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ON CERTAIN SEPARABLE AND CONNECTED REFINEMENTS OF THE EUCLIDEAN TOPOLOGY

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Abstract. Write c for the cardinality of the continuum and let η be the Euclidean topology on $\mathbb{R}$. Let Σ be the family of all σ-ideals $\mathcal{I}$ on $\mathbb{R}$ such that $\bigcup\mathcal{I}$ is dense and $\mathbb{Q}\cap\bigcup\mathcal{I}=\emptyset$. Then for each $\mathcal{I}\in\Sigma$ the family $\eta/\mathcal{I}$ of all sets $X\setminus Y$ with $X\in\eta$ and $Y\in\mathcal{I}$ is a topology on $\mathbb{R}$. Such a refinement of η always preserves separability and connectedness, but destroys metrizability (and first countability almost always) and makes the space totally pathwise disconnected. Nevertheless, the separable Hausdorff space $(\mathbb{R},\eta/\mathcal{I})$ still has the two metric properties that every point is reachable by a sequence of points within any fixed countable dense set and that (even in the absence of first countability) sequential continuity is strong enough to entail continuity. In detail we investigate further main properties in the four most interesting cases when the σ-ideal $\mathcal{I}$ consists of either all countable sets or all null sets or all meager sets or all sets contained in $\mathbb{R}\setminus\mathbb{Q}$. Finally we track down a subfamily Σ_1 of Σ with cardinality 2^{2^c} such that $(\mathbb{R},\eta/\mathcal{I})$ and $(\mathbb{R},\eta/\mathcal{J})$ are never homeomorphic for distinct $\mathcal{I},\mathcal{J}$ in Σ_1.

1. Introduction

As usual, c is the cardinality of $\mathbb{R}$ and 2^κ is the cardinality of the power set of a set of cardinality κ. (In particular, $c=2^{\aleph_0}$. Naturally, $2^\kappa>\kappa$.) Let $\eta:=\{X\subset\mathbb{R}\mid\forall x\in X\exists a,b\in\mathbb{R}:x\in]a,b[\subset X\}$ be the Euclidean topology and $\delta:=\{X\mid X\subset\mathbb{R}\}$ be the discrete topology on the real number line. In a very natural way we will construct five topologies $\rho,\vartheta,\lambda,\mu,\gamma$ strictly finer than η and strictly coarser than δ and check all properties given by the table below. In the table, **Y** means *Yes* and **N** means *No* and the set $\mathcal{X}$ is either $\mathbb{R}$ or $[a,b]\subset\mathbb{R}$ with $a<b$. If $\mathcal{X}=\mathbb{R}$ then replace **S** with **N**, if $\mathcal{X}=[a,b]$ then replace **S** with **Y**.

Beside the three special topologies ϑ,λ,μ we will construct 2^{2^c} refinements τ of η such that all spaces $(\mathbb{R},\tau)$ are mutually non-homeomorphic and have the property of being *separable* and *connected* and *totally pathwise disconnected* and *not first countable*. There is not one Hausdorff space with this property included in the famous catalogue [9] of 143 topological spaces.

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Further, all the separable and not first countable Hausdorff spaces $(\mathbb{R}, \tau)$ have the following *metric approximation property* and *metric continuity property.*

(MAP) *If D is a countable dense subset of the space, then every point in the space is the limit of a convergent sequence of points in D.*

(MCP) *If f is a function from the space into an arbitrary regular space, then f is continuous if and only if f is sequentially continuous.*

$\mathcal{X}$ *with the induced topology of*	η	ρ	ϑ	λ	μ	γ	δ
(countably) compact	S	N	N	N	N	N	N
locally compact	Y	N	N	N	N	N	Y
pseudocompact	S	S	S	S	S	S	N
σ-compact	Y	N	N	N	N	N	N
$\aleph_1$-compact	Y	Y	Y	N	N	N	N
Lindelöf	Y	Y	Y	N	N	N	N
connected	Y	Y	Y	Y	Y	Y	N
locally connected	Y	N	N	N	N	N	Y
path connected	Y	N	N	N	N	N	N
locally path connected	Y	N	N	N	N	N	Y
totally pathwise disconnected	N	Y	Y	Y	Y	Y	Y
first countable	Y	Y	N	N	N	Y	Y
second countable	Y	Y	N	N	N	N	N
separable	Y	Y	Y	Y	Y	Y	N
$\mathrm{T}_0 \wedge \mathrm{T}_1 \wedge \mathrm{T}_2 \wedge \mathrm{T}_{2\frac{1}{2}}$	Y	Y	Y	Y	Y	Y	Y
$\mathrm{T}_3 \vee \mathrm{T}_{3\frac{1}{2}} \vee \mathrm{T}_4 \vee \mathrm{T}_5$	Y	N	N	N	N	N	Y
first category	N	N	N	Y	N	Y	N
exactly c open sets	Y	Y	Y	N	N	N	N
exactly 2^c open sets	N	N	N	Y	Y	Y	Y

2. Four refinements of the Euclidean topology

For abbreviation, if τ is a topology on X, then we say that $Y \subset X$ is τ-closed when Y is closed in the space (X, τ). Similarly we speak of τ-open sets, τ-compact sets, τ-connected sets, τ-dense sets etc.

A σ-ideal $\mathcal{I}$ on a non-empty set X is any nonempty family of subsets of X which is closed under countable unions and where always $K \in \mathcal{I}$ when $K \subset L$ and $L \in \mathcal{I}$. (In particular, $\emptyset \in \mathcal{I}$ and $\mathcal{I}$ is closed under arbitrary intersections.) Note that for the sake of simplicity we follow [2] in not ruling out $X \in \mathcal{I}$. In other words, we also regard the power set $\mathcal{P}(X)$ of X to be a σ-ideal on X.

Theorem 1. *Let (X, τ) be a topological space and let $\mathcal{I}$ be a σ-ideal on the set X. If (X, τ) is second countable, then the family*

$$\tau/\mathcal{I} := \{U \setminus A \mid U \in \tau \wedge A \in \mathcal{I}\}$$

is a topology on X which is (not necessarily strictly) finer than τ.

Theorem 1 is an immediate consequence of [2, Theorems 4.2 and 4.4]. But alternatively, an elementary proof of Theorem 1 is a rather easy and beautiful exercise. It should be mentioned that if (X,τ) is not assumed to be second countable, then the statement in Theorem 1 is not necessarily true. (For a nice counterexample let (X,τ) be the first countable Hausdorff space of all countable ordinals equipped with the order topology and let $\mathcal{I}$ be the family of all countable subsets of X.)

Let $\mathcal{I}_c$ be the family of all *countable* sets of irrational numbers. Let $\mathcal{I}_n$ be the family of all *Lebesgue null* sets of irrational numbers and let $\mathcal{I}_m$ be the family of all subsets of $\mathbb{R}\setminus\mathbb{Q}$ which are *meager* in the Euclidean space $(\mathbb{R},\eta)$. (Of course, the three families are σ-ideals on $\mathbb{R}\setminus\mathbb{Q}$.) Referring to Theorem 1 we define the following topologies on $\mathbb{R}$.

$$\vartheta := \eta/\mathcal{I}_c = \big\{U\setminus A \mid U\subset\mathbb{R}\wedge U\eta\text{-open }\ \wedge A\subset\mathbb{R}\setminus\mathbb{Q}\wedge A\text{ countable }\big\}$$
$$\lambda := \eta/\mathcal{I}_n = \big\{U\setminus A \mid U\subset\mathbb{R}\wedge U\eta\text{-open }\ \wedge A\subset\mathbb{R}\setminus\mathbb{Q}\wedge A\text{ null }\big\}$$
$$\mu := \eta/\mathcal{I}_m = \big\{U\setminus A \mid U\subset\mathbb{R}\wedge U\eta\text{-open }\ \wedge A\subset\mathbb{R}\setminus\mathbb{Q}\wedge A\ \eta\text{-meager }\big\}$$

Since a *countable* set is always *null* and *meager*, both topologies λ and μ are finer than the topology ϑ. The topology ϑ is strictly finer than the Euclidean topology η because $\vartheta\supset\eta$ and, e.g. , the set $\{\pi/n \mid 0<n\in\mathbb{N}\}$ is ϑ-closed but not η-closed. Although ϑ is strictly finer than η, the following theorem shows that ϑ is not larger than η from the set theoretic point of view. Further, the theorem implies in a rather harsh way that both topologies λ and μ are strictly finer than ϑ.

Theorem 2. *The family ϑ is equipollent with η and hence ϑ has cardinality c. Both families λ and μ have cardinality 2^c.*

Proof. The statement on λ and μ is true because if $\mathbb{D}$ is the Cantor set, which has cardinality c and is both null and η-meager, then each of the 2^c subsets of $\mathbb{D}\setminus\mathbb{Q}$ is λ-closed and μ-closed. Since the family η has cardinality c, this is also true for the family ϑ because a set of cardinality c contains precisely c countable subsets. ∎

The reason for the special role of the set $\mathbb{Q}$ in the definitions of the topologies ϑ and λ and μ is so that $\mathbb{Q}$ is obviously kept as a *dense* subset of $\mathbb{R}$ and hence the spaces $(\mathbb{R},\vartheta)$ and $(\mathbb{R},\lambda)$ and $(\mathbb{R},\mu)$ are *separable Hausdorff spaces.* We choose the set $\mathbb{Q}$ merely for the sake of simply speaking. (Of course, our story would be the same if $\mathbb{Q}$ were replaced with any countable η-dense subset of $\mathbb{R}$.) In this connection it should be mentioned that, although considering ideals on a basic space X in order to construct refinements of its topology is an old idea [4] which has entailed some occasional investigations [3],[5],[7],[8], our approach of considering only σ-ideals on the basic space $X=\mathbb{R}$ which are also σ-ideals on the complement of a fixed countable dense subset of $\mathbb{R}$ leads to several new results.

Although the three spaces $(\mathbb{R},\vartheta)$ and $(\mathbb{R},\lambda)$ and $(\mathbb{R},\mu)$ take their stand between the Euclidean space $(\mathbb{R},\eta)$ and the discrete space $(\mathbb{R},\delta)$, which both are *metric spaces*, the following theorem shows that the three spaces $(\mathbb{R},\vartheta)$ and $(\mathbb{R},\lambda)$ and $(\mathbb{R},\mu)$ are *not first countable* whence they are *not metrizable.*

Theorem 3. *Every local ϑ-basis of ϑ-open sets at an arbitrary point has cardinality c. Every local λ-basis and every local μ-basis is uncountable.*

Proof. Fix $a \in \mathbb{R}$ and let $\mathcal{N}$ be a collection of ϑ-open ϑ-neighborhoods of a and write $\mathcal{N} = \{E_\lambda \setminus A_\lambda \mid \lambda \in \Lambda\}$ with η-open η-neighborhoods E_λ of a and countable sets A_λ of irrationals $\neq a$. Suppose that the cardinality of Λ is less than c. (By Theorem 2, the cardinality cannot be greater than c when the sets $E_\lambda \setminus A_\lambda (\lambda \in \Lambda)$ are distinct.) Then $S = \bigcup\{A_\lambda \mid \lambda \in \Lambda\}$ is a set of irrationals $\neq a$ with cardinality less than c as well. Thus for every $n \in \mathbb{N}$ we can choose a number a_n in $]a, a+2^{-n}] \setminus (S \cup \mathbb{Q})$. Naturally, every set $E_\lambda \setminus A_\lambda$ contains a certain number a_n. Therefore $\mathbb{R} \setminus \{a_n \mid n \in \mathbb{N}\}$ is a ϑ-open ϑ-neighborhood of a which does not contain any set from the collection $\mathcal{N}$. Thus $\mathcal{N}$ cannot be a local ϑ-basis at a. By an analogous argument, a local λ-basis and a local μ-basis can never be countable. ■

Remark. In standard set theory it is unprovable that a nonempty open interval cannot be covered by less than c null or meager sets respectively (cf. [1]). Therefore we cannot prove that the cardinality of a local λ-basis or a local μ-basis is not smaller than c. (Note that, in view of Theorem 2, there are local λ-bases of λ-open sets and local μ-bases of μ-open sets with cardinality 2^c. Of course, in the topology η and hence also in the topology ϑ there are local bases with cardinality 2^c provided that the definition of a local base does not exclude non-open sets.)

By Theorem 2, the space $(\mathbb{R}, \vartheta)$ cannot be homeomorphic to $(\mathbb{R}, \lambda)$ or $(\mathbb{R}, \mu)$. In view of the famous duality between *measure* and *category* (cf. [6]) the question arises whether the two spaces $(\mathbb{R}, \lambda)$ and $(\mathbb{R}, \mu)$ are homeomorphic. The following theorem shows that this is not the case.

Theorem 4. *The space* $(\mathbb{R}, \lambda)$ *is of first category. The spaces* $(\mathbb{R}, \vartheta)$ *and* $(\mathbb{R}, \mu)$ *are of second category. (In other words,* $\mathbb{R}$ *is* λ*-meager but not* ϑ*-meager and not* μ*-meager.)*

Proof. Recall that a subset M of a space X is meager if and only if it equals a countable union of nowhere dense subsets of X. Note further that $N \subset X$ is nowhere dense if and only if every nonempty open subset of X contains a nonempty open set disjoint from N.

In order to show that $\mathbb{R}$ is λ-meager take a bijective function $i \mapsto r_i$ from $\mathbb{N}$ onto $\mathbb{Q}$ and define η-open and hence λ-open sets

$$U_j := \bigcup \left\{ \left] r_i - \tfrac{1}{2^{i+j}}, r_i + \tfrac{1}{2^{i+j}} \right[\mid i \in \mathbb{N} \right\} \qquad (j \in \mathbb{N}).$$

Since $\mathbb{Q} \subset U_j$, every set U_j is λ-dense and hence the sets $\mathbb{R} \setminus U_j$ are all nowhere λ-dense. Naturally, $L = \bigcap\{U_j \mid j \in \mathbb{N}\}$ is a null set. Thus $L \setminus \mathbb{Q}$ is a null set and hence λ-closed. Therefore the set $L \setminus \mathbb{Q}$ is nowhere λ-dense because it cannot contain a nonempty λ-open set since such a set must have a positive Lebesgue measure. Thus $\mathbb{R}$ equals

$$\bigcup \left\{ \{x\} \mid x \in L \cap \mathbb{Q} \right\} \cup (L \setminus \mathbb{Q}) \cup \bigcup \left\{ \mathbb{R} \setminus U_j \mid j \in \mathbb{N} \right\}$$

which is a countable union of nowhere λ-dense sets.

In order to show that $\mathbb{R}$ is not μ-meager suppose indirectly that $\mathcal{F}$ is a countable family of nowhere μ-dense sets which covers $\mathbb{R}$. Now for a fixed set $N \in \mathcal{F}$ and arbitrary $r, s \in \mathbb{Q}$ with $r < s$ there exist an η-meager set $A_{rs} \subset \mathbb{R} \setminus \mathbb{Q}$ and numbers

$r', s' \in \mathbb{Q}$ so that both $\emptyset \neq]r', s'[\setminus A_{rs} \subset]r, s[$ and $]r', s'[\cap (N \setminus A_{rs}) = \emptyset$. We must have $r \leq r' < s' \leq s$ since an η-meager set cannot contain an interval $I \not\subset \{a\}$. Now put $A_N := \bigcup\{A_{rs} \mid r, s \in \mathbb{Q} \wedge r < s\}$. Then for every $N \in \mathcal{F}$ the set $N \setminus A_N$ is nowhere η-dense because every interval $]r, s[\neq \emptyset$ contains an interval $]r', s'[\neq \emptyset$ disjoint from $N \setminus A_N$. As a countable union of η-meager sets the set A_N is η-meager. Consequently, $\mathbb{R}$ equals the η-meager set $\bigcup\{N \setminus A_N \mid N \in \mathcal{F}\} \cup \bigcup\{A_N \mid N \in \mathcal{F}\}$ and this is impossible. Obviously, the same argument shows that $\mathbb{R}$ is not ϑ-meager and this concludes the proof of Theorem 4. ■

Remark. The first statement of Theorem 4 is very close to [5, Theorem 11] which says that the space $(\mathbb{R}, \eta/\mathcal{L})$ is of first category where $\mathcal{L}$ is the σ-ideal of all Lebesgue null sets $L \subset \mathbb{R}$. But our proof is different because $(\mathbb{R}, \eta/\mathcal{L})$ is not separable since naturally every countable subset of $\mathbb{R}$ is closed in the space $(\mathbb{R}, \eta/\mathcal{L})$.

What happens if we vary the definitions of the two topologies λ and μ by mixing them so that we consider the two topologies

$$\gamma_1 := \lambda/\mathcal{I}_m \quad \text{and} \quad \gamma_2 := \mu/\mathcal{I}_n \ ?$$

Since $(\mathbb{R}, \lambda)$ and $(\mathbb{R}, \mu)$ are not second countable we cannot apply Theorem 1 to realize that the two families γ_1 and γ_2 actually are topologies on $\mathbb{R}$. But since it is well-known that $\mathbb{R}$ can be written as a disjoint union of an η-meager set and a null set, we have $\gamma_1 = \gamma_2 = \gamma$ with $\gamma := \eta/\mathcal{I}_g$ where the σ-ideal $\mathcal{I}_g$ is the family of *all* subsets of $\mathbb{R} \setminus \mathbb{Q}$. By Theorem 1, γ is a topology on $\mathbb{R}$. But also without applying Theorem 1, γ is a topology on $\mathbb{R}$ evidently. And it is a very nice one because the subspace $\mathbb{R} \setminus \mathbb{Q}$ is obviously *discrete*. Since $\mathbb{Q}$ is still dense in $(\mathbb{R}, \gamma)$ the space $(\mathbb{R}, \gamma)$ is an example of a *separable Hausdorff space which has a non-separable subspace*. The prototype of a space with this property is the famous *Niemytzki space*. (No. 82 in [9].) But the construction of the Niemytzki space is much more complicated than the construction of $(\mathbb{R}, \gamma)$. Moreover, as the Niemytzki space, the space $(\mathbb{R}, \gamma)$ is *first countable* but *not second countable*. (Obviously, $\big\{\{x\} \cup (]r, s[\cap \mathbb{Q}) \mid r, s \in \mathbb{Q} \wedge r < x < s\big\}$ is a local γ-base at $x \in \mathbb{R}$, and $(\mathbb{R}, \gamma)$ is not second countable since the subspace $\mathbb{R} \setminus \mathbb{Q}$ is discrete and uncountable.)

As we will see later, although the large subspace $\mathbb{R} \setminus \mathbb{Q}$ is totally disconnected, the small subspace $\mathbb{Q}$ works as a sort of paste which takes care that the whole space $(\mathbb{R}, \gamma)$ is *connected*. (As a trivial consequence, the three spaces $(\mathbb{R}, \vartheta), (\mathbb{R}, \lambda), (\mathbb{R}, \mu)$ are connected as well.)

Since $(\mathbb{R}, \gamma)$ is first countable the space $(\mathbb{R}, \gamma)$ is not homeomorphic to one of the three spaces $(\mathbb{R}, \vartheta), (\mathbb{R}, \lambda), (\mathbb{R}, \mu)$. Finally we note that $(\mathbb{R}, \gamma)$ is of first category. In fact, $\mathbb{Q}$ is γ-open and γ-dense whence $\mathbb{R} \setminus \mathbb{Q}$ is nowhere γ-dense. Thus $\mathbb{R}$ is a countable union of nowhere γ-dense sets since $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \bigcup\{\{x\} \mid x \in \mathbb{Q}\}$.

3. General properties

Summarizing and generalizing our four topologies $\vartheta, \lambda, \mu, \gamma$ we consider an arbitrary σ-ideal $\mathcal{I}$ on $\mathbb{R}$ such that $\mathcal{U} := \bigcup \mathcal{I}$ is an η-dense set of irrational numbers. (This is clearly true for $\mathcal{I} = \mathcal{I}_c, \mathcal{I}_n, \mathcal{I}_m, \mathcal{I}_g$ with $\mathcal{U} = \mathbb{R} \setminus \mathbb{Q}$ in all four cases.) Certainly, we can fix a countable η-dense subset $\mathcal{D}$ of $\mathcal{U}$. Note that every countable subset A of $\mathcal{U}$ lies in $\mathcal{I}$. (Choose $I_a \in \mathcal{I}$ with $a \in I_a$ for every $a \in A$. Then $A \subset \bigcup\{I_a \mid a \in A\} \in \mathcal{I}$.) In particular, $\mathcal{D} \in \mathcal{I}$. Further, $\mathcal{I} \supset \mathcal{I}_c$ when $\mathcal{U} = \mathbb{R} \setminus \mathbb{Q}$. In the following we investigate main properties of the topology

$$\tau := \eta/\mathcal{I} = \{U \setminus A \mid U \in \eta \wedge A \in \mathcal{I}\},$$

which is finer than η and which is finer than ϑ if $\mathcal{I}_c \subset \mathcal{I}$. (Of course, τ is strictly coarser than the discrete topology δ.) It is trivial but useful to note that a set $X \subset \mathbb{R}$ is τ-closed if and only if $X = Y \cup A$ where Y is η-closed and $A \in \mathcal{I}$. In particular, $\mathcal{D}$ is τ-closed whence the topology τ is strictly finer than the Euclidean topology η.

Certainly, $\mathbb{Q}$ is a τ-dense subset of $\mathbb{R}$ whence $(\mathbb{R}, \tau)$ is a separable Hausdorff space. (More generally it is plain that if $\mathcal{J}$ is an arbitrary σ-ideal on a second countable space $(X, \tilde{\tau})$, then $(X, \tilde{\tau}/\mathcal{J})$ is separable if and only if there exists a countable $\tilde{\tau}$-dense subset of X disjoint from $\bigcup \mathcal{J}$.) But $(\mathbb{R}, \tau)$ is not regular or normal since, obviously, the two τ-closed sets $\{0\}$ and $\mathcal{D}$ cannot be separated by τ-open sets. (In view of $\tau \neq \eta$ it also follows from [7, Corollary 3] that $(\mathbb{R}, \tau)$ is not regular.) Another important property of the space $(\mathbb{R}, \tau)$ is *connectedness.*

Theorem 5. *A subset of $\mathbb{R}$ is τ-connected if and only if it is an interval.*

Since the closure of a connected set is always connected, Theorem 5 implies that for $a < b$ the interval $[a, b]$ is the τ-closure of the interval $]a, b[$. As a consequence, the space $(\mathbb{R}, \tau)$ satisfies the separation axiom $\mathrm{T}_{2\frac{1}{2}}$, i.e. two points can always be separated by open sets with disjoint closures. (But this is small wonder since τ is finer than η.)

The following theorem demonstrates that the space $(\mathbb{R}, \tau)$, although being connected, is extremely far from being path connected or locally path connected.

Theorem 6. *The space $(\mathbb{R}, \tau)$ is totally pathwise disconnected.*

In view of $]a, b[\cap \mathbb{Q} \neq \emptyset$ for $a < b$, a proof of Theorem 5 is straightforward. For a proof of Theorem 6 we need to know how the τ-compact sets look like.

Theorem 7. *A set $K \subset \mathbb{R}$ is τ-compact if and only if $K \cap \mathcal{U}$ is finite and K is η-compact.*

Since $A \subset \mathbb{R}$ is τ-closed if and only if A is η-closed provided that $A \cap \mathcal{U}$ is finite, on the one hand Theorem 7 yields to an *internal* characterization of the τ-compact sets: *A set $K \subset \mathbb{R}$ is τ-compact if and only if $K \cap \mathcal{U}$ is finite and K is bounded and τ-closed.* On the other hand, since τ is finer than η, Theorem 7 is an immediate consequence of

Theorem 8. *If $M \subset \mathbb{R}$ such that $M \cap \mathcal{U}$ is infinite, then M as a subspace of $(\mathbb{R}, \tau)$ is not countably compact and hence neither compact nor sequentially compact.*

Proof. Choose a sequence of distinct numbers $x_1, x_2, x_3, \ldots$ in $M \cap \mathcal{U}$. Then the τ-open sets $\mathbb{R} \setminus \{x_m \mid m \geq n\} (n = 1, 2, 3, \ldots)$ form a countable cover of M without a finite subcover. ∎

Remark. By Theorem 8, for $a < b$ the η-compact interval $[a,b]$ is never τ-compact. There is a deeper reason why. Actually, *compactness* is a *maximal* property of the class of all Hausdorff spaces. Therefore the Hausdorff space $([a,b],\tau)$ cannot be compact since the Hausdorff space $([a,b],\eta)$ is compact and τ is strictly finer than η.

Now we are going to prove Theorem 6. If τ_1 and τ_2 are two topologies on $\mathbb{R}$ and $A, B \subset \mathbb{R}$, then we call a function $f : A \to B$ τ_1-τ_2-*continuous* when f is a continuous mapping from the space A with the relative topology of τ_1 to the space B with the relative topology of τ_2.

Theorem 6 says that every η-τ-continuous function from $[0,1]$ to $\mathbb{R}$ is constant. Let $f : [0,1] \to \mathbb{R}$ be η-τ-continuous. Then, a fortiori, f is η-η-continuous, i.e. continuous in the common sense. Suppose that f is not constant. Then the image of $[0,1]$ under f certainly is an interval $[u,v]$ with $u < v$. Now consider the τ-closed set $A = [u,v] \cap \mathcal{D}$. Due to the η-τ-continuity of f the set $f^{-1}(A)$ is an η-closed and hence an η-compact subset of $[0,1]$. But then, contrary to Theorem 7, the set $A = f\big(f^{-1}(A)\big)$ is τ-compact.

In view of the previous proof it is clear that for an arbitrary interval $[a,b]$ every η-τ-continuous function from $[a,b]$ to $\mathbb{R}$ is *constant.* The following Theorem, which will be established in the next chapter, says that every τ-η-continuous function from $[a,b]$ to $\mathbb{R}$ is *bounded*.

Theorem 9. *For $a \le b$ the space $[a,b]$ with the relative topology of τ is pseudocompact.*

Concerning other compactness properties, it is clear that $(\mathbb{R},\tau)$ is not paracompact since the separation axiom T_3 is violated. Further, $(\mathbb{R},\tau)$ is not σ-compact, i.e. $\mathbb{R}$ is not a countable union of τ-compact subsets, since $(\mathbb{R},\eta)$ is of second category and a τ-compact set K is always nowhere η-dense. (Indeed, $\mathbb{R} \setminus K$ is η-open and η-dense since K is η-compact and $K \cap \mathcal{U}$ is finite and $\mathcal{U}$ is η-dense.) There is a simpler reason why $(\mathbb{R},\tau)$ is not σ-compact when the topology τ is one of our topologies $\vartheta, \lambda, \mu, \gamma$, because in this case, referring to Theorem 7, any τ-compact set is countable.

The question whether $(\mathbb{R},\tau)$ is a *Lindelöf space* (every open cover has a countable subcover) depends on the size of the sets in $\mathcal{I}$.

Theorem 10. *The space $(\mathbb{R},\tau)$ is Lindelöf if and only if every set in $\mathcal{I}$ is countable. In particular, $(\mathbb{R},\vartheta)$ is Lindelöf, whereas $(\mathbb{R},\lambda), (\mathbb{R},\mu), (\mathbb{R},\gamma)$ are not Lindelöf.*

Proof. Suppose firstly that every set in $\mathcal{I}$ is countable, which means that $\mathcal{I}$ equals the family of all countable subsets of $\mathcal{U}$. Let $\{E_\lambda \setminus A_\lambda \mid \lambda \in \Lambda\}$ be a τ-open cover of $\mathbb{R}$ where all sets E_λ are η-open and all sets A_λ are countable subsets of $\mathcal{U}$. Then $\{E_\lambda \mid \lambda \in \Lambda\}$ is an η-open cover of $\mathbb{R}$. Since the space $(\mathbb{R},\eta)$ is Lindelöf, we can choose a countable index set $\Lambda_1 \subset \Lambda$ such that $\mathbb{R} = \bigcup\{E_\lambda \mid \lambda \in \Lambda_1\}$. Now consider the countable set $A = \bigcup\{A_\lambda \mid \lambda \in \Lambda_1\}$ and choose for every $a \in A$ an index $\lambda(a) \in \Lambda$ so that a lies in $E_{\lambda(a)} \setminus A_{\lambda(a)}$. Then with $\Lambda_2 = \{\lambda(a) \mid a \in A\}$ the countable family $\{E_\lambda \setminus A_\lambda \mid \lambda \in \Lambda_1 \cup \Lambda_2\}$ covers $\mathbb{R}$.

Suppose secondly that Y is an uncountable set in $\mathcal{I}$. Then choose a set $X \subset Y$ with cardinality equal to the first uncountable cardinal $\aleph_1$. Certainly, $X \in \mathcal{I}$. Let $\prec$ be a strict well-ordering on X so that $\{x \in X \mid x \prec y\}$ is countable for every $y \in X$. (In particular, X has no maximum and may be identified with the collection of all countable ordinals.) Naturally, every nonempty countable subset of X has a $\prec$-supremum in X. Now let $V_z := \mathbb{R} \setminus \{x \in X \mid z \prec x\}$ for $z \in X$. Then $\{V_z \mid z \in X\}$ is a τ-open cover of $\mathbb{R}$ which is isotonic, $V_y \subset V_z$ when $y \prec z$. This cover has not a countable subcover because, if A is a countable subset of X, then $\bigcup\{V_z \mid z \in A\} = V_a$ where $a \in X$ is the $\prec$-supremum of A, and $V_a = \mathbb{R}$ is impossible since $\mathbb{R} \setminus V_a = \{x \in X \mid a \prec x\}$ is infinite. ∎

Concerning local properties, from Theorem 7 and Theorem 5 we immediately derive

THEOREM 11. *The space $(\mathbb{R}, \tau)$ is not locally compact and not locally connected.*

4. The metric properties

We continue our investigation of the space $(\mathbb{R}, \tau)$ where $\tau = \eta/\mathcal{I}$ for an arbitrary σ-ideal $\mathcal{I}$ on $\mathbb{R} \setminus \mathbb{Q}$ such that $\mathcal{U} := \bigcup \mathcal{I}$ is η-dense.

In order to establish the property (MAP) in the first chapter we characterize the countable τ-dense subsets of $\mathbb{R}$.

THEOREM 12. *For every countable set $D \subset \mathbb{R}$ the following statements are equivalent.*

(1) *D is τ-dense in $\mathbb{R}$.*
(2) *$D \setminus \mathcal{U}$ is τ-dense in $\mathbb{R}$.*
(3) *$D \setminus \mathcal{U}$ is η-dense in $\mathbb{R}$.*

Proof. Trivially, (2) implies (1) since $D \setminus \mathcal{U} \subset D$. Further, (2) and (3) are equivalent since the relative topologies of τ and η on $\mathbb{R} \setminus \mathcal{U}$ are identical. In order to show that (1) implies (3) suppose that D is τ-dense. Put $A = D \cap \mathcal{U}$. Then $A \in \mathcal{I}$ since A is a countable subset of $\mathcal{U}$. Then for every η-open set $E \neq \emptyset$ the τ-open set $E \setminus A$ is nonempty (since $E \setminus A \supset E \cap \mathbb{Q} \neq \emptyset$) and hence the set $E \cap (D \setminus \mathcal{U}) = (E \setminus A) \cap D$ is nonempty. Therefore, $D \setminus \mathcal{U}$ is η-dense. ∎

REMARK. Note that Theorem 12 is not necessarily true when D is not assumed to be countable. For instance, if $\tau \in \{\vartheta, \lambda, \mu\}$, then $D = \mathbb{R} \setminus \mathbb{Q}$ is certainly τ-dense but the set $D \setminus \mathcal{U}$ is not η-dense because it is empty. This example shows also that a τ-dense set need not contain a countable τ-dense subset.

Since the relative topologies of τ and η on $\mathbb{R} \setminus \mathcal{U}$ coincide, a sequence of numbers in $\mathbb{R} \setminus \mathcal{U}$ is τ-convergent if and only if it is η-convergent. Moreover, we obviously have

THEOREM 13. *A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is τ-convergent if and only if $(x_n)_{n\in\mathbb{N}}$ is η-convergent and either $(x_n)_{n\in\mathbb{N}}$ is eventually constant or $x_n \notin \mathcal{U}$ for almost all $n \in \mathbb{N}$.*

Further, in view of Theorem 12, the metric approximation property (MAP) is always true in the non-metrizable, separable space $(\mathbb{R}, \tau)$.

THEOREM 14. *If D is a countable τ-dense subset of $\mathbb{R}$, then every $x \in \mathbb{R}$ is the limit of a τ-convergent sequence of numbers in D.*

Again, the counterexample $\tau = \vartheta$ demonstrates that Theorem 14 is not necessarily true for the space $(\mathbb{R}, \tau)$ when the assumption of the countability of D is dropped.

In order to verify the metric continuity property (MCP) we need the following lemma.

LEMMA 1. *Let (X, d) be a metric space and S be a dense subset of X. Then a function f from X into any regular space Y must be continuous if for every sequence (a_n) in S which converges to $b \in X$ the sequence $\big(f(a_n)\big)$ converges to $f(b)$.*

Proof. Suppose indirectly that f is not continuous at $x \in X$ and choose a convergent sequence (x_n) in X with limit x such that $f(x_n)$ does not converge to $f(x)$. Then there is an open neighborhood U of $f(x)$ in the space Y such that $f(x_n) \notin U$ for all $n \in M$ where M is an infinite subset of $\mathbb{N}$. Since Y is regular, we can find disjoint open sets $U', V \subset Y$ such that $f(x_n) \in V$ for every $n \in M$ and $f(x) \in U' \subset U$. Now choose for every $n \in M$ a sequence $(s_{n,m})_{m\in\mathbb{N}}$ within S which converges to x_n, whence $\big(f(s_{n,m})\big)_{m\in\mathbb{N}}$ converges to $f(x_n)$. Thus for every $n \in M$ we can find an index $m(n)$ such that for $\tilde{s}_n := s_{n,m(n)}$ we both have $d(\tilde{s}_n, x_n) < \frac{1}{n}$ and $f(\tilde{s}_n) \in V$. Since $\lim_{n\to\infty} x_n = x$, certainly the sequence $(\tilde{s}_n)_{n\in M}$ tends to x and therefore the sequence $\big(f(\tilde{s}_n)\big)_{n\in M}$ tends to $f(x)$. But then $f(\tilde{s}_n) \in U'$ for almost all $n \in M$ repugnant with $f(\tilde{s}_n) \in V$ for every $n \in M$ and $V \cap U' = \emptyset$. ■

Obviously, the following theorem immediately implies Theorem 9.

THEOREM 15. *Let (Y, σ) be a regular space and $I \subset \mathbb{R}$ be an interval and $f : I \to Y$. Then f is τ-σ-continuous if and only if f is η-σ-continuous.*

Proof. Trivially, f is τ-σ-continuous if f is η-σ-continuous. Suppose conversely that f is τ-σ-continuous. In particular, f is sequentially τ-σ-continuous. Thus we conclude that f is η-σ-continuous by using Theorem 13 and by applying Lemma 1 with (X, d) being the Euclidean metric space I and $S = I \setminus \mathcal{U}$. ■

REMARK. Theorem 15 is also a consequence of [7, Corollary 2] and of [3, Corollary 6.12].

Finally, by applying Theorem 13 and Lemma 1 and the trivial implication in Theorem 15, it is plain to establish the metric continuity property (MCP) for the space $(\mathbb{R}, \tau)$.

Theorem 16. *Let (Y,σ) be a regular space and $I \subset \mathbb{R}$ be an interval and $f : I \to Y$. Then f is τ-σ-continuous if the sequence $\big(f(x_n)\big)$ σ-converges to $f(x)$ whenever (x_n) is a τ-convergent sequence of real numbers in I with limit $x \in I$.*

Remark. Although all spaces $(\mathbb{R},\tau)$ satisfy the two metric properties (MAP) and (MCP), if $\mathcal{I} \neq \mathcal{P}(\mathcal{U})$ then a space $(\mathbb{R},\tau)$ never satisfies the metric property that every limit point of any set A is the limit of a sequence of points in A. (For a counterexample choose any not τ-closed set $A \subset \mathcal{U}$. Such a choice is possible in view of Theorem 20 below.)

Recall that a space X is $\aleph_1$-*compact* if and only if every uncountable set has a limit point. Referring to Theorem 10, the following theorem shows that for all spaces $(\mathbb{R},\tau)$ the properties of being $\aleph_1$-*compact* and of being Lindelöf are equivalent.

Theorem 17. *The space $(\mathbb{R},\tau)$ is $\aleph_1$-compact if and only if every set in $\mathcal{I}$ is countable.*

Proof. If A is an uncountable set in $\mathcal{I}$, then A cannot have a τ-limit point $x \in \mathbb{R}$ since $\{x\} \cup (\mathbb{R} \setminus A)$ is a τ-neighborhood of x disjoint from $A \setminus \{x\}$. Suppose conversely that every set in $\mathcal{I}$ is countable. Hence every η-condensation point of any set $S \subset \mathbb{R}$ is a τ-limit point of S. Therefore, $(\mathbb{R},\tau)$ must be $\aleph_1$-compact because every uncountable set of real numbers contains uncountably many η-condensation points of itself. ■

5. Another concrete topology

What happens if we switch the two sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in the definition of the topology ϑ and consider the topology
$$\rho := \big\{U \setminus A \mid U \subset \mathbb{R} \wedge U \eta\text{-open} \wedge A \subset \mathbb{Q}\big\}$$
so that $\rho = \eta/\mathcal{I}_q$ where the σ-ideal $\mathcal{I}_q$ is the power set of $\mathbb{Q}$?

In a certain sense the topology ρ stands between η and ϑ because the space $(\mathbb{R},\rho)$ is obviously homeomorphic to $(\mathbb{R},\rho')$ with $\rho' = \eta/\mathcal{I}'_q$ where $\mathcal{I}'_q = \mathcal{P}(\{\pi + r \mid r \in \mathbb{Q}\})$. In particular, $\eta \subset \rho' \subset \vartheta$ and the space $(\mathbb{R},\rho')$ is only a very special example of the general space $(\mathbb{R},\tau)$ already discussed. Moreover, the space $(\mathbb{R},\rho)$ is less exotic than $(\mathbb{R},\vartheta)$ because $(\mathbb{R},\rho)$ is not only *first countable* but also *second countable* since the countable family of all sets $\{q\} \cup (]r,s[\setminus\mathbb{Q})$ with $q,r,s \in \mathbb{Q}$ and $r < q < s$ is obviously a base.

Whereas a τ-compact set must be countable in case that the topology τ is one of our topologies $\vartheta,\lambda,\mu,\gamma$, referring to Theorem 7, there certainly exist ρ-compact subsets of $\mathbb{R}$ with cardinality c.

The space $(\mathbb{R},\rho)$ is only a marginal expansion of a very prominent space, namely the Euclidean space $\mathbb{R} \setminus \mathbb{Q}$ (No. 31 in [9]) since the relative topologies of ρ and η coincide on $\mathbb{R} \setminus \mathbb{Q}$. The η-dense set $\mathbb{Q}$ becomes discrete in $(\mathbb{R},\rho)$. We leave it to the reader to check all properties of the topology ρ given by our table.

6. Infinitely many topologies

Let Σ be the family of all σ-ideals $\mathcal{I}$ on $\mathbb{R}$ where $\bigcup\mathcal{I}$ is an η-dense set of irrational numbers. Of course the family Σ is infinite. But Σ contains large subfamilies Σ' where the topologies $\eta/\mathcal{I}(\mathcal{I} \in \Sigma')$ coincide. For instance, put $\mathcal{I}_K = \{M \setminus K \mid M \subset \mathbb{R} \setminus \mathbb{Q}\}$. If we let K run through the nonempty subsets of $\{k + \pi \mid k \in \mathbb{Z}\}$, we obtain c distinct σ-ideals $\mathcal{I}_K$ in Σ such that all topologies $\eta/\mathcal{I}_K$ equal our topology γ.

Nevertheless, there are very many topologies on $\mathbb{R}$ induced by σ-ideals in Σ.

THEOREM 18. *There exists a subfamily Σ_1 of Σ with cardinality 2^{2^c} such that $\bigcup\mathcal{I} = \mathbb{R} \setminus \mathbb{Q}$ for every $\mathcal{I} \in \Sigma_1$ and two spaces $(\mathbb{R}, \eta/\mathcal{I}_1), (\mathbb{R}, \eta/\mathcal{I}_2)$ are never homeomorphic for distinct $\mathcal{I}_1, \mathcal{I}_2 \in \Sigma_1$.*

Remark. The family Σ_1 must contain a subfamily Σ_2 with cardinality 2^{2^c} such that for every $\mathcal{I} \in \Sigma_2$ the space $(\mathbb{R}, \eta/\mathcal{I})$ is not first countable because there cannot be more than 2^c topologies on $\mathbb{R}$ which make $\mathbb{R}$ first countable. In fact, a topology is completely determined by any base. If $\mathbb{R}$ is first countable, then there must be a base with cardinality not greater than c. All bases are subfamilies of $\mathcal{P}(\mathbb{R})$. There are exactly $(2^c)^c = 2^c$ such subfamilies with cardinality $\leq c$.

In order to get rid of homeomorphic topologies we will use the following lemma.

LEMMA 2. *Let $\mathcal{T}_\sigma$ be a family of topologies of the form $\eta/\mathcal{I}$ with $\mathcal{I} \in \Sigma$ and let κ be the cardinality of $\mathcal{T}_\sigma$. If κ is greater than c, then there exists a subfamily $\mathcal{T}$ of $\mathcal{T}_\sigma$ with cardinality κ such that $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$ are never homeomorphic for distinct topologies τ_1 and τ_2 in $\mathcal{T}$.*

Proof. Write $\tau_1 \sim \tau_2$ for $\tau_1, \tau_2 \in \mathcal{T}_\sigma$ when the two spaces $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$ are homeomorphic. Clearly, $\sim$ defines an equivalence relation on the family $\mathcal{T}_\sigma$. By choosing one representative in every equivalence class we get a family $\mathcal{T} \subset \mathcal{T}_\sigma$ where $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$ are never homeomorphic for distinct topologies τ_1 and τ_2 in $\mathcal{T}$. Naturally, the cardinality of $\mathcal{T}$ must equal κ provided that the cardinality of every equivalence class is not greater than c. This is actually true in view of the following theorem. ∎

THEOREM 19. *Let $\tau_i = \eta/\mathcal{I}_i$ for $\mathcal{I}_1, \mathcal{I}_2 \in \Sigma$. Then every homeomorphism from the space $(\mathbb{R}, \tau_1)$ onto the space $(\mathbb{R}, \tau_2)$ is a strictly monotonic, η-η-continuous function from $\mathbb{R}$ onto $\mathbb{R}$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be bijective and suppose that f^{-1} is τ_2-τ_1-continuous. Then by using Theorem 5 it is straightforward to realize that the image of any bounded η-open interval under f^{-1} must be a bounded η-open interval. Consequently, the bijection f is η-η-continuous and therefore f must be strictly monotonic. (Alternatively, if f is τ_1-τ_2-continuous, then f is τ_1-η-continuous, whence f is η-η-continuous by Theorem 15.) ∎

Remark. We do not really need Theorem 19 in order to verify that the cardinality of any equivalence class cannot be greater than c because if τ_1, τ_2 are two Hausdorff topologies on $\mathbb{R}$ and $A \subset \mathbb{R}$ is countable and τ_1-dense, then each τ_1-τ_2-continuous function $f : \mathbb{R} \to \mathbb{R}$ is completely determined by its values on A and naturally there are precisely c functions from A into $\mathbb{R}$. But Theorem 19 is nice and we need it in the next chapter anyway.

In order to achieve the enormous cardinality 2^{2^c} in Theorem 18 we need a statement on large families of σ-ideals.

LEMMA 3. *There exists a family $\Im$ of σ-ideals $\mathcal{I}$ on $\mathcal{U} := \mathbb{R} \setminus \mathbb{Q}$ such that the cardinality of $\Im$ is 2^{2^c}, and $\bigcup \mathcal{I} = \mathcal{U}$ for every $\mathcal{I} \in \Im$, and for distinct $\mathcal{I}_1, \mathcal{I}_2$ in $\Im$ there is always a set $X \subset \mathcal{U}$ with $X \in \mathcal{I}_1$ and $\mathcal{U} \setminus X \in \mathcal{I}_2$.*

Proof. By applying [2, Theorem 7.3] with $\kappa = \aleph_1$ and $\alpha = c$, there is a family Φ of 2^{2^c} $\aleph_1$-complete filters on $\mathcal{U}$ such that for distinct filters $\mathcal{F}_1, \mathcal{F}_2$ in Φ there is always a set $X \subset \mathcal{U}$ with $X \in \mathcal{F}_1$ and $\mathcal{U} \setminus X \in \mathcal{F}_2$. Thus $\Phi' = \big\{\{\mathcal{U} \setminus F \mid F \in \mathcal{F}\} \mid \mathcal{F} \in \Phi\big\}$ is a family of σ-ideals on $\mathcal{U}$ that is equipollent to Φ and has the same separation property. Now, if $\mathcal{I}_1, \mathcal{I}_2 \in \Phi'$ are distinct with identical unions $\bigcup \mathcal{I}_1 = \bigcup \mathcal{I}_2 = V$, then $V = \mathcal{U}$ because with $X \in \mathcal{I}_1$ and $\mathcal{U} \setminus X \in \mathcal{I}_2$ we have $V \supset X$ and $V \supset \mathcal{U} \setminus X$. Consequently, for every *proper* subset V of $\mathcal{U}$ there is at most one σ-ideal $\mathcal{I}$ in Φ' with $\bigcup \mathcal{I} = V$. Since there are only 2^c subsets of $\mathcal{U}$, the family $\Phi'' := \{\mathcal{I} \in \Phi' \mid \bigcup \mathcal{I} \neq \mathcal{U}\}$ has cardinality at most 2^c and hence the family $\Im := \Phi' \setminus \Phi''$ has all properties claimed in Lemma 3. ∎

Now we are going to prove Theorem 18. Let $\Im$ be as in Lemma 3. We claim that the two topologies $\eta/\mathcal{I}$ and $\eta/\mathcal{J}$ are always distinct for distinct σ-ideals $\mathcal{I}, \mathcal{J} \in \Im$. Then in view of Lemma 2 we are finished. Suppose indirectly that $\tau = \eta/\mathcal{I} = \eta/\mathcal{J}$ for two distinct $\mathcal{I}, \mathcal{J} \in \Im$. By Lemma 3, we can fix a set $X \subset \mathcal{U}$ with X in $\mathcal{I}$ and $X' = \mathcal{U} \setminus X$ in $\mathcal{J}$. Then for an arbitrary set $V \subset \mathcal{U}$ the set $V \cap X$ lies in $\mathcal{I}$ and the set $V \cap X'$ lies in $\mathcal{J}$. Therefore, $V = (V \cap X) \cup (V \cap X')$ must be τ-closed since $V \cap X$ and $V \cap X'$ are both τ-closed. Thus *every* subset of $\mathcal{U}$ is τ-closed. But then, in view of the following theorem, the ideals $\mathcal{I}$ and $\mathcal{J}$ are both equal to the power set of $\mathcal{U}$ whence they are not distinct. ∎

THEOREM 20. *Let (X, π) be a Polish space, i.e. a completely metrizable separable space. Let $\mathcal{I}$ be a σ-ideal on X and $\mathcal{U} = \bigcup \mathcal{I}$ and define a topology τ on X by $\tau = \pi/I$. If every subset of $\mathcal{U}$ is τ-closed, then $\mathcal{I} = \mathcal{P}(\mathcal{U})$.*

Proof. Since automatically $\mathcal{I} = \mathcal{P}(\mathcal{U})$ if X is countable, assume that X is uncountable. Let $B \subset X$ be a *Bernstein set*, i.e. neither B nor $X \setminus B$ contains any π-perfect set $\neq \emptyset$. In particular, $\mathcal{B} = B \cap \mathcal{U}$ and $\mathcal{B}' = \mathcal{U} \setminus B$ cannot contain uncountable π-closed sets. By assumption, $\mathcal{B}$ is τ-closed. Thus we have $\mathcal{B} = I \cup C$ with $I \in \mathcal{I}$ and π-closed $C \subset X$. Thus C must be a *countable* subset of $\mathcal{U}$ since $C \subset \mathcal{B}$. But then C must lie in $\mathcal{I}$ since $\bigcup \mathcal{I} = \mathcal{U}$. Thus, as the union of two sets in $\mathcal{I}$ the set $\mathcal{B}$ must lie in $\mathcal{I}$. Similarly, the τ-closed set $\mathcal{B}'$ must lie in $\mathcal{I}$. Hence $\mathcal{B} \cup \mathcal{B}' = \mathcal{U} \in \mathcal{I}$ and thus $\mathcal{I} = \mathcal{P}(\mathcal{U})$. ∎

7. A continuum of concrete topologies

Since the 2^{2^c} topologies in Theorem 18 are very far from being concrete, we conclude the article by constructing infinitely many concrete examples of σ-ideals $\mathcal{J}_i$ such that for the topologies $\tau_i = \eta/\mathcal{J}_i$ the separable and connected spaces $(\mathbb{R}, \tau_i)$ are mutually non-homeomorphic. In doing so we also take care that in every space $(\mathbb{R}, \tau_i)$ no point has a countable local base.

Let $X_1 = [0,1], X_2 = [0,1] \cup [2,3], X_3 = [0,1] \cup [2,3] \cup [4,5]$ etc. In general,

$$X_k := \bigcup_{0 \le m < k} [2m, 2m+1] \quad (k = 1, 2, 3, \dots).$$

Now take the two σ-ideals $\mathcal{I}_c, \mathcal{I}_n$ which induce the topologies ϑ, λ and define σ-ideals $\mathcal{J}_k$ for all $k = 1, 2, 3, \dots$ via

$$\mathcal{J}_k := \{A \cup B \mid A \in \mathcal{I}_c \wedge B \in \mathcal{I}_n \wedge B \subset X_k\}.$$

Then, with $\tau_k = \eta/\mathcal{J}_k$ it is characteristic of the space $(\mathbb{R}, \tau_k)$ to have a sequence $x_1 < y_1 < x_2 < y_2 < \cdots < x_k < y_k$ such that the relative topologies of τ_k and ϑ are identical on $]-\infty, x_1[\cup]y_1, x_2[\cup \cdots \cup]y_{k-1}, x_k[\cup]y_k, \infty[$ while τ_k and λ coincide on $]x_1, y_1[\cup \cdots \cup]x_k, y_k[$. Thus, in view of Theorem 2 and Theorem 19, for different indices k, k' the two spaces $(\mathbb{R}, \tau_k)$ and $(\mathbb{R}, \tau_{k'})$ cannot be homeomorphic. By Theorem 3, no point in any space $(\mathbb{R}, \tau_k)$ has a countable local base.

With the help of a third topology beside ϑ and λ, say μ, we can easily adapt this construction in order to get c concrete σ-ideal topologies on $\mathbb{R}$ where no point has a countable local base. Naturally, there are c sequences $s = (s_0, s_1, s_2, \dots)$ of the digits 0 and 1. For each sequence s define a topology $\tau(s)$ on $\mathbb{R}$ such that

(i) *The topologies $\tau(s)$ and ϑ coincide on* $]-\infty, 0[\cup]1, 2[\cup]3, 4[\cup]5, 6[\cup]7, 8[\cup \cdots$.

(ii) *For every $n = 0, 1, 2, \dots$ on the interval $]2n, 2n+1[$ the relative topologies of $\tau(s)$ and λ coincide when $s_n = 0$ while $\tau(s)$ and μ coincide when $s_n = 1$.*

Certainly, in view of Theorem 4 and Theorem 19, for two distinct sequences s, s' the spaces $(\mathbb{R}, \tau(s))$ and $(\mathbb{R}, \tau(s'))$ cannot be homeomorphic.

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