

A NOTE ON RATE OF APPROXIMATION FOR CERTAIN BÉZIER-DURRMEYER OPERATORS

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Abstract. The present paper deals with certain Bézier-Durrmeyer type sequence of linear positive operators $M_{n,\alpha}(f, x)$, having different basis functions in summation and integration. We estimate the rate of convergence of these operators $M_{n,\alpha}(f, x)$, for functions having derivatives of bounded variation.

1. Introduction

To approximate Lebesgue integrable functions on $[0, \infty)$, a new sequence of linear positive operators M_n introduced and studied in [2] is defined as

$$M_n(f, x) = \sum_{v=0}^{\infty} \Phi_{n,v}(f) p_{n,v}(x) \quad (1)$$

where

$$\Phi_{n,v}(f) = \begin{cases} n \int_0^{\infty} q_{n,v-1}(t) f(t) dt, & v > 0; \\ f(0), & v = 0 \end{cases}$$

and $p_{n,v}(x) = \binom{n+v-1}{v} x^v (1+x)^{-n-v}$ and $q_{n,v}(t) = e^{-nt} \frac{(nt)^v}{v!}$ are respectively Baskakov and Szász basis functions.

Very recently Abel et al. [1] introduced a more general sequence of summation-integral type operators, which for parameters $a \in R$ and $b \in Z$ is defined as

$$M_n^{a,b}(f, x) = n \sum_{v=\max\{0,-b\}}^{\infty} p_{n+a,v}(x) \int_0^{\infty} q_{n,v+b}(t) f(t) dt \quad (2)$$

We may remark here that for the case $a = 0$, $b = -1$ the operators (2) are very similar to the operators (1). The only difference is that the definition (1) contains an additional term $(1+x)^{-n} f(0)$ to ensure that the operators preserve constant functions. For the similar type of operators, we refer the readers to [7–9] where, some approximation properties of summation-integral type operators have been studied.

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It is observed that Bézier curves play an important role in Computer aided geometric design. Several researchers have considered the Bézier variants of well known operators and studied their approximation properties. We now consider the Bézier variant of the operators (1), which for $\alpha \geq 1$ or $0 < \alpha < 1$ are defined as

$$\begin{aligned} M_{n,\alpha}(f, x) &= n \sum_{v=1}^{\infty} Q_{n,v}^{(\alpha)}(x) \int_0^{\infty} q_{n,v-1}(t) f(t) dt + Q_{n,0}^{(\alpha)}(x) f(0) \\ &= \int_0^{\infty} W_{n,\alpha}(x, t) f(t) dt \end{aligned} \quad (3)$$

where $Q_{n,v}^{(\alpha)}(x) = (J_{n,v}(x))^\alpha - (J_{n,v+1}(x))^\alpha$, $J_{n,v}(x) = \sum_{j=v}^{\infty} p_{n,j}(x)$ and the kernel is given by

$$W_{n,\alpha}(x, t) = n \sum_{v=1}^{\infty} Q_{n,v}^{(\alpha)}(x) q_{n,v-1}(t) + Q_{n,0}^{(\alpha)}(x) \delta(t),$$

$\delta(t)$ being Dirac delta function. In case $\alpha = 1$ the operators defined by (3) reduce to the operators (1).

We denote by $DB_r(0, \infty)$, $r \geq 0$ the class of absolutely continuous functions f defined on $(0, \infty)$ satisfying:

- (i) $f(t) = O(t^r)$, $t \rightarrow \infty$,
- (ii) having a derivative f' on the interval $(0, \infty)$ coinciding a.e. with a function ψ which is of bounded variation on every finite subinterval of $(0, \infty)$. It can be observed that all functions $f \in BD_r(0, \infty)$ possess for each $c > 0$ the representation

$$f(x) = f(c) + \int_c^x \psi(t) dt.$$

The rate of convergence for functions of bounded variations for Bernstein and Bernstein-Durrmeyer operators were established in [4] and [10]. Bojanić and Cheng [3] obtained interesting results on the rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation. We now extend the study and here we estimate the rate of convergence of the operators $M_{n,\alpha}(f, x)$, defined by (3) for functions having derivatives of bounded variation.

2. Auxiliary results

In the sequel, we shall need the following lemmas:

LEMMA 1 [2] *Let the m -th order moment $\mu_{n,m}(x)$, $m \in N \cup \{0\}$ be defined as*

$$\mu_{n,m}(x) \equiv M_n((t-x)^m, x) = n \sum_{v=1}^{\infty} p_{n,v}(x) \int_0^{\infty} q_{n,v-1}(t) (t-x)^m dt + (-x)^m (1+x)^{-n}.$$

Then

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = 0, \quad \mu_{n,2}(x) = \frac{x(x+2)}{n}$$

and we have the recurrence relation:

$$n\mu_{n,m+1}(x) = x(1+x)\mu'_{n,m}(x) + m\mu_{n,m}(x) + mx(x+2)\mu_{n,m-1}(x), \quad m \geq 1,$$

where $\mu'_{n,m}(x) = \frac{d}{dx}\mu_{n,m}(x)$. Consequently, for each $x \in [0, \infty)$, we get

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}),$$

where $[\alpha]$ denotes the integral part of α .

REMARK 1. From Lemma 1, using Cauchy-Schwarz inequality, it follows that

$$M_n(|t-x|, x) \leq (\mu_{n,2}(x))^{1/2} = \sqrt{x(x+2)/n}.$$

REMARK 2. It is observed that for $\alpha \geq 1$ we have $Q_{n,v}^{(\alpha)}(x) \leq \alpha p_{n,v}(x)$, $x \in [0, \infty)$. We define $\beta_{n,\alpha}(x, t) = \int_0^t W_{n,\alpha}(x, s) ds$. In particular, we have

$$\beta_{n,\alpha}(x, \infty) = \int_0^\infty W_{n,\alpha}(x, s) ds = 1.$$

LEMMA 2. For $x \in (0, \infty)$ and $\alpha \geq 1$, we have

$$(i) \beta_{n,\alpha}(x, y) \leq \frac{\alpha x(x+2)}{n(x-y)^2}, \quad 0 \leq y < x,$$

$$(ii) 1 - \beta_{n,\alpha}(x, z) \leq \frac{\alpha x(x+2)}{n(z-x)^2}, \quad x < z < \infty.$$

Proof. First we prove (i). By using Lemma 1 and Remark 2, we have

$$\begin{aligned} \beta_{n,\alpha}(x, y) &= \int_0^y W_{n,\alpha}(x, t) dt \leq \int_0^y \frac{(x-t)^2}{(x-y)^2} W_{n,\alpha}(x, t) dt \\ &\leq (x-y)^{-2} \alpha \mu_{n,2}(x) = \frac{\alpha x(x+2)}{n(x-y)^2}. \end{aligned}$$

The proof of (ii) is similar, we omit the details. ■

3. Rate of convergence

Our main result is stated as follows:

THEOREM 1. Let $f \in DB_r(0, \infty)$, $r \in N$; also suppose $\alpha \geq 1$, and $x \in (0, \infty)$. Then for n sufficiently large, we have

$$\begin{aligned} |M_{n,\alpha}(f, x) - f(x)| &\leq \frac{\alpha(x+2)}{n} \sum_{v=1}^{[\sqrt{n}]} \bigvee_{x-x/v}^{x+x/v} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} ((f')_x) \\ &+ \frac{\alpha(x+2)}{nx} (|f(2x) - f(x) - xf'(x^+)| + |f(x)| + x|f'(x^+)|) \\ &+ \sqrt{\frac{\alpha(x+2)}{nx}} 2^r O(n^{-r/2}) + \frac{\alpha}{\alpha+1} \sqrt{\frac{\alpha x(x+2)}{n}} |f'(x^+) - f'(x^-)|, \end{aligned}$$

where the auxiliary function f_x is given by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \leq t < x \\ 0, & t = x \\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

$\bigvee_a^b f(x)$ denotes the total variation of f on $[a, b]$. Also by $f(x^-)$ and $f(x^+)$, we mean the left and right-hand limits respectively.

Proof. By the Mean value theorem, we have

$$\begin{aligned} M_{n,\alpha}(f, x) - f(x) &= \int_0^\infty W_{n,\alpha}(x, t)(f(t) - f(x)) dt \\ &= \int_0^\infty \int_x^t W_{n,\alpha}(x, t)(f'(u) du) dt. \end{aligned} \quad (4)$$

Using the following identity

$$\begin{aligned} f'(u) &= \frac{1}{\alpha+1}[f'(x^+) + \alpha f'(x^-)] + (f')_x(u) + \frac{1}{2}[f'(x^+) - f'(x^-)] \times \\ &\quad \times \left(\operatorname{sgn}(u-x) + \frac{\alpha-1}{\alpha+1} \right) + [f'(x) - \frac{1}{2}[f'(x^+) + f'(x^-)]] \chi_x(u) \end{aligned}$$

where $\chi_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$ Obviously, $M_{n,\alpha}(\chi_x, x) = 0$. Also

$$\begin{aligned} \int_0^\infty \left(\int_x^t \frac{1}{2}[f'(x^+) - f'(x^-)] \operatorname{sgn}(u-x) du \right) W_{n,\alpha}(x, t) dt \\ = \frac{1}{2}[f'(x^+) - f'(x^-)] M_{n,\alpha}(|t-x|, x) \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \left(\int_x^t \frac{1}{\alpha+1}[f'(x^+) + \alpha f'(x^-)] du \right) W_{n,\alpha}(x, t) dt \\ \leq \frac{\alpha}{\alpha+1}[f'(x^+) + \alpha f'(x^-)] M_n((t-x), x) = 0. \end{aligned}$$

Thus using the methods as given in [6], Lemma 1, Remark 1 and above values, we have

$$\begin{aligned} &|M_{n,\alpha}(f, x) - f(x)| \\ &\leq \left| \int_x^\infty \left(\int_x^t (f')_x(u) du \right) W_{n,\alpha}(x, t) dt - \int_0^x \left(\int_x^t (f')_x(u) du \right) W_{n,\alpha}(x, t) dt \right| \\ &\quad + \frac{\alpha}{\alpha+1} |f'(x^+) - f'(x^-)| M_{n,\alpha}(|t-x|, x) \\ &= |A_{n,\alpha}(f, x) - B_{n,\alpha}(f, x)| + \frac{\alpha}{\alpha+1} |f'(x^+) - f'(x^-)| M_{n,\alpha}(|t-x|, x) \\ &\leq |A_{n,\alpha}(f, x)| + |B_{n,\alpha}(f, x)| + \frac{\alpha}{\alpha+1} |f'(x^+) - f'(x^-)| \sqrt{\frac{\alpha x(x+2)}{n}}. \end{aligned} \quad (5)$$

Thus it suffice to estimate the terms $A_{n,\alpha}(f, x)$ and $B_{n,\alpha}(f, x)$. Integrating by parts, using Lemma 2 and taking $y = x - x/\sqrt{n}$, we have

$$\begin{aligned} |B_{n,\alpha}(f, x)| &= \left| \int_0^x \left(\int_x^t (f')_x(u) du \right) d_t \beta_{n,\alpha}(x, t) \right| \\ &= \left| \int_0^x \beta_{n,\alpha}(x, t) (f')_x(t) dt \right| \leq \left(\int_0^y + \int_y^x \right) |\beta_{n,\alpha}(x, t)| \bigvee_t^x ((f')_x) dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\alpha x(x+2)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \int_y^x \bigvee_t^x ((f')_x) dt \\
&\leq \frac{\alpha x(x+2)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x).
\end{aligned}$$

Let $u = \frac{x}{x-t}$. Then we have

$$\begin{aligned}
\frac{\alpha x(x+2)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt &= \frac{\alpha x(x+2)}{n} x^{-1} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x ((f')_x) du \\
&\leq \frac{\alpha(x+2)}{n} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-\frac{x}{u}}^x ((f')_x) du \leq \frac{\alpha(x+2)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x ((f')_x).
\end{aligned}$$

Thus,

$$|B_{n,\alpha}(f, x)| \leq \frac{\alpha(x+2)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x). \quad (6)$$

On the other hand, we have

$$\begin{aligned}
|A_{n,\alpha}(f, x)| &= \left| \int_x^\infty \left(\int_x^t (f')_x(u) du \right) W_{n,\alpha}(x, t) dt \right| \\
&\leq \left| \int_{2x}^\infty \left(\int_x^t (f')_x(u) du \right) W_{n,\alpha}(x, t) dt \right| + \left| \int_x^{2x} \left(\int_x^t (f')_x(u) du \right) dt (1 - \beta_{n,\alpha}(x, t)) \right| \\
&\leq \left| \int_{2x}^\infty (f(t) - f(x)) W_{n,\alpha}(x, t) dt \right| + |f'(x^+)| \left| \int_{2x}^\infty (t-x) W_{n,\alpha}(x, t) dt \right| \\
&\quad + \left| \int_x^{2x} (f')_x(u) du \right| |1 - \beta_{n,\alpha}(x, 2x)| + \int_x^{2x} |(f')_x(t)| |1 - \beta_{n,\alpha}(x, t)| dt \\
&\leq \frac{C}{x} \int_{2x}^\infty W_{n,\alpha}(x, t) t^r |t-x| dt + \frac{|f(x)|}{x^2} \int_{2x}^\infty W_{n,\alpha}(x, t) (t-x)^2 dt \\
&\quad + |f'(x^+)| \int_{2x}^\infty W_{n,\alpha}(x, t) |t-x| dt + \frac{\alpha(x+2)}{nx} |f(2x) - f(x) - x f'(x^+)| \\
&\quad + \frac{\alpha(x+2)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} ((f')_x). \quad (7)
\end{aligned}$$

For the estimation of the first two terms in the right-hand side of (7), we proceed as follows:

Applying Holder's inequality, Remark 1 and Lemma 1, we have

$$\begin{aligned}
&\frac{C}{x} \int_{2x}^\infty W_{n,\alpha}(x, t) t^r |t-x| dt + \frac{|f(x)|}{x^2} \int_{2x}^\infty W_{n,\alpha}(x, t) (t-x)^2 dt \\
&\leq \frac{C}{x} \left(\int_{2x}^\infty W_{n,\alpha}(x, t) t^{2r} dt \right)^{\frac{1}{2}} \left(\int_0^\infty W_{n,\alpha}(x, t) (t-x)^2 dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$+ \frac{|f(x)|}{x^2} \int_{2x}^{\infty} W_{n,\alpha}(x,t)(t-x)^2 dt \leq C2^r O(n^{-r/2}) \frac{\sqrt{\alpha(x+2)}}{\sqrt{nx}} + |f(x)| \frac{\alpha(x+2)}{nx}.$$

Also, by Remark 1, the third term of the right-hand side of (6) is given by

$$\begin{aligned} |f'(x^+)| \int_{2x}^{\infty} W_{n,\alpha}(x,t)|t-x| dt &\leq |f'(x^+)|x^{-1} \int_0^{\infty} W_{n,\alpha}(x,t)(t-x)^2 dt \\ &\leq |f'(x^+)| \frac{\alpha(x+2)}{n}. \end{aligned}$$

Combining the estimates (4)–(7), we get the desired result. This completes the proof of the theorem. ■

REMARK 3. We may note here that for $a = b = 0$, the operators reduce to the standard Baskakov-Szász operators studied in [5] and [8] which are defined by

$$M_n^{0,0}(f, x) = n \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} q_{n,v}(t) f(t) dt, \quad x \in [0, \infty)$$

We may consider the Bézier variant of such type of operators and the analogous results can be obtained along the lines similar to our main result.

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