МАТЕМАТИЧКИ ВЕСНИК
64, 2 (2012), 167–172
June 2012

оригинални научни рад
research paper

ON AN INTEGRAL OPERATOR OF MEROMORPHIC FUNCTIONS

B.A. Frasin

Abstract. New sufficient conditions are derived for the integral operator of mereomorphic functions defined by

$$H(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1}(uf_1(u))^{\gamma_1} \cdots (uf_n(u))^{\gamma_n}\, du,$$

to be in the class $\Sigma_N(\lambda)$ of meromorphic functions satisfying the condition $-\operatorname{Re}\{\frac{zf''(z)}{f'(z)}+1\} < \lambda$, where $\lambda > 1$.

1. Introduction and definitions

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

which are analytic in the punctured unit disk $\mathcal{U} = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma$ is said to be meromorphic starlike of order δ for some δ $(0 \le \delta < 1)$ if and only if

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta \qquad (z \in \mathcal{U}). \tag{1.1}$$

We denote by $\Sigma^*(\delta)$ the class of all meromorphic starlike functions of order δ.

The class $\Sigma^*(\delta)$ and various other subclasses of Σ have been studied rather extensively by Nehari and Netanyahu [15], Clunie [8], Pommerenke [18], Miller [12], Royster [19], and others. Analogous to the subclass $\mathcal{N}(\lambda)$ (see [20]) of analytic functions satisfying

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\} < \lambda \qquad (\lambda > 1,\ z \in \mathcal{U}),$$

2010 AMS Subject Classification: 30C45.
Keywords and phrases: Meromorphic functions; integral operator.

Wang et al. [22] (see also [15]) introduced and studied the subclass $\Sigma_N(\lambda)$ of Σ consisting of functions $f(z)$ satisfying

$$-\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\}<\lambda, \qquad (\lambda>1,\ z\in\mathcal{U}).$$

Recently, many authors introduced and studied various integral operators of analytic and univalent functions in open unit disk $\mathcal{U}$ (cf., e.g., [2–6, 9–10, 16–17]).

In the present paper, we derive new sufficient condition for the following new integral operator $H(z)$ of meromorphic functions Σ to be in the class $\Sigma_N(\lambda)$.

Definition 1.1. Let $f_j\in\Sigma$ and $\gamma_j>0$ for $j=1,\dots,n$, $n\in\mathbb{N}$. Let $H(z)\colon\Sigma^n\to\Sigma$ be the integral operator defined by

$$H(z)=\frac{c}{z^{c+1}}\int_0^z u^{c-1}(uf_1(u))^{\gamma_1}\cdots(uf_n(u))^{\gamma_n}\,du, \qquad (c>0,\ z\in\mathcal{U}). \tag{1.2}$$

Here and throughout in the sequel every many-valued function is taken with the principal branch.

Remark 1.2. We note that if $c=1$, then the integral operator $H(z)$ reduces to the integral operator

$$H(f_1,\dots,f_n,z)=\frac{1}{z^2}\int_0^z (uf_1(u))^{\gamma_1}\cdots(uf_n(u))^{\gamma_n}\,du, \qquad (z\in\mathcal{U}),$$

introduced by Mohammed and Darus [14]. If $n=1$, $\gamma_1=\gamma$ and $f_1=f$, then the integral operator $H(z)$ reduces to the integral operator

$$I_{\gamma,c}(z)=\frac{c}{z^{c+1}}\int_0^z u^{c-1}(uf(u))^{\gamma}\,du, \qquad (\gamma,c>0,\ z\in\mathcal{U}).$$

In particular, for $\gamma=1$, we have the integral operator $I_c(z)=\frac{c}{z^{c+1}}\int_0^z u^c f(u)\,du$, which was studied by many authors (cf., e.g., [1, 11, 13]).

In order to derive our main results, we have to recall here the following

Lemma 1.3. [7] *If $f(z)\in\Sigma$ satisfies $f(z)f'(z)\neq 0$ in $\mathcal{D}$ and*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right\}<2-\beta \qquad (z\in\mathcal{U}),$$

then

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\frac{1}{3-2\beta} \qquad (z\in\mathcal{U}),$$

where $1/2\le\beta<1$.

2. Results

Theorem 2.1. *Let $f_j(z)\in\Sigma$ and $\gamma_j>0$ for $j=1,\dots,n$, with*

$$1<\sum_{j=1}^{n}\gamma_j<c+1, \qquad (c>0). \tag{2.1}$$

If $H(z)\in\Sigma^*(\delta)$ and $zH'(z)/H(z)\neq 0$ in $\mathcal{D}$, then $H(z)\in\Sigma_N(\lambda)$, where $\lambda>1$.

Proof. From (1.2) it follows that

$$\left(\frac{z^{c+1}H(z)}{c}\right)' = z^{c-1}\left(zf_1(z)\right)^{\gamma_1}\cdots\left(zf_n(z)\right)^{\gamma_n} \tag{2.2}$$

Differentiating both sides of (2.2) logarithmically and multiplying by z, we obtain

$$\frac{z^2H''(z)+2(c+1)zH'(z)+c(c+1)H(z)}{zH'(z)+(c+1)H(z)} = c-1+\sum_{j=1}^{n}\gamma_j\left(\frac{zf_j'(z)}{f_j(z)}+1\right)$$

which is equivalent to

$$\frac{\frac{zH''(z)}{H'(z)}+c(c+1)\frac{H(z)}{zH'(z)}+2(c+1)}{1+(c+1)\frac{H(z)}{zH'(z)}} = c-1+\sum_{j=1}^{n}\gamma_j\left(\frac{zf_j'(z)}{f_j(z)}+1\right).$$

Therefore, we have

$$\frac{-\left(\frac{zH''(z)}{H'(z)}+1\right)-c(c+1)\frac{H(z)}{zH'(z)}-(2c+1)}{1+(c+1)\frac{H(z)}{zH'(z)}} = 1-c-\sum_{j=1}^{n}\gamma_j\left(\frac{zf_j'(z)}{f_j(z)}+1\right). \tag{2.3}$$

From (2.3), we easily get

$$\begin{aligned}-\left(\frac{zH''(z)}{H'(z)}+1\right) &= \sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right)\left(1+\frac{(c+1)H(z)}{zH'(z)}\right)\\ &\quad -\frac{(c+1)H(z)}{zH'(z)}\Big(\sum_{j=1}^{n}\gamma_j-1\Big)+c+2-\sum_{j=1}^{n}\gamma_j.\end{aligned} \tag{2.4}$$

Taking real part of both sides of (2.4), we obtain

$$\begin{aligned}-\operatorname{Re}\left(\frac{zH''(z)}{H'(z)}+1\right) &= \operatorname{Re}\Big[\sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right)\left(1+\frac{(c+1)H(z)}{zH'(z)}\right)\Big]\\ &\quad +(c+1)\Big(\sum_{j=1}^{n}\gamma_j-1\Big)\operatorname{Re}\left(\frac{-H(z)}{zH'(z)}\right)+c+2-\sum_{j=1}^{n}\gamma_j.\end{aligned}$$

Thus, we have

$$\begin{aligned}-\operatorname{Re}\left(\frac{zH''(z)}{H'(z)}+1\right) &= \operatorname{Re}\Big[\sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right)\left(1+\frac{(c+1)H(z)}{zH'(z)}\right)\Big]\\ &\quad +(c+1)\Big(\sum_{j=1}^{n}\gamma_j-1\Big)\operatorname{Re}\left(-\frac{1}{\frac{zH'(z)}{H(z)}}\right)+c+2-\sum_{j=1}^{n}\gamma_j\\ &\le \Big|\sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right)\left(1+\frac{(c+1)H(z)}{zH'(z)}\right)\Big|\\ &\quad +(c+1)\Big(\sum_{j=1}^{n}\gamma_j-1\Big)\frac{\operatorname{Re}\left(\frac{-zH'(z)}{H(z)}\right)}{\left|\frac{zH'(z)}{H(z)}\right|^2}+c+2-\sum_{j=1}^{n}\gamma_j.\end{aligned}$$

Let

$$\lambda = \Big|\sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right)\left(1+\frac{(c+1)H(z)}{zH'(z)}\right)\Big| + (c+1)\Big(\sum_{j=1}^{n}\gamma_j - 1\Big)\frac{\mathrm{Re}\left(\frac{-zH'(z)}{H(z)}\right)}{\left|\frac{zH'(z)}{H(z)}\right|^2} + c + 2 - \sum_{j=1}^{n}\gamma_j.$$

Since $\left|\sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right)\left(1+\frac{(c+1)H(z)}{zH'(z)}\right)\right| > 0$ and $H(z)\in\Sigma^*(\delta)$, then we have

$$\lambda > (c+1)\Big(\sum_{j=1}^{n}\gamma_j - 1\Big)\frac{\delta}{\left|\frac{zH'(z)}{H(z)}\right|^2} + c + 2 - \sum_{j=1}^{n}\gamma_j > c + 2 - \sum_{j=1}^{n}\gamma_j$$

which, in light of the hypothesis (2.1), yields $\lambda > 1$. Therefore, $H(z)\in\Sigma_N(\lambda)$, where $\lambda > 1$. This completes the proof. ■

Letting $n = 1$, $\gamma_1 = \gamma$ and $f_1 = f$ in Theorem 2.1, we have

COROLLARY 2.2. *Let $f\in\Sigma$ and $1 < \gamma < c+1$, $c > 0$. If $I_{\gamma,c}(z)\in\Sigma^*(\delta)$ and $zI'_{\gamma,c}(z)/I_{\gamma,c}(z)\neq 0$ in $\mathcal{D}$ then $I_{\gamma,c}(z)\in\Sigma_N(\lambda)$, where $\lambda > 1$.*

With the help of Lemma 1.3, we now derive the following theorem

THEOREM 2.3. *Let $f_j(z)\in\Sigma$ satisfies $f_j(z)f_j'(z)\neq 0$ in $\mathcal{D}$, for all $j = 1,\dots,n$, $1/2\le\beta<1$ and*

$$\mathrm{Re}\left\{\frac{zf_j'(z)}{f_j(z)} - \frac{zf_j''(z)}{f_j'(z)}\right\} < 2-\beta \qquad (j = 1,\dots,n,\ z\in\mathcal{U}).$$

If $\gamma_j > 0$ for all $j = 1,\dots,n$, with

$$\sum_{j=1}^{n}\gamma_j > \frac{(1+c)(3-2\beta)}{2-2\beta}, \qquad (c>0). \tag{2.5}$$

then $H(z)\in\Sigma_N(\lambda)$, where $\lambda > 1$.

Proof. From (2.4), we obtain

$$-\left(\frac{zH''(z)}{H'(z)}+1\right) = \left(\frac{(c+1)H(z)}{zH'(z)}\right)\Big[\sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right) - \Big(\sum_{j=1}^{n}\gamma_j - 1\Big)\Big] + \sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right) + c + 2 - \sum_{j=1}^{n}\gamma_j.$$

Thus, we have

$$-\mathrm{Re}\left(\frac{zH''(z)}{H'(z)}+1\right) = \mathrm{Re}\left\{\left(\frac{(c+1)H(z)}{zH'(z)}\right)\Big[\sum_{j=1}^{n}\gamma_j\left(-\frac{zf_j'(z)}{f_j(z)}\right) - \Big(\sum_{j=1}^{n}\gamma_j - 1\Big)\Big]\right\} + \sum_{j=1}^{n}\gamma_j\,\mathrm{Re}\left(-\frac{zf_j'(z)}{f_j(z)}\right) + c + 2 - \sum_{j=1}^{n}\gamma_j$$

$$\leq \left| \left(\frac{(c+1)H(z)}{zH'(z)} \right) \left[\sum_{j=1}^{n} \gamma_j \left(-\frac{zf_j'(z)}{f_j(z)} \right) - \left(\sum_{j=1}^{n} \gamma_j - 1 \right) \right] \right| + \sum_{j=1}^{n} \gamma_j \operatorname{Re} \left(-\frac{zf_j'(z)}{f_j(z)} \right) + c + 2 - \sum_{j=1}^{n} \gamma_j.$$

Let

$$\lambda = \left| \left(\frac{(c+1)H(z)}{zH'(z)} \right) \left[\sum_{j=1}^{n} \gamma_j \left(-\frac{zf_j'(z)}{f_j(z)} \right) - \left(\sum_{j=1}^{n} \gamma_j - 1 \right) \right] \right| + \sum_{j=1}^{n} \gamma_j \operatorname{Re} \left(-\frac{zf_j'(z)}{f_j(z)} \right) + c + 2 - \sum_{j=1}^{n} \gamma_j. \quad (2.6)$$

Then, applying Lemma 1.3 and since $\left| \left(\frac{(c+1)H(z)}{zH'(z)} \right) \left[\sum_{j=1}^{n} \gamma_j \left(-\frac{zf_j'(z)}{f_j(z)} \right) - \left(\sum_{j=1}^{n} \gamma_j - 1 \right) \right] \right| > 0$, we have

$$\lambda > \sum_{j=1}^{n} \gamma_j \left(\frac{1}{3-2\beta} \right) + c + 2 - \sum_{j=1}^{n} \gamma_j = c + 2 + \left(\frac{2\beta - 2}{3-2\beta} \right) \sum_{j=1}^{n} \gamma_j$$

which, in light of the hypothesis (2.5), yields $\lambda > 1$. Therefore, $H(z) \in \Sigma_N(\lambda)$, where $\lambda > 1$. ■

Letting $n = 1$, $\gamma_1 = \gamma$ and $f_1 = f$ in Theorem 2.3, we have

Corollary 2.4. *Let $f(z) \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$, $1/2 \leq \beta < 1$, and*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right\} < 2 - \beta \qquad (z \in \mathcal{U}).$$

If

$$\gamma > \frac{(1+c)(3-2\beta)}{2-2\beta}, \qquad (c > 0),$$

then $I_{\gamma,c}(z) \in \Sigma_N(\lambda)$, where $\lambda > 1$.

Letting $\beta = 1/2$ in Corollary 2.4, we immediately have

Corollary 2.5. *Let $f(z) \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$, and*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right\} < 3/2 \qquad (z \in \mathcal{U}).$$

If $\gamma > 2(1+c)$, $c > 0$, then $I_{\gamma,c}(z) \in \Sigma_N(\lambda)$, where $\lambda > 1$.

Making use of (2.6) and definition (1.1), we can prove

Theorem 2.6. *Let $f_j(z) \in \Sigma$ and $\gamma_j > 0$, for $j = 1, \ldots, n$, with*

$$\sum_{j=1}^{n} \gamma_j > \frac{c+1}{1-\delta}, \qquad (c > 0,\ 0 \leq \delta < 1).$$

If $f(z) \in \Sigma^*(\delta)$ then $H(z) \in \Sigma_N(\lambda)$, where $\lambda > 1$.

Letting $n = 1$, $\gamma_1 = \gamma$ and $f_1 = f$ in Theorem 2.6, we have

COROLLARY 2.7. *Let* $f(z) \in \Sigma$ *and* $\gamma > \frac{c+1}{1-\delta}$ $(c > 0)$*, where* $0 \leq \delta < 1$*. If* $f(z) \in \Sigma^*(\delta)$ *then* $I_{\gamma,c}(z) \in \Sigma_N(\lambda)$*, where* $\lambda > 1$*.*

ACKNOWLEDGEMENTS. The author would like to thank the referee for his helpful comments and suggestions.

REFERENCES

[**1**] S.K. Bajpai, *A note on a class of meromorphic univalent functions*, Rev. Roumaine Math. Pures Appl. **22** (1977), 295–297.

[**2**] Á. Baricz, B.A. Frasin, *Univalence of integral operators involving Bessel functions*, Applied Math. Letters **23** (2010), 371–376.

[**3**] D. Blezu, R.N. Pascu, *Univalence criteria for integral operators*, Glasnik Mat. **36**(56) (2001), 241–245.

[**4**] D. Breaz, N. Breaz, *Two integral operators*,Studia Universitatis Babes-Bolyai, Mathematica, Cluj-Napoca, **3** (2002), 13–21.

[**5**] D. Breaz, S. Owa, N. Breaz, *A new integral univalent operator*, Acta Univ. Apul. **16** (2008), 11–16.

[**6**] D. Breaz, V. Pescar, *Univalence conditions for some general integral operators*, Banach J. Math. Anal. **2** (2008), 53–58.

[**7**] N.E. Cho, S. Owa, *Sufficient conditions for meromorphic starlikeness and close-to-convexity of order* α, Intern. J. Math. Math. Sci. **26** (2001), 317–319.

[**8**] J. Clunie, *On meromorphic schlicht functions*, J. London Math. Soc. **34** (1959), 215–216.

[**9**] B.A. Frasin, *Univalence of two general integral operators*, Filomat **23** (2009), 223–229.

[**10**] B.A. Frasin, *New criteria for univalence of certain integral operators*, Acta Math. Acad. Paed. Nyir, in press.

[**11**] R.M. Goel, N.S. Sohi, *On a class of meromorphic functions*, Glasnik Mat. **17** (1981), 19–28.

[**12**] J. Miller, *Convex meromorphic mappings and related functions*, Proc. Amer. Math. Soc. **25** (1970), 220–228.

[**13**] M.L. Mogra, T.R. Reddy, O.P. Juneja, *Meromorphic univalent functions with positive coefficients*, Bull. Austral. Math. Soc. **32** (1985), 161–176.

[**14**] A. Mohammed, M. Darus, *A new integral operator for meromorphic functions*, Acta Univ. Apul. **24** (2010), 231–238.

[**15**] Z. Nehari, E. Netanyahu, *On the coefficients of meromorphic schlicht functions*, Proc. Amer. Math. Soc. **8** (1957), 15–23.

[**16**] V. Pescar, *A new generalization of Ahlfor's and Becker's criterion of univalence*, Bull. Malaysian Math. Soc. **19** (1996), 53–54.

[**17**] V. Pescar, *Univalence of certain integral operators*, Acta Univ. Apul. **12** (2006), 43-48.

[**18**] Ch. Pommerenke, *Über einige Klassen meromorpher schlichter Funktionen*, Math. Z. **78** (1962), 263–284.

[**19**] W.C. Royster, *Meromorphic starlike multivalent functions*, Trans. Amer. Math. Soc. 107 (1963), 300–308.

[**20**] B.A. Uralegaddi, M.D. Ganigi, S.M. Sarangi, *Univalent functions positive coefficients*, Tamkang J. Math. **25** (1994), 225–230.

[**21**] Zhi-Gang Wang, Yong Sun, Zhi-Hua Zhang, *Certain classes of meromorphic multivalent functions*, Computers Math. Appl. **58** (2009), 1408–1417.

(received 08.12.2010; in revised form 31.05.2011; available online 01.07.2011)

Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box: 130095 Mafraq, Jordan

E-mail: bafrasin@yahoo.com