МАТЕМАТИЧКИ ВЕСНИК
63, 3 (2011), 201–217
September 2011

оригинални научни рад
research paper

PROPERTIES OF NEW CLASSES OF MULTIVALENT ANALYTIC FUNCTIONS

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Abstract. By means of the subordination, we introduce and investigate classes of multivalent functions with varying argument of coefficients and two fixed points. The results obtained here for each of these classes include coefficients estimates, growth and distortion theorems, radii of starlikeness and convexity, subordination theorems and integral means inequalities. Some consequences of the results for well-known classes of functions are also pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} = \mathcal{U}(1)$, where

$$\mathcal{U}(r) = \{z \in \mathbb{C} : |z| < r\}$$

and let $\mathcal{A}(p,k)$ $(p,k \in \mathbb{N} = \{1,2,3\dots\},\ p < k)$ denote the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = a_p z^p + \sum_{n=k}^{\infty} a_n z^n \quad (z \in \mathcal{U};\ a_p > 0). \tag{1}$$

For multivalent function $f \in \mathcal{A}(p,k)$ the normalization

$$\left.\frac{f(z)}{z^{p-1}}\right|_{z=0} = 0 \text{ and } \left.\frac{f(z)}{z^p}\right|_{z=0} = 1 \tag{2}$$

is classical. One can obtain interesting results by applying normalization of the form

$$\left.\frac{f(z)}{z^{p-1}}\right|_{z=0} = 0 \text{ and } \left.\frac{f'(z)}{z^{p-1}}\right|_{z=\rho} = p, \tag{3}$$

where ρ is a fixed point of the unit disk $\mathcal{U}$. In particular, for $p = 1$ we obtain Montel's normalization (cf. [9]).

We denote by $\mathcal{A}_\rho(p,k)$ the classes of functions $f \in \mathcal{A}(p,k)$ with the normalization (3). It will be called the class of functions with two fixed points.

2010 AMS Subject Classification: 30C45, 30C50, 30C55.

Keywords and phrases: Analytic functions; varying arguments; fixed points; subordination; Hadamard product; integral means inequalities.

Also, by $\mathcal{T}(p,k;\eta)$ $(\eta \in \mathbb{R})$ we denote the class of functions $f \in \mathcal{A}(p,k)$ of the form (1) for which all of non-vanishing coefficients satisfy the condition

$$\arg(a_n) = \pi + (p-n)\eta \quad (n = k, k+1, \dots). \tag{4}$$

For $\eta = 0$ we obtain the class $\mathcal{T}(p,k;0)$ of functions with negative coefficients. Moreover, we define

$$\mathcal{T}(p,k) := \bigcup_{\eta\in\mathbb{R}} \mathcal{T}(p,k;\eta). \tag{5}$$

The classes $\mathcal{T}(p,k)$ and $\mathcal{T}(p,k;\eta)$ are called the classes of functions with varying argument of coefficients. The class $\mathcal{T}(1,2)$ was introduced by Silverman [15] (see also [21]).

Let $\alpha \in \langle 0,p)$, $r \in (0,1\rangle$. A function $f \in \mathcal{A}(p,k)$ is said to be *convex of order* α in $\mathcal{U}(r)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ \ (z \in \mathcal{U}(r)).$$

A function $f \in \mathcal{A}(p,k)$ is said to be *starlike of order* α in $\mathcal{U}(r)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{U}(r)). \tag{6}$$

We denote by $\mathcal{S}^c(\alpha)$ the class of functions $f \in \mathcal{A}(p,p+1)$, which are convex of order α in $\mathcal{U}$ and by $\mathcal{S}_p^*(\alpha)$ we denote the class of functions $f \in \mathcal{A}(p,p+1)$, which are starlike of order α in $\mathcal{U}$. We also set

$$\mathcal{S}^c = \mathcal{S}_1^c(0) \text{ and } \mathcal{S}^* = \mathcal{S}_1^*(0).$$

It is easy to show that for a function $f \in \mathcal{T}(p,k)$ the condition (6) is equivalent to the following

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \alpha \quad (z \in \mathcal{U}(r)). \tag{7}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{A}(p,k)$. We define *the radius of starlikeness of order* α and *the radius of convexity of order* α for the class $\mathcal{B}$ by

$$R_\alpha^*(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left(\sup\{r \in (0,1] : f \text{ is starlike of order } \alpha \ \text{ in } \mathcal{U}(r)\}\right),$$
$$R_\alpha^c(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left(\sup\{r \in (0,1] : f \text{ is convex of order } \alpha \text{ in } \mathcal{U}(r)\}\right),$$

respectively.

We say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function $\omega \in \mathcal{A}$ $(|\omega(z)| \le |z|,\ z \in \mathcal{U})$, such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence.

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

For functions $f, g \in \mathcal{A}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote the *Hadamard product* (or *convolution* of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

Let γ, δ be real parameters, $0 \leq \gamma < 1$, $\delta \geq 0$, and let $\varphi, \phi \in \mathcal{A}_0(p,k)$. By $\mathcal{W}(p,k;\phi,\varphi;\gamma,\delta)$ we denote the class of functions $f \in \mathcal{A}(p,k)$ such that

$$(\varphi * f)(z) \neq 0 \;\; (z \in \mathcal{U} \setminus \{0\}) \tag{8}$$

and

$$\operatorname{Re}\left\{\frac{(\phi * f)(z)}{(\varphi * f)(z)} - \gamma\right\} > \delta \left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| \quad (z \in \mathcal{U}). \tag{9}$$

Also, let us denote

$$\begin{aligned}
\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta) &:= \mathcal{T}(p,k) \cap \mathcal{W}(p,k;\phi,\varphi;\gamma,\delta), \\
\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta) &:= \mathcal{T}(p,k;\eta) \cap \mathcal{W}(p,k;\phi,\varphi;\gamma,\delta), \\
\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta;\eta) &:= \mathcal{A}_\rho(p,k) \cap \mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta), \\
\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta) &:= \mathcal{A}_\rho(p,k) \cap \mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta).
\end{aligned}$$

For the presented investigations we assume that φ, ϕ are the functions of the form

$$\varphi(z) = z^p + \sum_{n=k}^{\infty} \alpha_n z^n, \quad \phi(z) = z^p + \sum_{n=k}^{\infty} \beta_n z^n \quad (z \in \mathcal{U}), \tag{10}$$

where $0 \leq \alpha_n < \beta_n$ $(n = k, k+1, \dots)$. Moreover, let us put

$$d_n := (\delta + 1)\beta_n - (\delta + \gamma)\alpha_n \quad (n = k, k+1, \dots). \tag{11}$$

The families $\mathcal{W}_\rho(p,k;\phi,\varphi;\gamma,\delta;\eta)$ and $\mathcal{W}_\rho(p,k;\phi,\varphi;\gamma,\delta)$ unify various new and well-known classes of analytic functions. In particular, the class

$$\mathcal{W}_\rho(\varphi;\gamma,\delta;\eta) := \mathcal{W}_\rho\left(p,k;\frac{z\varphi'(z)}{p},\varphi(z);\gamma,\delta;\eta\right),$$

contains functions $f \in \mathcal{A}_\rho(p,k)$, such that

$$\operatorname{Re}\left\{\frac{z(\varphi * f)'(z)}{p(\varphi * f)(z)} - \gamma\right\} > \delta \left|\frac{z(\varphi * f)'(z)}{p(\varphi * f)(z)} - 1\right| \quad (z \in \mathcal{U}).$$

The class

$$\mathcal{H}_{\mathcal{T}}(\varphi;\gamma,\delta) := \mathcal{TW}_0(1,2;\varphi;\gamma,\delta;0)$$

was introduced and studied by Raina and Bansal [11]. If we set

$$h(\alpha_1, z) := z\, {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

where ${}_qF_s$ is the generalized hypergeometric function (see for details [18] and [10]), then we obtain the class

$$\mathcal{UH}\,(q, s, \lambda, \gamma, \delta) := \mathcal{TW}_0\,(1, 2; \lambda h(\alpha_1 + 1, z) + (1-\lambda)\, h(\alpha_1, z); \gamma, \delta; 0) \quad (0 \le \lambda \le 1)$$

defined by Srivastava et al. [12]. The classes

$$\delta - UST\,(\gamma) = \mathcal{W}_0\left(1, 2; \frac{z}{1-z}; \gamma, \delta\right),$$

$$\delta - UCV\,(\gamma) = \mathcal{W}_0\left(1, 2; \frac{z}{(1-z)^2}; \gamma, \delta\right),$$

are the well-known classes of δ-starlike functions of order γ and δ-uniformly convex functions of order γ, respectively. In particular, the classes $UCV := UCV\,(1, 0)$, $\delta - UCV := UCV(\delta, 0)$ were introduced by Goodman [4] (see also [8, 13, 17]), and Kanas and Wisniowska [5], respectively.

Many other classes, are also particular cases of the class investigated here, see for example [6, 19, 22].

The object of the present paper is to investigate the coefficients estimates, distortion properties, the radii of starlikeness and convexity, subordination theorems, partial sums and integral means inequalities for the classes of functions.

2. Coefficients estimates

We first mention a sufficient condition for a function to belong to the class $\mathcal{W}\,(p, k; \phi, \varphi; \gamma, \delta)$.

THEOREM 1. *Let $\{d_n\}$ be defined by* (11) *and let $0 \le \gamma < 1$. If a function f of the form* (1) *satisfies the condition*

$$\sum_{n=k}^{\infty} d_n\,|a_n| \le (1-\gamma)\, a_p, \tag{12}$$

then f belongs to the class $\mathcal{W}\,(p, k; \phi, \varphi; \gamma, \delta)$.

Proof. By definition of the class $\mathcal{W}\,(p, k; \phi, \varphi; \gamma, \delta)$, it suffices to show that

$$\delta \left| \frac{(\phi * f)\,(z)}{(\varphi * f)\,(z)} - 1 \right| - \operatorname{Re}\left\{ \frac{(\phi * f)\,(z)}{(\varphi * f)\,(z)} - 1 \right\} \le 1 - \gamma \quad (z \in \mathcal{U}). \tag{13}$$

Simple calculations give

$$\delta \left| \frac{(\phi * f)\,(z)}{(\varphi * f)\,(z)} - 1 \right| - \operatorname{Re}\left\{ \frac{(\phi * f)\,(z)}{(\varphi * f)\,(z)} - 1 \right\}$$

$$\le (\delta + 1) \left| \frac{(\phi * f)\,(z)}{(\varphi * f)\,(z)} - 1 \right| \le (\delta + 1)\, \frac{\sum_{n=k}^{\infty} (\beta_n - \alpha_n)\,|a_n||z|^{n-p}}{a_p - \sum_{n=k}^{\infty} \alpha_n |a_n||z|^{n-p}}.$$

Now, if (12) holds, then the last expression is bounded by $(1-\gamma)$. Whence $f \in \mathcal{W}(p,k;\phi,\varphi;\gamma,\delta)$. ■

Our next theorem shows that the condition (12) is necessary as well for functions of the form (1), with (4) to belong to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$.

THEOREM 2. *Let f be a function of the form* (1)*, satisfying the argument property* (4)*. Then f belongs to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ if and only if the condition* (12) *holds true.*

Proof. In view of Theorem 1 we need only show that each function f from the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ satisfies the coefficient inequality (12). Let a function f of the form (1), satisfying the argument property (4) belongs to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$. Then by (9),

$$\delta\left|\frac{a_pz^p+\sum\limits_{n=k}^{\infty}\beta_na_nz^n}{a_pz^p+\sum\limits_{n=k}^{\infty}\alpha_na_nz^n}-1\right|<\operatorname{Re}\left\{\frac{a_pz^p+\sum\limits_{n=k}^{\infty}\beta_na_nz^n}{a_pz^p+\sum\limits_{n=k}^{\infty}\alpha_na_nz^n}-\gamma\right\},$$

or equivalently

$$\delta\left|\frac{\sum\limits_{n=k}^{\infty}(\beta_n-\alpha_n)\,a_nz^{n-p}}{a_p+\sum\limits_{n=k}^{\infty}\alpha_na_nz^{n-p}}\right|<\operatorname{Re}\left\{\frac{(1-\gamma)a_p+\sum\limits_{n=k}^{\infty}(\beta_n-\gamma\alpha_n)\,a_nz^{n-p}}{a_p+\sum\limits_{n=k}^{\infty}\alpha_na_nz^{n-p}}\right\}.$$

In view of (4), we set $z=re^{i\eta}$ $(0\le r<1)$ in the above inequality to obtain

$$\frac{\sum\limits_{n=k}^{\infty}\delta\,(\beta_n-\alpha_n)\,|a_n|\,r^{n-p}}{a_p-\sum\limits_{n=k}^{\infty}\alpha_n\,|a_n|\,r^{n-p}}<\frac{(1-\gamma)a_p-\sum\limits_{n=k}^{\infty}(\beta_n-\gamma\alpha_n)\,|a_n|\,r^{n-p}}{a_p-\sum\limits_{n=k}^{\infty}\alpha_n\,|a_n|\,r^{n-p}}.$$

Thus, by (8) we have

$$\sum_{n=k}^{\infty}\left[(\delta+1)\,\beta_n-(\delta+\gamma)\,\alpha_n\right]|a_n|r^{n-p}<(1-\gamma)a_p,$$

which, upon letting $r\to1^-$, readily yields the assertion (12). ■

By applying Theorem 2 we can deduce following result.

THEOREM 3. *Let f be a function of the form (1), satisfying the argument property* (4)*. It belongs to the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta;\eta)$ if and only if it satisfies* (3) *and*

$$\sum_{n=k}^{\infty}\left(pd_n-(1-\gamma)\,n\,|\rho|^{n-p}\right)|a_n|\le1-\gamma, \tag{14}$$

where $\{d_n\}$ is defined by (11).

Proof. For the function f with the normalization (3), we have

$$a_p = 1 + \sum_{n=k}^{\infty} \tfrac{n}{p} \left|a_n\right| \left|\rho\right|^{n-p}. \tag{15}$$

Applying the equality (15) to (12), we obtain the assertion (14). ■

Since the condition (14) is independent of η, Theorem 3 yields the following theorem.

THEOREM 4. *Let f be a function of the form* (1), *satisfying the argument property* (4). *Then f belongs to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;\gamma,\delta)$ if and only if the condition* (14) *holds true.*

By applying Theorem 3 we obtain the following lemma.

LEMMA 1. *Let $\{d_n\}$ be defined by* (11), *$\rho \in \mathcal{U}$, and let us assume, that there exists an integer n_0 $(n_0 \in \mathbb{N}_k = \{k, k+1, \dots\})$ such that*

$$pd_{n_0} - (1-\gamma)\, n_0 \left|\rho\right|^{n_0-p} \le 0. \tag{16}$$

Then the function

$$f_{n_0}(z) = \left(1 + a\frac{n_0}{p}\rho^{n_0-p}\right) z^p - ae^{i(p-n_0)\eta} z^{n_0}$$

belongs to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;\gamma,\delta;\eta)$ for all positive real numbers a. Moreover, for all n $(n \in \mathbb{N}_k)$ such that

$$pd_n - (1-\gamma)\, n \left|\rho\right|^{n-p} > 0, \tag{17}$$

the functions

$$f_n(z) = \left(1 + a\frac{n_0}{p}\rho^{n_0-p} + b\frac{n}{p}z^{n-p}\right) z^p - ae^{i(p-n_0)\eta} z^{n_0} - be^{i(p-n)\eta} z^n,$$

where

$$b = \frac{p\,(1-\gamma) + \left((1-\gamma)\, n_0 \left|\rho\right|^{n_0-p} - pd_{n_0}\right) a}{pd_n - (1-\gamma)\, n \left|\rho\right|^{n-p}},$$

belong to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;\gamma,\delta;\eta)$.

By Lemma 1 and Theorem 3, we have following corollary.

COROLLARY 1. *Let a function f of the form* (1) *belongs to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;\gamma,\delta;\eta)$ and let $\{d_n\}$ be defined by* (11). *Then all of the coefficients a_n for which*

$$pd_n - (1-\gamma)\, n \left|\rho\right|^{n-p} = 0$$

are unbounded. Moreover, if there exists an integer n_0 $(n_0 \in \mathbb{N}_k = \{k, k+1, \dots\})$ such that

$$pd_{n_0} - (1-\gamma)\, n_0 \left|\rho\right|^{n_0-p} < 0,$$

then all of the coefficients of the function f are unbounded. In the remaining cases

$$|a_n| \leqq \frac{p\,(1-\gamma)}{pd_n - (1-\gamma)\,n\,|\rho|^{n-p}}. \tag{18}$$

The result is sharp, the functions f_n of the form

$$f_{n,\eta}(z) = \frac{pd_n z^p - p\,(1-\gamma)\,e^{i(p-n)\eta} z^n}{pd_n - (1-\gamma)\,n\,|\rho|^{n-p}} \quad (z \in \mathcal{U};\ n = k, k+1, \dots) \tag{19}$$

are the extremal functions.

Remark 1. The coefficients estimates for the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;\gamma,\delta)$ are the same as for the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;\gamma,\delta;\eta)$.

By putting $\rho = 0$ in Theorems 3 and 4, and Corollary 1, we have the corollaries listed below.

Corollary 2. *Let f be a function of the form* (1) *satisfying the argument property* (4). *It belongs to the class $\mathcal{TW}_0\,(p,k;\phi,\varphi;\gamma,\delta;\eta)$ if and only if*

$$\sum_{n=k}^{\infty} d_n\,|a_n| \leq 1-\gamma, \tag{20}$$

where $\{d_n\}$ is defined by (11).

Corollary 3. *Let f be a function of the form* (1) *satisfying the argument property* (4). *Then f belongs to the class $\mathcal{TW}_0\,(p,k;\phi,\varphi;\gamma,\delta)$ if and only if the condition* (20) *holds true.*

Corollary 4. *If a function f of the form* (1) *belongs to the class $\mathcal{TW}_0\,(p,k;\phi,\varphi;\gamma,\delta;\eta)$ then*

$$|a_n| \leq \frac{1-\gamma}{d_n} \quad (n = k, k+1, \dots), \tag{21}$$

where d_n is defined by (11). *The result is sharp. The functions $f_{n,\eta}$ of the form*

$$f_{n,\eta}(z) = z^p - \frac{1-\gamma}{d_n} e^{i(p-n)\eta} z^n \quad (z \in \mathcal{U};\ n = k, k+1, \dots) \tag{22}$$

are the extremal functions.

Corollary 5. *If a function f of the form* (1) *belongs to the class $\mathcal{TW}_0\,(p,k;\phi,\varphi;\gamma,\delta)$, then the coefficients estimates* (21) *hold true. The result is sharp. The functions $f_{n,\eta}$ $(\eta \in \mathbb{R})$ of the form* (22) *are the extremal functions.*

3. Growth and distortion theorems

From Theorem 2 we have the following lemma.

Lemma 2. *Let a function f of the form (1) belong to the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta;\eta)$. If the sequence $\{d_n\}$ defined by (11) satisfies the inequality*

$$0 < d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p} \leq d_n - (1-\gamma)\frac{n}{p}|\rho|^{n-p} \quad (n = k, k+1, \dots), \tag{23}$$

then

$$\sum_{n=k}^{\infty} |a_n| \leq \frac{1-\gamma}{d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p}}.$$

Moreover, if

$$0 < \frac{d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p}}{k} \leq \frac{d_n - (1-\gamma)\frac{n}{p}|\rho|^{n-p}}{n} \quad (n = k, k+1, \dots), \tag{24}$$

then

$$\sum_{n=k}^{\infty} n|a_n| \leq \frac{k(1-\gamma)}{d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p}}.$$

Remark 2. The second part of Lemma 2 can be rewritten in terms of σ-neighborhood N_σ defined by

$$N_\sigma = \left\{ f(z) = a_p z^p + \sum_{n=k}^{\infty} a_n z^n \in \mathcal{T}(p,k;\eta) : \quad \sum_{n=k}^{\infty} n|a_n| \leq \sigma \right\}$$

in the following form: *if the sequence $\{d_n\}$ defined by* (11) *satisfies* (24), *then* $\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta;\eta) \subset N_\sigma$. *where*

$$\delta = \frac{k(1-\gamma)}{d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p}}.$$

Theorem 5. *Let a function f belong to the class $\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta;\eta)$ and let $|z| = r < 1$. If the sequence $\{d_n\}$ defined by* (11) *satisfies* (23), *then*

$$pa_p r^p - \frac{1-\gamma}{d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p}} r^k \leq |f(z)| \leq \frac{d_k r^p + (1-\gamma) r^k}{d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p}}, \tag{25}$$

Moreover, if (24) *holds, then*

$$\phi'(r) \leq |f'(z)| \leq \frac{d_k r^p + k(1-\gamma) r^{k-1}}{d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p}}. \tag{26}$$

where

$$\phi(r) := \begin{cases} r^p & (r \leqq \rho) \\ \frac{d_k r^p - (1-\gamma) r^k}{d_k - (1-\gamma)\frac{k}{p}|\rho|^{k-p}} & (r > \rho). \end{cases} \tag{27}$$

The result is sharp, with the extremal functions $f_{k,\eta}$ of the form (22) *and* $f(z) = z$ $(z \in \mathcal{U})$.

Proof. Suppose that the function f of the form (1) belongs to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;\gamma,\delta;\eta)$. By Lemma 2 we have

$$\begin{aligned}|f'(z)| &= \left|pa_p z^{p-1} + \sum_{n=k}^{\infty} na_n z^{n-1}\right| \leq r^{p-1}\left(pa_p + \sum_{n=k}^{\infty} n\,|a_n|\,r^{n-p}\right)\\ &\leq r^{p-1}\left(p + \sum_{n=k}^{\infty} n\,|a_n|\,|\rho|^{n-p} + \sum_{n=k}^{\infty} n\,|a_n|\,r^{n-p}\right)\\ &\leq r^{p-1}\left(p + (|\rho|^{k-p} + r^{k-p})\sum_{n=k}^{\infty} n\,|a_n|\right) \leq \frac{d_k r^{p-1} + k\,(1-\gamma)\,r^k}{d_k - (1-\gamma)\,\frac{k}{p}\,|\rho|^{k-p}},\end{aligned}$$

and

$$|f'(z)| \geq r^{p-1}\left(pa_p - \sum_{n=k}^{\infty} n\,|a_n|\,r^{n-p}\right) = r^{p-1}\left(p + \sum_{n=k}^{\infty}(|\rho|^{n-p} - r^{n-p})n\,|a_n|\right). \tag{28}$$

If $r \leq \rho$, then we obtain $|f'(z)| \geq r^{p-1}$. If $r > \rho$, then the sequence $\{(\rho^{n-p} - r^{n-p})\}$ is decreasing and negative. Thus, by (28), we obtain

$$|f'(z)| \geq r^{p-1}\left(p - (r^{k-p} - |\rho|^{k-p})\sum_{n=2}^{\infty} a_n\right) \geq \frac{pd_k r^p - k\,(1-\gamma)\,r^k}{d_k - (1-\gamma)\,\frac{k}{p}\,|\rho|^{k-p}},$$

and we have the assertion (26). Making use of Lemma 2, in conjunction with (15), we readily obtain the assertion (25) of Theorem 2. ∎

Theorem 5 implies the following results.

COROLLARY 6. *Let a function f belong to the class $\mathcal{TW}_\rho\,(p,k;\phi,\varphi;\gamma,\delta)$. If the sequence $\{d_n\}$ defined by* (11) *satisfies* (23)*, then the assertions* (25) *hold true. Moreover, if we assume* (24)*, then the assertions* (26) *hold true. The result is sharp, with the extremal functions $f_{k,\eta}$ $(\eta \in \mathbb{R})$ of the form* (22) *and $f(z) = z$ $(z \in \mathcal{U})$.*

COROLLARY 7. *Let a function f belong to the class $\mathcal{TW}_0\,(p,k;\phi,\varphi;\gamma,\delta;\eta)$ and let the sequence $\{d_n\}$ be defined by* (11)*. If*

$$d_k \leq d_n \quad (n = k, k+1, \dots)\,, \tag{29}$$

then

$$r^p - \frac{1-\gamma}{d_k} r^k \leq |f(z)| \leq r^p + \frac{1-\gamma}{d_k} r^k \quad (|z| = r < 1)\,. \tag{30}$$

Moreover, if

$$nd_k \leq kd_n \quad (n = k, k+1, \dots)\,, \tag{31}$$

then

$$pr^{p-1} - \frac{k\,(1-\gamma)}{d_k} r^{k-1} \leq |f'(z)| \leq pr^{p-1} + \frac{k\,(1-\gamma)}{d_k} r^{k-1} \quad (|z| = r < 1)\,. \tag{32}$$

The result is sharp, with the extremal function $f_{k,\eta}$ of the form (22)*.*

Corollary 8. *Let a function f belong to the class $\mathcal{TW}_0(p,k;\phi,\varphi;\gamma,\delta)$. If the sequence $\{d_n\}$ defined by* (11) *satisfies* (29)*, then the assertions* (30) *hold true. Moreover, if we assume* (31)*, then the assertions* (32) *hold true. The result is sharp, with the extremal functions $f_{k,\eta}$ $(\eta \in \mathbb{R})$ of the form* (22) *and $f(z) = z$ $(z \in \mathcal{U})$.*

4. The radii of convexity and starlikeness

Theorem 6. *The radius of starlikeness of order α for the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ is given by*

$$R_\alpha^*(\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)) = \inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n-p}}, \tag{33}$$

where d_n is defined by (11)*. The functions $f_{n,\eta}$ of the form*

$$f_{n,\eta}(z) = a_p\left(z^p - \frac{1-\gamma}{d_n}e^{i(p-n)\eta}z^n\right) \quad (z\in\mathcal{U};\ \ n=k,k+1,\dots;\ \ a_p>0) \tag{34}$$

are the extremal functions.

Proof. A function $f \in \mathcal{T}(p,k;\eta)$ of the form (1) is starlike of order α in the disk $\mathcal{U}(r)$, $0<r\leq 1$, if and only if it satisfies the condition (7). Since

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{\sum_{n=k}^{\infty}(n-p)a_nz^n}{a_pz^p + \sum_{n=k}^{\infty}a_nz^n}\right| \leq \frac{\sum_{n=k}^{\infty}(n-p)\,|a_n|\,|z|^{n-p}}{a_p - \sum_{n=k}^{\infty}|a_n|\,|z|^{n-p}},$$

putting $|z|=r$ the condition (7) is true if

$$\sum_{n=k}^{\infty}\frac{n-\alpha}{p-\alpha}\,|a_n|\,r^{n-p}\ \leq a_p. \tag{35}$$

By Theorem 2, we have

$$\sum_{n=k}^{\infty}\frac{d_n}{1-\gamma}\,|a_n| \leq a_p, \tag{36}$$

Thus, the condition (35) holds if

$$\frac{n-\alpha}{p-\alpha}r^{n-p} \leq \frac{d_n}{1-\gamma} \quad (n=k,k+1,\dots),$$

that is, if

$$r \leq \left(\frac{(p-\alpha)\,d_n}{(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n-p}} \qquad (n=k,k+1,\dots). \tag{37}$$

It follows that each function $f \in \mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ is starlike of order α in the disk $\mathcal{U}(r)$, where

$$r = \inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n-p}}$$

The functions $f_{n,\eta}$ of the form (34) realize equality in (36), and the radius r can not be larger. Thus we have (33). ■

The following result may be proved in much the same way as Theorem 6.

THEOREM 7. *The radius of convexity of order α for the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ is given by*

$$R_{\alpha}^{c}\left(\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)\right)=\inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{n\,(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n-p}},$$

where d_n is defined by (11). *The functions $f_{n,\eta}$ of the form* (34) *are the extremal functions.*

It is clear that for

$$a_p=\frac{d_n}{d_n-(1-\gamma)\,\frac{k}{p}\,|\rho|^{n-p}}>0$$

the extremal functions $f_{n,\eta}$ of the form (34) belong to the class $\mathcal{TW}_{\rho}(p,k;\phi,\varphi;\gamma,\delta;\eta)$. Moreover, we have

$$\mathcal{TW}_{\rho}(p,k;\phi,\varphi;\gamma,\delta;\eta)\subset\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta).$$

Thus, by Theorems 6 and 7 we have the following results.

COROLLARY 9. *Let the sequence $\left\{d_n-(1-\gamma)\,\frac{n}{p}\,|\rho|^{n-p}\right\}$, where d_n is defined by* (11), *be positive. Then*

$$R_{\alpha}^{*}\left(\mathcal{TW}_{\rho}(p,k;\phi,\varphi;\gamma,\delta;\eta)\right)=\inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n-p}}.$$

COROLLARY 10. *Let the sequence $\left\{d_n-(1-\gamma)\,\frac{n}{p}\,|\rho|^{n-p}\right\}$, where d_n is defined by* (11), *be positive. Then*

$$R_{\alpha}^{c}\left(\mathcal{TW}_{\rho}(p,k;\phi,\varphi;\gamma,\delta;\eta)\right)=\inf_{n\geq k}\left(\frac{(p-\alpha)\,d_n}{n\,(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n-p}}.$$

5. Subordination results

Before stating and proving our subordination theorems for the classes $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ and $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta)$ we need the following definition and lemma.

DEFINITION 1. A sequence $\{b_n\}$ of complex numbers is said to be a subordinating factor sequence if for each function f of the form (1) from the class $\mathcal{S}^c$ we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z)\quad (a_1=1). \tag{38}$$

Lemma 3. [24] *The sequence* $\{b_n\}$ *is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \quad (z \in \mathcal{U})\,. \tag{39}$$

Theorem 8. *Let the sequence* $\{d_n\}$ *defined by* (11) *satisfy the inequality* (23). $g \in \mathcal{S}^c$ *and* $f \in \mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$, *then*

$$\varepsilon \frac{f(z)}{z^{p-1}} * g(z) \prec g(z) \tag{40}$$

and

$$\operatorname{Re} \frac{f(z)}{z^{p-1}} > -\frac{1}{2\varepsilon} \quad (z \in \mathcal{U})\,, \tag{41}$$

where

$$\varepsilon = \frac{d_k}{2a_p\,(1-\gamma+d_k)}. \tag{42}$$

If p *and* $(k-p)$ *are odd, and* $\eta = 0$, *then the constant factor* ε *cannot be replaced by a larger number.*

Proof. Let a function f of the form (1) belong to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ and suppose that a function g of the form

$$g(z) = \sum_{n=1}^{\infty} c_n z^n \quad (c_1 = 1;\ z \in \mathcal{U})$$

belongs to the class $\mathcal{S}^c$. Then

$$\varepsilon \frac{f(z)}{z^{p-1}} * g(z) = \sum_{n=1}^{\infty} b_n c_n z^n \quad (z \in \mathcal{U})\,,$$

where

$$b_n = \begin{cases} \varepsilon a_p & \text{if } n = 1 \\ 0 & \text{if } 2 \le n \le k-p \\ \varepsilon a_{n+p-1} & \text{if } n > k-p. \end{cases}$$

Thus, by Definition 1 the subordination result (40) holds true if $\{b_n\}$ is the subordinating factor sequence. By (23) we have

$$\begin{aligned}\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\} &= \operatorname{Re}\left\{1+2\varepsilon a_p z + \sum_{n=k}^{\infty} \frac{d_k}{1-\gamma+d_k} a_n z^{n-p}\right\} \\ &\ge 1-2\varepsilon r - \frac{r}{(1-\gamma+d_k)\,a_p}\sum_{n=k}^{\infty} d_n\,|a_n| \quad (|z| = r < 1)\,,\end{aligned}$$

Thus, by using Theorem 2 we obtain

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\} \ge 1 - \frac{d_k}{1-\gamma+d_k} r - \frac{1-\gamma}{1-\gamma+d_k} r > 0.$$

This evidently proves the inequality (39) and hence the subordination result (40). The inequality (41) follows from (40) by taking

$$g(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \quad (z \in \mathcal{U}) .$$

Next, we observe that the function $f_{k,\eta}$ of the form (34) belongs to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$. If p and $(k-p)$ are odd, and $\eta = 0$, then

$$\left.\frac{f_{k,\eta}(z)}{z^{p-1}}\right|_{z=-1} = -\frac{1}{2\varepsilon},$$

and the constant (42) can not be replaced by any larger one. ∎

Directly from Theorem 8 we obtain

Theorem 9. *Let the sequence* $\{d_n\}$ *defined by* (11) *satisfy the inequality* (23). *If* $g \in \mathcal{S}^c$ *and* $f \in \mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta)$, *then conditions* (40) *and* (41) *hold true. If* p *and* $(k-p)$ *are odd, then the constant factor* ε *in* (40) *cannot be replaced by a larger number.*

Remark 3. By using (15) in Theorems 8 and 9 we obtain the results related to the classes $\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta)$ and $\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta;\eta)$. Moreover, by putting $\rho = 0$ we have the following corollary.

Corollary 11. *Let the sequence* $\{d_n\}$ *defined by* (11) *satisfy the inequality* (23). *If* $g \in \mathcal{S}^c$ *and* $f \in \mathcal{TW}_0(p,k;\phi,\varphi;\gamma,\delta;\eta)$, *then conditions* (40) *and* (41), *where*

$$\varepsilon = \frac{d_k}{2(1-\gamma+d_k)} \tag{43}$$

hold true. If p *and* $(k-p)$ *are odd, and* $\eta = 0$, *then the constant factor* ε *in* (43) *cannot be replaced by a larger number.*

6. Integral mean inequalities

Due to Littlewood [7] we obtain integral means inequalities for the functions from the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$.

Lemma 4. [7] *Let* $f, g \in \mathcal{A}$. *If* $f \prec g$, *then*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \le \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \quad (0 < r < 1,\ \eta > 0) . \tag{44}$$

Applying Lemma 4 and Theorem 2 we prove the following result.

Theorem 10. *Let the sequence* $\{d_n\}$, *defined by* (11), *satisfy the inequality* (23). *If* $f \in \mathcal{TW}(p,p+1;\phi,\varphi;\gamma,\delta;\eta)$, *then*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\lambda} d\theta \le \int_0^{2\pi} \left|f_{p+1,\eta}(re^{i\theta})\right|^{\lambda} d\theta \quad \left(0 < r < 1,\ \lambda > 0;\ z = re^{i\theta}\right), \tag{45}$$

where $f_{p+1,\eta}(z)$ *is defined by* (34).

Proof. For function f of the form (1), the inequality (45) is equivalent to the following

$$\int_0^{2\pi} \left| a_p + \sum_{n=p+1}^{\infty} a_n z^{n-p} \right|^\lambda d\theta \leq \int_0^{2\pi} \left| a_p - \frac{1-\gamma}{d_{p+1}} e^{-i\eta} z \right|^\lambda d\theta \quad \left(z = re^{i\theta}\right).$$

By Lemma 4, it suffices to show that

$$\sum_{n=p+1}^{\infty} a_n z^{n-p} \prec -\frac{1-\gamma}{d_{p+1}} e^{-i\eta} z. \tag{46}$$

Setting

$$w(z) = -\sum_{n=p+1}^{\infty} \frac{d_{p+1} e^{i\eta}}{1-\gamma} a_n \; z^{n-p} \quad (z \in \mathcal{U})$$

and using (23) and Theorem 2 we obtain

$$|w(z)| = \left| \sum_{n=p+1}^{\infty} \frac{d_{p+1}}{1-\gamma} a_n \; z^{n-p} \right| \leq |z| \sum_{n=p+1}^{\infty} \frac{d_n}{1-\gamma} |a_n| \leq |z| \quad (z \in \mathcal{U}).$$

Since

$$\sum_{n=p+1}^{\infty} a_n z^{n-p} = -\frac{1-\gamma}{d_{p+1}} e^{-i\eta} w(z) \quad (z \in \mathcal{U}),$$

by definition of subordination we have (46) and this completes the proof. ∎

We can rewrite Theorem 10 in the following form.

THEOREM 11. *Let the sequence* $\{d_n\}$ *defined by* (11) *satisfy the inequality* (23). *If a function* f *of the form* (1) *with* (4) *belongs to the class* $\mathcal{TW}(p, p+1; \phi, \varphi; \gamma, \delta)$, *then the integral means inequality* (45) *holds true.*

By using (15) in Theorem 10 we have the following corollary.

COROLLARY 12. *Let the sequence* $\{d_n\}$ *defined by* (11) *satisfy the inequality* (23). *If* $f \in \mathcal{TW}_\rho(p, p+1; \phi, \varphi; \gamma, \delta; \eta)$, *then*

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^\lambda d\theta \leq \int_0^{2\pi} \left| f_{p+1,\eta}(re^{i\theta}) \right|^\lambda d\theta \quad \left(0 < r < 1,\ \lambda > 0;\ z = re^{i\theta}\right).$$

where $f_{p+1,\eta}(z)$ *is defined by* (7).

7. Partial sums

Let f be a function of the form (1). Due to Silverman [14] and Silvia [16] we investigate the partial sums f_m of the function f defined by

$$f_{k-1}(z) = a_p z^p; \text{ and } f_m(z) = a_p z^p + \sum_{n=k}^{m} a_n z^n, \ (m = k, k+1, \dots) \tag{47}$$

In this section we consider partial sums of functions in the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$ and obtain sharp lower bounds for the ratios of real part of f to f_m and f' to f'_m.

THEOREM 12. *Let the sequence* $\{d_n\}$, *defined by* (11), *be increasing and let*

$$d_k \geq 1-\gamma. \tag{48}$$

If a function f *of the form* (1) *belongs to the class* $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1-\frac{1-\gamma}{d_{m+1}} \quad (z\in\mathcal{U},\ m=k-1,k,\dots) \tag{49}$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{d_{m+1}}{1-\gamma+d_{m+1}} \quad (z\in\mathcal{U},\ m=k-1,k,\dots). \tag{50}$$

The bounds are sharp, with the extremal functions $f_{m+1,\eta}$ *defined by* (34).

Proof. Since

$$\frac{d_{n+1}}{1-\gamma} > \frac{d_n}{1-\gamma} > 1 \quad (n=k,k+1,\dots),$$

by Theorem 1 we have

$$\sum_{n=k}^{m}|a_n| + \frac{d_{m+1}}{1-\gamma}\sum_{n=m+1}^{\infty}|a_n| \leq \sum_{n=k}^{\infty}\frac{d_n}{1-\gamma}|a_n| \leq a_p. \tag{51}$$

Let

$$g(z)=\frac{d_{m+1}}{1-\gamma}\left\{\frac{f(z)}{f_m(z)}-\left(1-\frac{1-\gamma}{d_{m+1}}\right)\right\} = 1+\frac{\frac{d_{m+1}}{1-\gamma}\sum\limits_{n=m+1}^{\infty}a_n z^{n-p}}{a_p+\sum\limits_{n=k}^{m}a_n z^{n-p}} \quad (z\in\mathcal{U}). \tag{52}$$

Applying (51), we find that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{d_{m+1}}{1-\gamma}\sum\limits_{n=m+1}^{\infty}|a_n|}{2a_p-2\sum\limits_{n=2}^{n}|a_n|-\frac{d_{m+1}}{1-\gamma}\sum\limits_{n=m+1}^{\infty}|a_n|} \leq 1 \quad (z\in\mathcal{U}).$$

Thus we have $\operatorname{Re} g(z)\geq 0 \quad (z\in U)$, which by (52) readily yields the assertion (49) of Theorem 12. Similarly, if we take

$$h(z)=\left(1+\frac{d_{m+1}}{1-\gamma}\right)\left\{\frac{f_m(z)}{f(z)}-\frac{d_{m+1}}{1-\gamma+d_{m+1}}\right\} \quad (z\in\mathcal{U})$$

and making use of (51), we can deduce that

$$\left|\frac{h(z)-1}{h(z)+1}\right| \leq \frac{\left(1+\frac{d_{m+1}}{1-\gamma}\right)\sum\limits_{n=m+1}^{\infty}|a_n|}{2a_p-2\sum\limits_{n=k}^{m}|a_n|-\left(\frac{d_{m+1}}{1-\gamma}-1\right)\sum\limits_{n=m+1}^{\infty}|a_n|} \leq 1 \quad (z\in\mathcal{U}),$$

which leads us immediately to the assertion (50) of Theorem 12. In order to see that the function $f_{m+1,\eta}$ of the form (22) gives the results sharp, we observe that

$$\frac{f_{m+1,\eta}(z)}{\left(f_{m+1,\eta}\right)_m(z)} = 1 - \frac{1-\gamma}{d_{m+1}} \quad \left(z = e^{i\eta}\right),$$
$$\frac{\left(f_{m+1,\eta}\right)_m(z)}{f_{m+1,\eta}(z)} = \frac{d_{m+1}}{1-\gamma+d_{m+1}} \quad \left(z = e^{i\left(\eta+\frac{\pi}{m-p+1}\right)}\right).$$

This completes the proof. ■

Theorem 13. *Let the sequence $\{d_n\}$, defined by* (11), *be increasing and let $d_k > (m+1)(1-\gamma)$. If a function f of the form* (1) *belongs to the class $\mathcal{TW}(p,k;\phi,\varphi;\gamma,\delta;\eta)$, then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{(m+1)(1-\gamma)}{d_{m+1}} \quad (z \in \mathcal{U},\ m = k-1, k, \dots),$$
$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{d_{m+1}}{(m+1)(1-\gamma)+d_{m+1}} \quad (z \in \mathcal{U},\ m = k-1, k, \dots).$$

The bounds are sharp, with the extremal functions $f_{m+1,\eta}$ defined by (34).

Proof. By setting

$$g(z) = \frac{d_{m+1}}{1-\gamma}\left\{\frac{f'(z)}{f_m'(z)} - \left(1 - \frac{(m+1)(1-\gamma)}{d_{m+1}}\right)\right\} \quad (z \in \mathcal{U}),$$

and

$$h(z) = \left(m+1+\frac{d_{m+1}}{1-\gamma}\right)\left\{\frac{f_m'(z)}{f'(z)} - \frac{d_{m+1}}{(m+1)(1-\gamma)+d_{m+1}}\right\} \quad (z \in \mathcal{U}),$$

the proof is analogous to that of Theorem 12, and we omit the details. ■

Remark 4. By using (15) in Theorems 12 and 13 we obtain the results related to the classes $\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta)$ and $\mathcal{TW}_\rho(p,k;\phi,\varphi;\gamma,\delta;\eta)$.

Remark 5. We conclude this paper by observing that, in view of the definitions of investigated classes which is expressed in terms of the convolution of the functions in (10), involving arbitrary sequences, our main results can lead to several additional new results. In fact, by appropriately selecting these arbitrary sequences, the results presented in this paper would find further applications for the class of analytic functions which would incorporate linear operators. Some of these results were obtained in earlier works, see for example [1, 2, 3, 20, 23].

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(received 14.04.2010)

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