 - \theta)}} \right)^2$$

satisfies $0 < h(\theta) < 1$ so that the zero of $\mu(x)$, which we defined in the statement of Theorem 3.1, will never be smaller than the zero of $\phi(x)$ (or $p'(x)$) on the interval (ξ_2, ξ_1).

(2) Completely analogous results can be obtained for an overshoot from an initial point to the left of the smallest zero of $p(x)$.

(3) From Lemma 2.3 we know that the function $\psi_{\theta,0}(x)$ achieves its maximum on $1 \leq x < \theta < 2$ at $x = 1$, yielding a function value of $-(\theta-1)^{-1}$. With Lemma 2.2 this means that

$$\phi(y) = \phi_{\theta,0}(y) \leq -\phi(z)/(\theta - 1) .$$

From this we have

$$y - \frac{p(y)}{p'(y)} = y - \frac{1}{\phi(y)} = y - \frac{1}{\phi_{\theta,0}(y)} \leq z - \frac{\theta}{\phi(z)} + \frac{\theta - 1}{\phi(z)} = z - \frac{1}{\phi(z)} = z - \frac{p(z)}{p'(z)} .$$

In other words, taking a regular Newton step from the overshoot yields an iterate that is not worse than a regular Newton step from the last iterate before the overshoot. This is the same property that holds for $\theta = 2$ as demonstrated in [3] and [4].

(4) It is a tedious but straightforward exercise to show that the lower bound on the overshoot we derived here is at least as good as the much cruder one obtained in [2].

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