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THE SCHUR-HARMONIC-CONVEXITY OF DUAL FORM OF THE HAMY SYMMETRIC FUNCTION

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Abstract. We prove that the dual form of the Hamy symmetric function

$$H_n(x,r) = H_n(x_1, x_2, \dots, x_n; r) = \prod_{1\le i_1<\cdots<i_r\le n} \Big(\sum_{j=1}^{r} x_{i_j}^{\frac{1}{r}}\Big)$$

is Schur harmonic convex in $\mathbf{R}^n_+$. As applications, some inequalities are established by use of the theory of majorization.

1. Introduction

Throughout this paper, we denote by $\mathbf{R}^n$ $(n \ge 2)$ the $n-$dimensional Euclidean space, $\mathbf{R} = \mathbf{R}^1$ and $\mathbf{R}^n_+ = \{(x_1, x_2, \dots, x_n) : x_i > 0, i = 1, 2, \dots, n\}$.

For $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots,) \in \mathbf{R}^n_+$ and $\alpha > 0$, we denote by

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),\\ xy &= (x_1y_1, x_2y_2, \dots, x_ny_n),\\ \alpha x &= (\alpha x_1, \alpha x_2, \dots, \alpha x_n), \text{and}\\ \frac{1}{x} &= (\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}). \end{aligned}$$

For $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n_+$ and $r \in \{1, 2, \dots, n\}$, the Hamy symmetric function [1–3] was defined as

$$F_n(x,r) = F_n(x_1, x_2, \dots, x_n; r) = \sum_{1\le i_1<i_2<\cdots<i_r\le n} \Big(\prod_{j=1}^{r} x_{i_j}\Big)^{\frac{1}{r}}.$$

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Corresponding to this is the r-order Hamy mean

$$\sigma_n(x,r) = \frac{1}{C_n^r} \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left(\prod_{j=1}^{r} x_{i_j} \right)^{\frac{1}{r}},$$

where $C_n^r = \frac{n!}{(n-r)!r!}$. T. Hara et al. [1] established the following refinement of the classical arithmetic and geometric mean inequality:

$$G_n(x) = \sigma_n(x,n) \le \sigma_n(x,n-1) \le \cdots \le \sigma_n(x,2) \le \sigma_n(x,1) = A_n(x).$$

Here $A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $G_n(x) = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ denote the classical arithmetic and geometric means, respectively.

The paper [4] by H. T. Ku et al. contains some interesting inequalities including the fact that $(\sigma_n(x,r))^r$ is log-concave, the more results can be found in the book [5] by P. S. Bullen. In [2], the Schur convexity of the Hamy symmetric function and its generalization was discussed. In [3], W. D. Jiang defined the dual form of the Hamy symmetric function as follows

$$H(x,r) = H_n(x_1, x_2, \ldots, x_n; r) = \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left(\sum_{j=1}^{r} x_{i_j}^{\frac{1}{r}} \right) \tag{1.1}$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$ and $r \in \{1, 2, \ldots, n\}$. The Schur concavity and the Schur geometrically convexity of $H_n(x,r)$ were discussed, and some analytic inequalities were established by use of the theory of majorization.

Our main aim in what follows is to introduce the notion of Schur-harmonic-convexity and prove that $H_n(x,r)$ is Schur harmonic convex in $\mathbf{R}_+^n$.

For convenience of readers, we recall and introduce some definitions as follows.

Definition 1.1. A set $E_1 \subseteq \mathbf{R}^n$ is called a convex set if $\frac{x+y}{2} \in E_1$ whenever $x, y \in E_1$. A set $E_2 \subseteq \mathbf{R}_+^n$ is called a harmonic convex set if $\frac{2xy}{x+y} \in E_2$ whenever $x, y \in E_2$.

It is easy to see that $E \subseteq \mathbf{R}_+^n$ is a harmonic convex set if and only if $\frac{1}{E} = \{\frac{1}{x} : x \in E\}$ is a convex set.

Definition 1.2. Let $E \subseteq \mathbf{R}^n$ be a convex set, a real-valued function f is said to be a convex function on E if $f(\frac{x+y}{2}) \le \frac{f(x)+f(y)}{2}$ for all $x, y \in E$. And f is called a concave function if $-f$ is a convex function.

Definition 1.3. Let $E \subseteq \mathbf{R}_+^n$ be a harmonic convex set, a function $f : E \longrightarrow \mathbf{R}_+$ is called a harmonic convex (or concave, respectively)function on E if $f(\frac{2xy}{x+y}) \le$ (or $\ge$, respectively) $\frac{2f(x)f(y)}{f(x)+f(y)}$ for all $x, y \in E$.

Definition 1.2 and 1.3 have the following consequences.

Remark 1.1. If $E_1 \subseteq \mathbf{R}_+^n$ is a harmonic convex set and $f : E_1 \longrightarrow \mathbf{R}_+$ is a harmonic convex function, then

$$F(x) = \frac{1}{f(\frac{1}{x})} : \frac{1}{E_1} \longrightarrow \mathbf{R}_+$$

is a concave function. Conversely, if $E_2 \subseteq \mathbf{R}_+^n$ is a convex set and $F : E_2 \longrightarrow \mathbf{R}_+$ is a convex function, then

$$f(x) = \frac{1}{F(\frac{1}{x})} : \frac{1}{E_2} \longrightarrow \mathbf{R}_+$$

is a harmonic concave function.

Definition 1.4. Let $E \subseteq \mathbf{R}^n$ be a set, a function $F : E \longrightarrow \mathbf{R}$ is called a Schur convex function on E if

$$F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n)$$

for each pair of n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in E, such that $x \prec y$, i. e.

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \ k = 1, 2, \ldots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the i-th largest component in x. And F is called Schur concave if $-F$ is Schur convex.

Definition 1.5. Let $E \subseteq \mathbf{R}_+^n$ be a set. A function $F : E \longrightarrow \mathbf{R}_+$ is called a Schur harmonic convex (or concave, respectively) function on E if

$$F(x_1, x_2, \ldots, x_n) \leq (\text{or} \geq, \text{respectively}) F(y_1, y_2, \ldots, y_n)$$

for each pair of n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$.

Definition 1.4 and 1.5 have the following consequences.

Remark 1.2. Let $E \subseteq \mathbf{R}_+^n$ be a set and $H = \frac{1}{E} = \{\frac{1}{x} : x \in E\}$. Then $f : E \longrightarrow \mathbf{R}_+$ is a Schur harmonic convex (or concave, respectively) function on E if and only if $\frac{1}{f(\frac{1}{x})}$ is a Schur concave (or convex, respectively) function on H.

Schur convexity was introduced by I. Schur in 1923 [6], it has many important applications in analytic inequalities [7–12], linear regression [13], graphs and matrices [14], combinatorial optimization [15], information-theoretic topics [16], Gamma functions [17], stochastic orderings [18], reliability [19] and other related fields. But no one has ever researched the Schur harmonic convexity. In the present paper, the Schur harmonic convexity of the dual form of the Hamy symmetric function will be discussed.

2. Lemmas

In this section, we introduce and establish several lemmas, which are used in the proof of our main result in next sections.

LEMMA 2.1 [6] *Let $E \subseteq \mathbf{R}^n$ be a symmetric convex set with nonempty interior $\mathrm{int}E$ and $\varphi : E \longrightarrow \mathbf{R}$ be a continuous symmetry function on E. If φ is differentiable on $\mathrm{int}E$, then φ is Schur convex on E if and only if*

$$(x_i - x_j)(\frac{\partial\varphi}{\partial x_i} - \frac{\partial\varphi}{\partial x_j}) \geq 0 \tag{2.1}$$

for all $i, j = 1, 2, \ldots, n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathrm{int}E$. Here, E is a symmetric set which means that $x \in E$ implies $Px \in E$ for any $n \times n$ permutation matrix P.

REMARK 2.1. Since φ is symmetric, the Schur's condition in Lemma 2.1, i. e. (2.1) can be reduced as

$$(x_1 - x_2)(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}) \geq 0.$$

The following Lemma 2.2 can easily be derived from Remark 1.2, Lemma 2.1 and Remark 2.1 together with elementary computation.

LEMMA 2.2. *Let $E \subseteq \mathbf{R}_+^n$ be a symmetric harmonic convex set with nonempty interior $\mathrm{int}E$ and $\varphi : E \longrightarrow \mathbf{R}_+$ be a continuous symmetry function on E. If φ is differentiable on $\mathrm{int}E$, then φ is Schur harmonic convex on E if and only if*

$$(x_1 - x_2)(x_1^2\frac{\partial\varphi}{\partial x_1} - x_2^2\frac{\partial\varphi}{\partial x_2}) \geq 0$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathrm{int}E$.

LEMMA 2.3 [2, Lemma 2.2(i); 3, Lemma 2.5; 20, Lemma 2.3] *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geq s$, then*

$$\frac{c-x}{\frac{nc}{s}-1} = \left(\frac{c-x_1}{\frac{nc}{s}-1}, \frac{c-x_2}{\frac{nc}{s}-1}, \ldots, \frac{c-x_n}{\frac{nc}{s}-1}\right) \prec (x_1, x_2, \ldots, x_n) = x.$$

LEMMA 2.4 [20, Lemma 2.4] *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geq 0$, then*

$$\frac{c+x}{\frac{nc}{s}+1} = \left(\frac{c+x_1}{\frac{nc}{s}+1}, \frac{c+x_2}{\frac{nc}{s}+1}, \ldots, \frac{c+x_n}{\frac{nc}{s}+1}\right) \prec (x_1, x_2, \ldots, x_n) = x.$$

3. Main result

THEOREM 3.1. *For $r \in \{1, 2, \ldots, n\}$, the function $H_n(x, r)$ is Schur harmonic convex in $\mathbf{R}_+^n$.*

Proof. The proof is divided into four cases.

Case 1. $r = 1$.

In this case, we see that

$$H_n(x, 1) = H_n(x_1, x_2, \ldots, x_n; 1) = \prod_{i=1}^{n} x_i. \tag{3.1}$$

(3.1) leads to

$$(x_1 - x_2)\left(x_1^2 \frac{\partial H_n(x, 1)}{\partial x_1} - x_2^2 \frac{\partial H_n(x, 1)}{\partial x_2}\right) = (x_1 - x_2)^2 \prod_{i=1}^{n} x_i \geq 0.$$

Case 2. $r = n$.

In this case, we see that

$$H_n(x, n) = H_n(x_1, x_2, \ldots, x_n; n) = \sum_{i=1}^{n} x_i^{\frac{1}{n}}. \tag{3.2}$$

(3.2) leads to

$$(x_1 - x_2)\left(x_1^2 \frac{\partial H_n(x, n)}{\partial x_1} - x_2^2 \frac{\partial H_n(x, n)}{\partial x_2}\right) = \frac{1}{n}(x_1 - x_2)(x_1^{1+\frac{1}{n}} - x_2^{1+\frac{1}{n}}) \geq 0.$$

Case 3. $n \geq 3$ and $r = 2$.

The definition of $H_n(x, r)$ in (1.1) implies

$$\begin{aligned} H_n(x, 2) &= H_n(x_1, x_2, \ldots, x_n; 2) \\ &= H_{n-1}(x_2, x_3, \ldots, x_n; 2)(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}) \prod_{j=3}^{n} (x_1^{\frac{1}{2}} + x_j^{\frac{1}{2}}). \end{aligned} \tag{3.3}$$

(3.3) leads to

$$\frac{\partial H_n(x, 2)}{\partial x_1} = \frac{1}{2} H_n(x, 2) \left(\frac{x_1^{-\frac{1}{2}}}{x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}} + \sum_{j=3}^{n} \frac{x_1^{-\frac{1}{2}}}{x_1^{\frac{1}{2}} + x_j^{\frac{1}{2}}} \right). \tag{3.4}$$

Similarly, we have

$$\frac{\partial H_n(x, 2)}{\partial x_2} = \frac{1}{2} H_n(x, 2) \left(\frac{x_2^{-\frac{1}{2}}}{x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}} + \sum_{j=3}^{n} \frac{x_2^{-\frac{1}{2}}}{x_2^{\frac{1}{2}} + x_j^{\frac{1}{2}}} \right). \tag{3.5}$$

(3.4) and (3.5) yield

$$
\begin{aligned}
&(x_1-x_2)\left(x_1^2\frac{\partial H_n(x,2)}{\partial x_1}-x_2^2\frac{\partial H_n(x,2)}{\partial x_2}\right)\\
&\quad=\frac{1}{2}(x_1-x_2)H_n(x,2)\left[\frac{x_1^{\frac{3}{2}}-x_2^{\frac{3}{2}}}{x_1^{\frac{1}{2}}+x_2^{\frac{1}{2}}}+\sum_{j=3}^{n}\frac{x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}(x_1-x_2)+x_j^{\frac{1}{2}}(x_1^{\frac{3}{2}}-x_2^{\frac{3}{2}})}{(x_1^{\frac{1}{2}}+x_j^{\frac{1}{2}})(x_2^{\frac{1}{2}}+x_j^{\frac{1}{2}})}\right]\\
&\quad\geq 0.
\end{aligned}
$$

Case 4. $n\geq 4$ and $3\leq r\leq n-1$.

The definition of $H_n(x,r)$ in (1.1) implies

$$
\begin{aligned}
H_n(x,r)&=H_n(x_1,x_2,\ldots,x_n;r)\\
&=H_{n-1}(x_2,x_3,\ldots,x_n;r)\prod_{3\leq i_1<i_2<\cdots<i_{r-1}\leq n}(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}})\times\\
&\quad\times\prod_{3\leq i_1<i_2<\cdots<i_{r-2}\leq n}(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}). \qquad (3.6)
\end{aligned}
$$

Differentiating $H_n(x,r)$ with respect to x_1 and making use of (3.6) yield

$$
\begin{aligned}
\frac{\partial H_n(x,r)}{\partial x_1}&=\frac{1}{r}H_n(x,r)\Bigg(\sum_{3\leq i_1<i_2<\cdots<i_{r-1}\leq n}\frac{x_1^{\frac{1}{r}-1}}{x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}}\\
&\quad+\sum_{3\leq i_1<i_2<\cdots<i_{r-2}\leq n}\frac{x_1^{\frac{1}{r}-1}}{x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}}\Bigg). \qquad (3.7)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{\partial H_n(x,r)}{\partial x_2}&=\frac{1}{r}H_n(x,r)\Bigg(\sum_{3\leq i_1<i_2<\cdots<i_{r-1}\leq n}\frac{x_2^{\frac{1}{r}-1}}{x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}}+\\
&\quad+\sum_{3\leq i_1<i_2<\cdots<i_{r-2}\leq n}\frac{x_2^{\frac{1}{r}-1}}{x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}}\Bigg). \qquad (3.8)
\end{aligned}
$$

(3.7) and (3.8) lead to

$$
\begin{aligned}
&(x_1-x_2)(x_1^2\frac{\partial H_n(x,r)}{\partial x_1}-x_2^2\frac{\partial H_n(x,r)}{\partial x_2})=\frac{1}{r}(x_1-x_2)H_n(x,r)\times\\
&\quad\times\Bigg[\sum_{3\leq i_1<i_2<\cdots<i_{r-1}\leq n}\frac{x_1^{\frac{1}{r}}x_2^{\frac{1}{r}}(x_1-x_2)+(x_1^{1+\frac{1}{r}}-x_2^{1+\frac{1}{r}})\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}}{(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}})(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}})}\\
&\quad+\sum_{3\leq i_1<i_2<\cdots<i_{r-2}\leq n}\frac{x_1^{1+\frac{1}{r}}-x_2^{1+\frac{1}{r}}}{x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}}\Bigg]\geq 0.
\end{aligned}
$$

Therefore, Theorem 3.1 follows from cases 1–4 and Lemma 2.2. ∎

4. Applications

In this section, we establish some inequalities by using Theorem 3.1 and the theory of majorization.

THEOREM 4.1. *If $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$ and $r \in \{1, 2, \ldots, n\}$, then*

$$H_n(x_1, x_2, \ldots, x_n; r) \geq \left[r \left(\frac{n}{\sum_{i=1}^n \frac{1}{x_i}} \right)^{\frac{1}{r}} \right]^{\frac{n!}{r!\,(n-r)!}}.$$

Proof. We clearly see that

$$\left(\frac{\sum_{i=1}^n \frac{1}{x_i}}{n}, \frac{\sum_{i=1}^n \frac{1}{x_i}}{n}, \ldots, \frac{\sum_{i=1}^n \frac{1}{x_i}}{n} \right) \prec (\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}). \tag{4.1}$$

Therefore, Theorem 4.1 follows from Theorem 3.1, (4.1) and the definition of $H_n(x, r)$ in (1.1). ∎

REMARK 4.1. W. D. Jiang [3, Corollary 3.1] proved that

$$H_n(x_1, x_2, \ldots, x_n; r) \leq \left[r \left(\frac{\sum_{i=1}^n x_i}{n} \right)^{\frac{1}{r}} \right]^{\frac{n!}{r!\,(n-r)!}}$$

for any $(x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$ and $r \in \{1, 2, \ldots, n\}$.

The following Theorems 4.2 and 4.3 can be derived directly from Theorem 3.1, Lemma 2.3 and 2.4, and (1.1).

THEOREM 4.2. *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$, $\sum_{i=1}^n x_i = s$. If $c \geq s$ and $r \in \{1, 2, \ldots, n\}$, then*

$$\frac{H_n(\frac{1}{x}, r)}{H_n(\frac{1}{c-x}, r)} \geq \left(\frac{nc}{s} - 1 \right)^{\frac{n!}{r!\,(n-r)!r}}.$$

THEOREM 4.3. *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$, $\sum_{i=1}^n x_i = s$. If $c \geq 0$ and $r \in \{1, 2, \ldots, n\}$, then*

$$\frac{H_n(\frac{1}{x}, r)}{H_n(\frac{1}{c+x}, r)} \geq \left(\frac{nc}{s} + 1 \right)^{\frac{n!}{r!\,(n-r)!r}}.$$

REMARK 4.3. If we take $c = s = r = 1$ in Theorem 4.2 and 4.3, respectively, then the following two Weierstrass inequalities [21] are obtained:

(i) $\prod_{i=1}^n (x_i^{-1} - 1) \geq (n-1)^n$;

(ii) $\prod_{i=1}^n (x_i^{-1} + 1) \geq (n+1)^n$.

If we take $c = s = 1$ and $r = n$ in Theorem 4.2 and 4.3, respectively, then we have the following corollary 4.1.

COROLLARY 4.1. *Suppose that* $(x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$. *If* $\sum_{i=1}^n x_i = 1$, *then*

$$\frac{\sum_{i=1}^n \left(\frac{1}{x_i}\right)^n}{\sum_{i=1}^n \left(\frac{1}{1-x_i}\right)^n} \geq (n-1)^{\frac{1}{n}}$$

and

$$\frac{\sum_{i=1}^n \left(\frac{1}{x_i}\right)^n}{\sum_{i=1}^n \left(\frac{1}{1+x_i}\right)^n} \geq (n+1)^{\frac{1}{n}}.$$

THEOREM 4.4. *Let A be an n-dimensional simplex in $\mathbf{R}^n$ ($n \geq 3$) and $\{A_1, A_2, \ldots, A_{n+1}\}$ be the set of vertices. Let P be an arbitrary point in the interior of A. If B_i is the intersection point of the extension line of A_iP and the $(n-1)$-dimensional hyperplane opposite to the point A_i, then*

$$H_{n+1}\left(\frac{A_1B_1}{PB_1}, \frac{A_2B_2}{PB_2}, \ldots, \frac{A_{n+1}B_{n+1}}{PB_{n+1}}; r\right) \geq \left[r(n+1)^{\frac{1}{r}}\right]^{\frac{(n+1)!}{r!\,(n-r+1)!}}$$

and

$$H_{n+1}\left(\frac{A_1B_1}{A_1P}, \frac{A_2B_2}{A_2P}, \ldots, \frac{A_{n+1}B_{n+1}}{A_{n+1}P}; r\right) \geq \left[r(\frac{n+1}{n})^{\frac{1}{r}}\right]^{\frac{(n+1)!}{r!\,(n-r+1)!}}$$

for $r \in \{1, 2, \ldots, n, n+1\}$.

PROOF. Theorem 4.4 follows from Theorem 3.1 and the fact that $\sum_{i=1}^{n+1} \frac{PB_i}{A_iB_i} = 1$ and $\sum_{i=1}^{n+1} \frac{A_iP}{A_iB_i} = n$.

THEOREM 4.5. *Let $A = (a_{ij})_{n\times n}$ ($n \geq 2$) be a complex matrix, $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of A, and I denotes $n \times n$ unit matrix. If A is a positive definite Hermitian matrix and $r \in \{1, 2, \ldots, n\}$, then*

$$\prod_{1\leq i_1<i_2<\cdots<i_r\leq n} \left[\sum_{j=1}^r \left(\log(1+\lambda_{i_j})\right)^{-\frac{1}{r}}\right] \geq \left[r\left(\frac{n}{\log\det(I+A)}\right)^{\frac{1}{r}}\right]^{\frac{n!}{r!\,(n-r)!}}.$$

Proof. Since λ_i ($i = 1, 2, \ldots, n$) is an eigenvalue of matrix A, we clearly see that $1 + \lambda_i$ is an eigenvalue of matrix $I + A$ and $\prod_{i=1}^n (1 + \lambda_i) = \det(I + A)$. This leads to

$$\left(\frac{\log\det(I+A)}{n}, \frac{\log\det(I+A)}{n}, \ldots, \frac{\log\det(I+A)}{n}\right) \prec \left(\log(1+\lambda_1), \log(1+\lambda_2), \ldots, \log(1+\lambda_n)\right). \quad (4.2)$$

Therefore, Theorem 4.5 follows from Theorem 3.1 and (4.2) together with the definition of $H_n(x, r)$ in (1.1). ∎

THEOREM 4.6. *Let $A = (a_{ij})_{n\times n}$ $(n \geq 2)$ be a complex matrix, $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of A. If A is a positive definite Hermitian matrix and $r \in \{1, 2, \ldots, n\}$, then*

$$\prod_{1\leq i_1<i_2<\cdots<i_r\leq n} \left(\sum_{j=1}^{r} \lambda_{i_j}^{-\frac{1}{r}}\right) \geq \left[r\left(\frac{n}{trA}\right)^{\frac{1}{r}}\right]^{\frac{n!}{r!\,(n-r)!}}.$$

Proof. Theorem 4.6 follows from (1.1) and Theorem 3.1 together with the fact that $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \lambda_i = trA$. ■

THEOREM 4.7. *If $(x_1, x_2, \ldots, x_n) \in \mathbf{R}_+^n$, then*

$$H_n\left(\frac{1}{1+x_1}, \frac{1}{1+x_2}, \ldots, \frac{1}{1+x_n}\right) \leq H_n\left(\frac{1}{1+\sum_{i=1}^{n} x_i}, 1, 1, \ldots, 1\right).$$

Proof. Theorem 4.7 follows from Theorem 3.1 and the fact that

$$(1+x_1, 1+x_2, \ldots, 1+x_n) \prec (1+\sum_{i=1}^{n} x_i, 1, 1, \ldots, 1). \quad ■$$

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