МАТЕМАТИЧКИ ВЕСНИК
63, 3 (2011), 191–199
September 2011

оригинални научни рад
research paper

LAMBERT MULTIPLIERS OF THE RANGE OF COMPOSITION OPERATORS

M. R. Jabbarzadeh and S. Khalil Sarbaz

Abstract. In this note Lambert multipliers of the range of composition operators acting between different L^p spaces are characterized by using some properties of conditional expectation operators. Also, necessary conditions for Fredholmness and normality of these type operators are investigated.

1. Introduction and preliminaries

Let (X, Σ, μ) be a sigma finite measure space. For any complete sub-sigma finite algebra $\mathcal{A} \subseteq \Sigma$ with $1 \le p \le \infty$, the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$. We view $L^p(\mathcal{A})$ as a closed subspace of $L^p(\Sigma)$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. For $D \in \Sigma$, we define $\mathcal{A}_D = \{A \cap D : A \in \mathcal{A}\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$.

For a sub-sigma algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \mapsto E^{\mathcal{A}} f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma)$, $1 \le p \le \infty$, where $E^{\mathcal{A}} f$ is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f \, d\mu = \int_A E^{\mathcal{A}} f \, d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. This operator will play major role in our work, and we list here some of its useful properties:

- If g is $\mathcal{A}$-measurable then $E^{\mathcal{A}}(fg) = E^{\mathcal{A}}(f)g$.
- $|E^{\mathcal{A}}(f)|^p \le E^{\mathcal{A}}(|f|^p)$.
- If $f \ge 0$ then $E^{\mathcal{A}}(f) \ge 0$; if $f > 0$ then $E^{\mathcal{A}}(f) > 0$.

2010 AMS Subject Classification: 47B20, 47B38.

Keywords and phrases: Conditional expectation; multipliers; multiplication operators; weighted composition operators.

- $E^{\mathcal{A}}(|f|^2) = |E^{\mathcal{A}}(f)|^2$ if and only if $f \in L^p(\mathcal{A})$.
- $\sigma(E^{\mathcal{A}}(|f|))$ is the smallest $\mathcal{A}$-measurable set containing $\sigma(f)$.

A detailed discussion and verification of most of these properties may be found in [2] and [4]. Let $\varphi : X \to X$ be a non-singular measurable transformation, namely, a mapping from X into itself with the properties that the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, and $\varphi^{-1}(\Sigma)$ is sub-sigma finite algebra of Σ. We set $h = d\mu \circ \varphi^{-1}/d\mu$. Recall that an $\mathcal{A}$-atom of the measure μ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space (X, Σ, μ) with no atoms is called non-atomic measure space. It is well-known fact that every σ-finite measure space $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ can be partitioned uniquely as $X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and $B \in \mathcal{A}$, being disjoint from each A_n, is non-atomic (see [6]).

Let $w \in L^0(\Sigma)$. Then w is said to be conditionable with respect to $E^{\mathcal{A}}$ if $w \in \mathcal{D}(E^{\mathcal{A}})$, where $\mathcal{D}(E^{\mathcal{A}})$ denotes the domain of $E^{\mathcal{A}}$. For w and f in $L^0(\Sigma)$ such that $\{w, f\circ\varphi\} \in \mathcal{D}(E^{\mathcal{A}})$, we define $w \diamond f := wE^{\mathcal{A}}(f\circ\varphi) + E^{\mathcal{A}}(w) f\circ\varphi - E^{\mathcal{A}}(w)E^{\mathcal{A}}(f\circ\varphi)$. Let $1 \le p,\ q \le \infty$. Since for each $f \in L^p(\Sigma)$, $f \circ \varphi$ is conditionable, a measurable function $w \in \mathcal{D}(E^{\mathcal{A}})$ for which $w \diamond f \in L^q(\Sigma)$ is called Lambert multiplier of the range of composition operator C_φ. An easy consequence of the closed graph theorem assures us that $w \in \mathcal{D}(E^{\mathcal{A}})$ is the Lambert multiplier of the range of composition operator C_φ if and only if the corresponding $\diamond$-multiplication operator $K_w^\varphi : L^p(\Sigma) \to L^q(\Sigma)$ defined as $K_w^\varphi f = w \diamond f$ is bounded. Note that if $\mathcal{A} = \Sigma$ or $\varphi^{-1}(\Sigma) = \mathcal{A}$, then $K_w^\varphi = M_w C_\varphi = wC_\varphi$, where wC_φ is a weighted composition operator.

If φ is the identity on X, these operators were initially introduced in [3] by A. Lambert and T. G. Lucas where, some operator properties of them are also studied in [1]. In the next section, weighted conditional multipliers acting between two different $L^p(\Sigma)$ spaces are characterized by using some properties of conditional expectation operator. Also we give a necessary condition for Fredholmness and normality of K_w^φ.

2. Characterization of Lambert multipliers of the range of C_φ

Let $1 \le p, q \le \infty$. Define $\mathcal{K}_{p,q}^\varphi$, the set of all Lambert multipliers of the range of composition operator C_φ from $L^p(\Sigma)$ into $L^q(\Sigma)$, as follows

$$\mathcal{K}_{p,q}^\varphi = \{w \in \mathcal{D}(E) : w \diamond \mathcal{R}(C_\varphi) \subset L^q(\Sigma)\},$$

where $\mathcal{R}(C_\varphi)$ is the range of C_φ. Note that $\mathcal{K}_{p,q}^\varphi$ is a vector subspace of $L^0(\Sigma)$. Put $\mathcal{K}_{p,q}^\varphi = \mathcal{K}_p^\varphi$ for $1 \le p = q \le \infty$. Suppose that $X = \left(\bigcup_{n\in\mathbb{N}} C_n\right) \cup C$, where $\{C_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint Σ-atoms and $C \in \Sigma$, being disjoint from each C_n, is non-atomic. Note that $(\bigcup_{n\in\mathbb{N}} C_n) \cap \mathcal{A} \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and $B \subseteq C$.

By making use of the methods, which are used in the proofs of the results in [1], in the following theorem we similarly characterize the elements of the $\mathcal{K}_{p,q}^\varphi$, $1 \le p, q \le \infty$ in the various cases.

From now on we assume that $w \in \mathcal{D}(E)$, $E^{\mathcal{A}} = E$ and $\varphi^{-1}(\mathcal{A})$ is a sub-sigma finite algebra of $\mathcal{A}$.

THEOREM 2.1. *Let φ be a non-singular measurable transformation on X such that $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ and $S_p := h[E^{\varphi^{-1}(\Sigma)}E(|w|^p)] \circ \varphi^{-1}$. Then we have*

(a) If $1 \le p = q < \infty$, then $w \in \mathcal{K}_p^{\varphi}$ if and only if $S_p \in L^{\infty}(\Sigma)$.

(b) If $1 \le q < p < \infty$, then $w \in \mathcal{K}_{p,q}^{\varphi}$ if and only if $\sqrt[q]{S_q} \in L^r(\Sigma)$, where $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

(c) Let $1 \le p < q < \infty$ and let $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$. If $S_q = 0$ on C and $\sup_{n \in \mathbb{N}} \frac{S_q(C_n)}{(\mu(C_n))^{\frac{q}{r}}} < \infty$, then $w \in \mathcal{K}_{p,q}^{\varphi}$. On the other hand, let $\Sigma_B = \mathcal{A}_B$. If $w \in \mathcal{K}_{p,q}^{\varphi}$, then $S_q = 0$ on B and $\sup_{n \in \mathbb{N}} \frac{S_q(A_n)}{(\mu(A_n))^{\frac{q}{r}}} < \infty$.

(d) If $p = q = \infty$, then $w \in \mathcal{K}_{\infty}^{\varphi}$ if and only if $w \in L^{\infty}(\Sigma)$.

(e) If $1 \le q < \infty = p$, then $w \in \mathcal{K}_{\infty,q}^{\varphi}$ if and only if $S_q \in L^1(\Sigma)$.

Proof. (a) As an application of the properties of the conditional expectation operator and using the change of variable formula, for each $f \in L^p(\Sigma)$, we have

$$\begin{aligned}
\|wE(f\circ\varphi)\|_p^p &= \int_X |wE(f\circ\varphi)|^p\,d\mu \le \int_X E(E(|w|^p)|f|^p\circ\varphi)\,d\mu \\
&= \int_X (E(|w|^p)|f|^p\circ\varphi)\,d\mu = \int_X E^{\varphi^{-1}(\Sigma)}(E(|w|^p)|f|^p\circ\varphi)\,d\mu \\
&= \int_X E^{\varphi^{-1}(\Sigma)}(E(|w|^p))|f|^p\circ\varphi\,d\mu = \int_X h[E^{\varphi^{-1}(\Sigma)}E(|w|^p)]\circ\varphi^{-1}|f|^p\,d\mu \\
&= \int_X S_p|f|^p\,d\mu \le \|S_p\|_\infty \int_X |f|^p\,d\mu = \|S_p\|_\infty \|f\|_p^p.
\end{aligned}$$

Hence we have that $\|wE(f\circ\varphi)\|_p \le \|S_p\|_\infty^{\frac{1}{p}}\|f\|_p$. Similar computations show that $\|K_w^{\varphi}f\|_p \le 3\|S_p\|_\infty^{\frac{1}{p}}\|f\|_p$. It follows that $w \diamond f \in L^p(\Sigma)$ and hence $w \in \mathcal{K}_p^{\varphi}$.

Now, suppose only that $w \in \mathcal{K}_p^{\varphi}$. Define a linear functional ψ on $L^1(\mathcal{A})$ by

$$\psi(f) = \int_X S_p f\,d\mu, \quad f \in L^1(\mathcal{A}).$$

We shall show that ψ is bounded linear functional on $L^1(\mathcal{A})$. Note that since $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$, $f\circ\varphi$ is an $\mathcal{A}$-measurable whenever f is an $\mathcal{A}$-measurable function. Hence we have that

$$\begin{aligned}
|\psi(f)| &\le \int_X h[E^{\varphi^{-1}(\Sigma)}E(|w|^p)]\circ\varphi^{-1}|f|\,d\mu = \int_X E^{\varphi^{-1}(\Sigma)}(|f|\circ\varphi E(|w|^p))\,d\mu \\
&= \int_X |f|\circ\varphi E(|w|^p)\,d\mu = \int_X E(|w|^p|f|\circ\varphi)\,d\mu = \int_X |w|^p|f|\circ\varphi\,d\mu \\
&= \int_X (|w||f|^{\frac{1}{p}}\circ\varphi)^p\,d\mu = \int_X |w|f|^{\frac{1}{p}}\circ\varphi|^p\,d\mu = \int_X |K_w^{\varphi}(|f|^{\frac{1}{p}})|^p\,d\mu \\
&= \|K_w^{\varphi}(|f|^{\frac{1}{p}})\|_p^p \le \|K_w^{\varphi}\|^p \,\|\, |f|^{\frac{1}{p}}\|_p^p = \|K_w^{\varphi}\|^p\|f\|_1.
\end{aligned}$$

Now, by the Hahn-Banach theorem we can assume that ψ is a bounded linear functional on $L^1(\Sigma)$ and $\|\psi\| \leq \|K_w^{\varphi}\|^p$. By the Riesz representation theorem, there exists a unique function $g \in L^{\infty}(\Sigma)$ such that

$$\psi(f) = \int_X gf\,d\mu, \quad f \in L^1(\Sigma).$$

Therefore, we must have $g = S_p$ a.e. on X and hence $S_p \in L^{\infty}(\Sigma)$.

(b) Suppose $\sqrt[q]{S_q} \in L^r(\Sigma)$ and $f \in L^p(\Sigma)$. By using the same method used in the proof of part (a), we have

$$\begin{aligned}\|wE(f\circ\varphi)\|_q^q &= \int_X |wE(f\circ\varphi)|^q\,d\mu \leq \int_X h[E^{\varphi^{-1}(\Sigma)}E(|w|^q)]\circ\varphi^{-1}|f|^q\,d\mu \\ &= \int_X S_q|f|^q\,d\mu = \|\sqrt[q]{S_q}f\|_q^q \leq \|\sqrt[q]{S_q}\|_r^q\ \|f\|_p^q.\end{aligned}$$

By a similar computation we obtain $\|K_w^{\varphi}f\|_q \leq 3\|\sqrt[q]{S_q}\|_r\|f\|_p$, and so $\|K_w^{\varphi}\| \leq 3\|\sqrt[q]{S_q}\|_r$. Consequently, K_w^{φ} is bounded and hence $w \in \mathcal{K}_{p,q}^{\varphi}$.

Conversely, suppose that $w \in \mathcal{K}_{p,q}^{\varphi}$. Define $\psi : L^{\frac{p}{q}}(\mathcal{A}) \to \mathbb{C}$ as

$$\psi(f) = \int_X S_q f\,d\mu, \quad f \in L^{\frac{p}{q}}(\mathcal{A}).$$

Clearly ψ is a linear functional. We shall show that ψ is bounded. Since $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$, $E^{\varphi^{-1}(\Sigma)}(|f|\circ\varphi) = E(|f|\circ\varphi) = |f|\circ\varphi$ for all $\mathcal{A}$-measurable function f. It follows that

$$|\psi(f)| \leq \|K_w^{\varphi}(|f|^{\frac{1}{q}})\|_q^q \leq \|K_w^{\varphi}\|^q\|f\|_{\frac{p}{q}}.$$

Thus $\|\psi\| \leq \|K_w^{\varphi}\|^q$ and hence ψ is bounded. By the Hahn-Banach theorem we can assume that ψ is a bounded linear functional on $L^{\frac{p}{q}}(\Sigma)$ with $\|\psi\| \leq \|K_w^{\varphi}\|^q$. By the Riesz-representation theorem, there exists a unique $g \in L^{\frac{r}{q}}(\mathcal{A})$ such that $\psi(f) = \int_X gf\,d\mu$ for each $f \in L^{\frac{p}{q}}(\Sigma)$. Hence $g = S_q$ a.e. on X. That is $\sqrt[q]{S_q} \in L^r(\Sigma)$ and hence the proof is complete.

(c) Suppose that $S_q = 0$ on C and $M := \sup_{n\in\mathbb{N}} \frac{S_q(C_n)}{(\mu(C_n))^{\frac{q}{r}}} < \infty$. Then, for each $f \in L^p(\Sigma)$ with $\|f\|_p \leq 1$ we have

$$\begin{aligned}\|wE(f\circ\varphi)\|_q^q &\leq \int_X S_q|f|^q\,d\mu = (\int_C + \int_{\cup_{n=1}^{\infty}C_n})(S_q|f|^q)\,d\mu \\ &= 0 + \sum_{n=1}^{\infty}\int_{C_n} S_q|f|^q\,d\mu = \sum_{n=1}^{\infty} S_q(C_n)|f(C_n)|^q\mu(C_n) \\ &= \sum_{n=1}^{\infty}\frac{S_q(C_n)}{\mu(C_n)^{\frac{q}{r}}}\,(|f(C_n)|^p\mu(C_n))^{\frac{q}{p}} \leq M\|f\|_p^q \leq M < \infty,\end{aligned}$$

where we have used the fact that $(S_q|f|^q)$ is constant function on each C_n. Consequent, we get $\|wE(f\circ\varphi)\|_q \leq \sqrt[q]{M}$. Similar computations show that $\|K_w^{\varphi}\| \leq 3\sqrt[q]{M} < \infty$ and hence $w \in \mathcal{K}_{p,q}^{\varphi}$.

Now, suppose that $w \in \mathcal{K}^{\varphi}_{p,q}$. First we show that $S_q = 0$ on B. Assuming the contrary, we can find some $\delta > 0$ such that $0 < \mu(\{x \in B : S_q(x) \geq \delta\}) < \infty$. Set $K = \{x \in B : S_q(x) \geq \delta\}$. Note that $K \in \Sigma_B = \mathcal{A}_B$, $B \subseteq C$ and $\mathcal{A}$ is sigma finite. Then for all $n \in \mathbb{N}$, there exists $K_n \subseteq K$ such that $K_n \in \mathcal{A}$ with $\mu(K_n) = \frac{\mu(K)}{2^n}$. For any $n \in \mathbb{N}$, put $f_n = \frac{1}{(\mu(K_n))^{1/p}}\, \chi_{K_n}$. It is clear that $f_n \in L^p(\mathcal{A})$ and $\|f_n\|_p = 1$. Since $\frac{q}{p} - 1 > 0$ and $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$, we obtain

$$\begin{aligned}\infty > \|K^{\varphi}_w\|^q \geq \|K^{\varphi}_w f_n\|_q^q = \|w(f_n \circ \varphi)\|_q^q = \int_X S_q |f_n|^q \, d\mu \\ = \frac{1}{\mu(K_n)^{\frac{q}{p}}} \int_{K_n} S_q \, d\mu \geq \frac{\delta \mu(K_n)}{\mu(K_n)^{\frac{q}{p}}} = \delta \left(\frac{\mu(K)}{2^n}\right)^{\frac{q}{p}-1} \quad \to \infty \text{ as } n \to \infty,\end{aligned}$$

which is a contradiction. Hence we conclude that $\mu(\{x \in B : S_q(x) \neq 0\}) = 0$. Next, we exam the supremum. For any $n \in \mathbb{N}$, put $f_n = \frac{1}{(\mu(A_n))^{1/p}}\, \chi_{A_n}$, Then it is clear that $f_n \in L^p(\mathcal{A})$ and $\|f_n\|_p = 1$. Then we have

$$\begin{aligned}\infty > \|K^{\varphi}_w\|^q &\geq \|K^{\varphi}_w f_n\|_q^q = \|w(f_n \circ \varphi)\|_q^q \\ &= \frac{1}{(\mu(A_n))^{\frac{q}{p}}} \int_{A_n} S_q \, d\mu = \frac{1}{(\mu(A_n))^{\frac{q}{p}}} S_q(A_n)\mu(A_n) = \frac{S_q(A_n)}{(\mu(A_n))^{\frac{q}{r}}}.\end{aligned}$$

Since this holds for any $n \in \mathbb{N}$, we get that $\sup_{n \in \mathbb{N}} \frac{S_q(A_n)}{(\mu(A_n))^{\frac{q}{r}}} < \infty$.

(d) Suppose that for each $f \in L^{\infty}(\Sigma)$, $K^{\varphi}_w f \in L^{\infty}(\Sigma)$. Then

$$\|w\|_{\infty} = \|w(\chi_X \circ \varphi)\|_{\infty} = \|K^{\varphi}_w \chi_X\|_{\infty} \leq \|K^{\varphi}_w\| \|\chi_X\|_{\infty} = \|K^{\varphi}_w\| < \infty.$$

Conversely, suppose that $w \in L^{\infty}(\Sigma)$. Since E is a contraction operator, then for each $f \in L^{\infty}(\Sigma)$, we have

$$\|K^{\varphi}_w f\|_{\infty} \leq 3\|w\|_{\infty} \|f \circ \varphi\|_{\infty} \leq 3\|w\|_{\infty} \|f\|_{\infty}.$$

Thus $\|K^{\varphi}_w\| \leq 3\|w\|_{\infty}$, and so $w \in \mathcal{K}^{\varphi}_{\infty}$.

(e) Suppose $S_q \in L^1(\Sigma)$ and $f \in L^{\infty}(\Sigma)$. Then we have

$$\|wE(f \circ \varphi)\|_q^q \leq \int_X S_q |f|^q \, d\mu \leq \|f\|_{\infty}^q \|S_q\|_1.$$

It follows that $\|K^{\varphi}_w\| \leq 3\|S_q\|_1^{1/q}$, and so $w \in \mathcal{K}^{\varphi}_{\infty,q}$. Conversely, suppose that $w \in \mathcal{K}^{\varphi}_{\infty,q}$. Since $\chi_X \in L^{\infty}(\mathcal{A})$, thus $K^{\varphi}_w \chi_X \in L^q(\Sigma)$, and so

$$\|S_q\|_1 = \int_X S_q \, d\mu = \|K^{\varphi}_w \chi_X\|_q^q < \infty.$$

This completes the proof. ∎

Corollary 2.2. *Let $w \in L^0(\Sigma)$ and let $wC_{\varphi} : L^p(\Sigma) \to L^q(\Sigma)$ be a weighted composition operator. Put $J_p = hE^{\varphi^{-1}(\Sigma)}(|w|^p) \circ \varphi^{-1}$. Then the following hold.*

(a) If $1 \le p = q < \infty$, then wC_φ is bounded if and only if $J_p \in L^\infty(\Sigma)$.

(b) If $1 \le q < p < \infty$, then wC_φ is bounded if and only if $\sqrt[q]{J_q} \in L^r(\Sigma)$, where $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

(c) If $1 \le p < q < \infty$, then wC_φ is bounded if and only if w satisfies the following conditions:

i) $J_q = 0$ on C;

ii) $\sup_{n\in\mathbb{N}} \frac{J_q(C_n)}{(\mu(C_n))^{\frac{q}{r}}} < \infty$, where $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$.

(d) If $p = q = \infty$, then wC_φ is bounded if and only if $w \in L^\infty(\Sigma)$.

(e) If $1 \le q < \infty = p$, then wC_φ is bounded if and only if $J_q \in L^1(\Sigma)$.

Proof. Put $\mathcal{A} = \Sigma$ in the previous theorem. Then we have $K_w^\varphi = wC_\varphi$ and $S_q = J_q$. Thus the proof holds. ∎

Corollary 2.3. *Let φ be the identity transformation on X and $w \in \mathcal{D}(E)$. Put $T_w f = wE(f) + fE(w) - Ew)E(f)$. Then the following hold.*

(a) If $1 \le p < \infty$, then $T_w : L^p(\Sigma) \to L^p(\Sigma)$ is bounded linear operator if and only if $E(|w|^p) \in L^\infty(\mathcal{A})$.

(b) If $1 \le q < p < \infty$, then $T_w : L^p(\Sigma) \to L^q(\Sigma)$ is bounded linear operator if and only if $(E(|w|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$ where $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

(c) Let $1 \le p < q < \infty$ and let $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$. If $E(|w|^q) = 0$ on C and $\sup_{n\in\mathbb{N}} \frac{E(|w|^q)(C_n)}{(\mu(C_n))^{\frac{q}{r}}} < \infty$, then $T_w : L^p(\Sigma) \to L^q(\Sigma)$ is bounded linear operator. On the other hand if T_w is bounded, then $E(|w|^q) = 0$ on B and $\sup_{n\in\mathbb{N}} \frac{E(|w|^q)(A_n)}{(\mu(A_n))^{\frac{q}{r}}} < \infty$.

(d) If $p = q = \infty$, then $T_w : L^\infty(\Sigma) \to L^\infty(\Sigma)$ is bounded linear operator if and only if $w \in L^\infty(\Sigma)$.

(e) If $1 \le q < \infty = p$, then $T_w : L^\infty(\Sigma) \to L^q(\Sigma)$ is bounded linear operator if and only if $E(|w|^q) \in L^1(\Sigma)$.

Proof. Put $\varphi = id$ in the previous theorem. Then we have $K_w^{id} = T_w$ and $S_p = E(|w|^p)$. ∎

Example 2.4. Let $X = [-1, 1]$, $d\mu = \frac{1}{2}dx$ and Σ the Lebesgue sets. Define the non-singular transformations $\varphi_i : X \to X$ by $\varphi_1(x) = \sqrt[3]{3x}$ and $\varphi_2(x) = (\sqrt{1+x} - 1)\chi_{[-1,0]} + (1 - \sqrt{1-x})\chi_{(0,1]}$. Put $h_{\varphi_i} = d\mu \circ \varphi_i / d\mu$ and $\mathcal{A} = \varphi_2^{-1}(\Sigma)$. It is easy to see that $E^{\varphi_1^{-1}(\Sigma)} = I$ and $E^{\mathcal{A}}(f) = (f(x) + f(-x))/2$, for all positive measurable function f on X. Put $w(x) = \sqrt{x^2 + x + 1}$. Direct computations show that $h_{\varphi_1}(x) = x^2$, $h_{\varphi_2}(x) = (2+2x)\chi_{[-1,0]} + (2-2x)\chi_{(0,1]}$ and $E^{\mathcal{A}}(w^2)(x) = x^2 + 1$. Therefore we get that

$$S_2(x) = h_{\varphi_1}(x)[E^{\varphi_1^{-1}(\Sigma)} E^{\mathcal{A}}(w^2)] \circ \varphi_1^{-1}(x) = x^2 + \frac{1}{9}x^8,$$

$$J_1(x) = x^2 + \frac{1}{3}x^5 + \frac{1}{9}x^8,$$
$$J_2(x) = (2+2x)\left((2x+x^2)^2+1\right)\chi_{[-1,0]} + (2-2x)\left((2x-x^2)^2+1\right)\chi_{(0,1]},$$

where $J_i := h_{\varphi_i} E^{\varphi_i^{-1}(\Sigma)}(w^2) \circ \varphi_i^{-1}$. If $W_i = w.f \circ \varphi_i$, then we get that

$$\|K_w^{\varphi_1}\|_{L^2(\Sigma)\to L^2(\Sigma)} \le \sqrt{10}, \ \|W_1\|_{L^2(\Sigma)\to L^2(\Sigma)} = \frac{\sqrt{13}}{3}, \ \|W_2\|_{L^2(\Sigma)\to L^2(\Sigma)} = 2\sqrt{10}. \ \blacksquare$$

In what follows we use the symbols $\mathcal{N}(K_w^\varphi)$ and $\mathcal{R}(K_w^\varphi)$ to denote the kernel and the range of K_w^φ, respectively. Recall that K_w^φ is a Fredholm operator on $L^p(\Sigma)$ if $\mathcal{R}(K_w^\varphi)$ is closed, $\dim \mathcal{N}(K_w^\varphi) < \infty$, and $\operatorname{codim} \mathcal{R}(K_w^\varphi) < \infty$.

In the following we give a necessary condition for K_w^φ on $L^p(\Sigma)$ to be a Fredholm operator. This is a generalization of the result obtained in [5] for multiplication operators.

Lemma 2.5. *Suppose that $w \in \mathcal{K}_p^\varphi$ and $\mathcal{A}$ is a non-atomic measure space. If K_w^φ is a Fredholm operator on $L^p(\Sigma)$ $(1 \le p < \infty)$, then it is onto and $E^{\varphi^{-1}(\mathcal{A})}(w) \neq 0$ almost everywhere on X.*

Proof. Suppose that K_w^φ is a Fredholm operator. We first claim that K_w^φ is onto. Suppose the contrary. Then there exists $f_0 \in L^p(\Sigma) \setminus \mathcal{R}(K_w^\varphi)$. Since $\mathcal{R}(K_w^\varphi)$ is closed, by the Hahn-Banach theorem there exists a bounded functional $F_{g_0} : L^p(\Sigma) \to \mathbb{C}$, corresponding to $g_0 \in L^q(\Sigma)$, such that

$$F_{g_0}(f_0) = \int_X \bar{f}_0 g_0 \, d\mu = 1 \tag{2.1}$$

and

$$F_{g_0}(\mathcal{R}(K_w^\varphi)) = 0 \tag{2.2}$$

Now (2.1) yields that the set $B_r = \{x \in X : |E^{\varphi^{-1}(\mathcal{A})}(\bar{f}_0 g_0)(x)| \ge r\}$ has positive and finite measure for some $r > 0$. Since $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ is sigma finite and $\mathcal{A}$ is non-atomic, we can choose a sequence of pairwise disjoint sets $\{A_n\}$ of $\mathcal{A}$ such that $\varphi^{-1}(A_n) \subseteq B_r$ and $\mu(\varphi^{-1}(A_n)) > 0$ for all $n \in \mathbb{N}$. Put $g_n = \chi_{\varphi^{-1}(A_n)} g_0$. Clearly, $g_n \in L^q(\Sigma)$ and is nonzero, because

$$\begin{aligned}\int_X |\bar{f}_0 g_n| \, d\mu &\ge \int_{\varphi^{-1}(A_n)} |\bar{f}_0 g_n| \, d\mu = \int_{\varphi^{-1}(A_n)} E^{\varphi^{-1}(\mathcal{A})}(|\bar{f}_0 g_0|) \\ &\ge \int_{\varphi^{-1}(A_n)} |E^{\varphi^{-1}(\mathcal{A})}(\bar{f}_0 g_0)| \, d\mu \ge r\mu(\varphi^{-1}(A_n)) > 0\end{aligned}$$

for each n. Also, for each $f \in L^p(\Sigma), \chi_{A_n} f \in L^p(\Sigma)$ and so (2.2) implies that

$$\begin{aligned}((K_w^\varphi)^* g_n, f) = (g_n, K_w^\varphi f) &= \int_X g_0 (\chi_{A_n} \circ \varphi) \overline{K_w^\varphi f} \, d\mu \\ &= \int_X g_0 \overline{K_w^\varphi(\chi_{A_n} f)} \, d\mu = (g_0, K_w^\varphi(\chi_{A_n} f)) = 0,\end{aligned}$$

which implies that $(K_w^\varphi)^* g_n = 0$ and so $g_n \in \mathcal{N}((K_w^\varphi)^*)$. Since all the sets in $\{\varphi^{-1}(A_n)\}$ are disjoint, the sequence $\{g_n\}$ forms a linearly independent subset of $\mathcal{N}((K_w^\varphi)^*)$. This contradicts the fact that $\dim \mathcal{N}((K_w^\varphi)^*) = \operatorname{codim} \mathcal{R}(K_w^\varphi) < \infty$. Hence K_w^φ is onto. Put $D = \{x \in X : E^{\varphi^{-1}(\mathcal{A})}(u)(x) = 0\}$. If $\mu(D) > 0$, there is a $\varphi^{-1}(\mathcal{A})$-measurable set $F \subseteq D$ with $0 < \mu(F) < \infty$. If $\chi_F \in \mathcal{R}(K_w^\varphi)$, then there exists $f \in L^p(\Sigma)$ such that $K_w^\varphi f = \chi_F$. Since F is also an $\mathcal{A}$-measurable set and $\sigma(E(w)) \subseteq \sigma(E^{\varphi^{-1}(\mathcal{A})}(w))$, we get that

$$\mu(F) = \int_X \chi_F \, d\mu = \int_F K_w^\varphi f \, d\mu = \int_F E(K_w^\varphi f) \, d\mu = \int_F E(w) E(f \circ \varphi) \, d\mu = 0,$$

and this is a contradiction. So $\chi_F \in L^p(\Sigma) \backslash \mathcal{R}(K_w^\varphi)$, which again contradicts the fact that K_w^φ is onto. ■

The proof of the following theorem can be obtained by Lemma 2.5 and adapting the proof of Theorem 3.2 in [1].

Theorem 2.6. *Let $w \in \mathcal{K}_p^\varphi$, $h \in L^\infty(\Sigma)$ and let $\mathcal{A}$ be a non-atomic measure space. If K_w^φ is a Fredholm operator on $L^p(\Sigma)$ $(1 \le p < \infty)$, then $|E^{\varphi^{-1}(\mathcal{A})}(w)| \ge \delta$ almost everywhere on X for some $\delta > 0$.*

Now, we consider the particular case when $p = 2$. For $w \in \mathcal{D}(E)$, define $T_w : L^2(\Sigma) \to L^2(\Sigma)$ as $T_w f = wE(f) + fE(w) - Ew)E(f)$. It is easy to see that $T_w^* f = E(\bar{w} f) + \overline{E(w)}(f - E(f))$ and $K_w^\varphi = T_w C_\varphi$. Also we have

$$(K_w^\varphi)^* f = C_\varphi^*(T_w^* f) = h E^{\varphi^{-1}(\Sigma)}(T_w^* f) \circ \varphi^{-1}$$
$$K_w^\varphi (K_w^\varphi)^* = T_w C_\varphi C_\varphi^* T_w = T_w M_{h\circ\varphi} E^{\varphi^{-1}(\Sigma)} T_w$$

and $(K_w^\varphi)^* K_w^\varphi = C_\varphi^* T_w^* T_w C_\varphi$. For the study of the Lambert multiplication operator T_w on L^p-spaces, see [1] and the references therein. By using these facts we have the following lemma.

Lemma 2.7. *Let $w \in \mathcal{K}_2^\varphi$. Then we have:*

(a) $(K_w^\varphi)^* f = h E^{\varphi^{-1}(\Sigma)}(E(\bar{w} f) + \overline{E(w)}(f - E(f))) \circ \varphi^{-1}$.

(b) $K_w^\varphi (K_w^\varphi)^* f = wE(h \circ \varphi E^{\varphi^{-1}(\Sigma)}(T_w^* f)) + E(w) h \circ \varphi E^{\varphi^{-1}(\Sigma)}(T_w^* f) - E(w) E(h \circ \varphi E^{\varphi^{-1}(\Sigma)}(T_w^* f))$.

(c) $(K_w^\varphi)^* K_w^\varphi f = h E^{\varphi^{-1}(\Sigma)}\{E(f \circ \varphi) E(|w|^2) + E(w) E(\bar{w} f \circ \varphi) + w E(\bar{w}) E(f \circ \varphi) + |E(w)|^2 f \circ \varphi - 3|E(w)|^2 E(f \circ \varphi)\} \circ \varphi^{-1}$.

Proposition 2.8. *Let $w \in \mathcal{K}_2^\varphi$ and let $\mathcal{A} \subseteq \varphi^{-1}(\Sigma)$. If K_w^φ is normal on $L^2(\Sigma)$, then $(w \diamond (h \circ \varphi))\overline{E(w)} = h E(|w|^2) \circ \varphi^{-1}$.*

Proof. Since K_w^φ is normal, then $K_w^\varphi (K_w^\varphi)^* f = (K_w^\varphi)^* K_w^\varphi f$ for all $f \in L^2(\Sigma)$. In particular if $f \in L^2(\mathcal{A})$, by Lemma 2.7 we have

$$K_w^\varphi (K_w^\varphi)^* f = (w \diamond h \circ \varphi)\overline{E(w)} f,$$
$$(K_w^\varphi)^* K_w^\varphi f = h\{E(|w|^2) \circ \varphi^{-1} - |E(w)|^2 \circ \varphi^{-1} + (E^{\varphi^{-1}(\Sigma)}(w)\overline{E(w)}) \circ \varphi^{-1}\} f.$$

Since $\mathcal{A} \subseteq \varphi^{-1}(\Sigma)$, for each $A \in \mathcal{A}$ we get that

$$\int_A E^{\varphi^{-1}(\Sigma)}(w)\overline{E(w)}\,d\mu = \int_A E^{\varphi^{-1}(\Sigma)}(w\overline{E(w)})\,d\mu = \int_A w\overline{E(w)}\,d\mu = \int_A |E(w)|^2\,d\mu.$$

Therefore, we get that $(K_w^\varphi)^* K_w^\varphi f = hE(|w|^2) \circ \varphi^{-1} f$. ■

Note that if φ is the identity on X, then by Proposition 2.8 normality of bounded operator $K_w^\varphi = T_w$ implies that $w\overline{E(w)} = E(|w|^2)$. Hence we obtain that $E(|w|^2) = |E(w)|^2$ and thus $w \in L^\infty(\mathcal{A})$. On the other hand, if $w \in L^\infty(\mathcal{A})$ then it is easy to see that $T_w^* T_w f = T_w T_w^* f = |w|^2 f$ for all $f \in L^2(\Sigma)$ and hence T_w is normal.

ACKNOWLEDGEMENT. The authors would like to express their deep gratitude to the referee(s) for his/her careful reading of the paper and helpful comments which improved the presentation of it.

REFERENCES

[**1**] M.R. Jabbarzadeh, S. Khalil Sarbaz, *Lambert multipliers between L^p spaces*, Czechoslovak Math. J. **60** (2010), 31–43.

[**2**] A. Lambert, *L^p multipliers and nested sigma-algebras*, Oper. Theory Adv. Appl. **104** (1998), 147–153.

[**3**] A. Lambert, Thomas G. Lucas, *Nagata's principle of idealization in relation to module homomorphisms and conditional expectations*, Kyungpook Math. J. **40** (2000), 327–337.

[**4**] M.M. Rao, *Conditional Measure and Applications*, Marcel Dekker, New York, 1993.

[**5**] H. Takagi, *Fredholm weighted composition operators*, Integral Equations Operator Theory **16** (1993), 267–276.

[**6**] A.C. Zaanen, *Integration*, 2nd ed., North-Holland, Amsterdam, 1967.

(received 17.03.2010; in revised form 17.01.2011)

Faculty of Mathematical Sciences, University of Tabriz, P. O. Box: 5166615648, Tabriz, Iran

E-mail: mjabbar@tabrizu.ac.ir, skhsarbaz@tabrizu.ac.ir