МАТЕМАТИЧКИ ВЕСНИК
64, 3 (2012), 223–231
September 2012

оригинални научни рад
research paper

SOLUTION OF NONLINEAR INTEGRAL EQUATIONS VIA FIXED POINT OF GENERALIZED CONTRACTIVE CONDITION

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Abstract. The main aim of our paper is to prove the existence of a solution of a system of simultaneous Voltera-Hammerstein nonlinear integral equations by the help of a common fixed point theorem satisfying a generalized contractive condition. For, we have used a common fixed point result of generalized contractive condition in a complete metric space for two pairs of weakly compatible mappings.

1. Introduction

In 2002, Branciari introduced the notion of contractions of integral type and proved fixed point theorem for this class of mappings. Further results on this class of mappings were obtained by [2, 3, 4, 12]. Zhang [13] and Abbas and Rhoades [1] replaced the integral operator by a monotone nondecreasing function. By $\mathcal{F}$ we denote the set of all continuous, monotone nondecreasing real-valued function $F : [0,\infty) \to [0,\infty)$ such that $F(x) = 0$ if and only if $x = 0$. In [13], following results were proved:

LEMMA 1.1. (Zhang [13]) *Let* $F \in \mathcal{F}$ *and* $\epsilon_n \subseteq [0,\infty)$. *Then* $\lim_{n\to\infty} F(\epsilon_n) = 0$ *implies* $\lim_{n\to\infty} \epsilon_n = 0$.

Let $a \in (0,+\infty]$, $R_a^+ = [0,a)$ and $\psi : R_a^+ \to R_+$. Then define the family $\Psi[0,a)$ of ψ by: $\Psi[0,a) := \{\psi : \psi \text{ satisfies (i)-(iii)}\}$, where:

(i) $\psi(t) < t$ for each $t \in (0,a)$,

(ii) ψ is non-decreasing and right upper semi-continuous,

(iii) $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \in (0,a)$.

LEMMA 1.2. (Zhang [13]). *If* $\psi \in \Psi[0,a)$, *then* $\psi(0) = 0$.

We extend the following theorem of Zhang for a quadruple of mappings:

2010 AMS Subject Classification: 47H10, 54H25.

Keywords and phrases: Common fixed point; generalized contractive condition; nonlinear integral equation; weakly compatible mappings.

THEOREM 1.3. (Zhang [13]) *Let (X, d) be a complete metric space and let $D = \sup\{d(x, y) : x, y \in X\}$. Set $a = D$, if $D = \infty$ and $a > D$, if $D < \infty$. Suppose that $A, B : X \to X$, $F \in \Im[0, a)$ and $\psi \in \Psi[0, F(a - 0))$ satisfy:*

$$F(d(Ax, By)) \le \psi(F(m(x, y))), \quad \forall x, y \in X,$$

where $m(x, y) = \max\{d(x, y), d(Ax, x), d(By, y), \frac{1}{2}[d(Ax, y) + d(By, x)]\}$. Then A and B have a unique common fixed point in X. Moreover for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Ax_{2n}$ and $x_{2n+2} = Bx_{2n+1}$ converges to the common fixed point of A and B.

Besides, Jungck [5] introduced compatible mappings, defined below, in a metric space as a generalization of commuting mappings and weakly commuting mappings [11]. This was further generalized to weakly compatible mappings by Jungck [6].

DEFINITION 1. Let A and S be two self-maps of a metric space (X, d). The pair (A, S) is said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

DEFINITION 2. Let $A, S : X \to X$, then the pair (A, S) is said to be weakly compatible if they commute at their coincidence points; i.e., $ASu = SAu$ whenever $Au = Su$, for some $u \in X$.

Compatible mappings are weakly compatible, but the converse need not be true.

2. Main results

The following Theorem 2.1 is a special case of Theorem 1 of Jungck and Rhoades [7]. First we use the following theorem, then we apply this theorem to prove the existence solution of a system of nonlinear Voltera-Hammerstein integral equation.

THEOREM 2.1. *Let (X, d) be a complete metric space and $A, B, S, T : X \to X$ be four maps. Suppose $F \in \Im[0, a)$ and $\psi \in \Psi[0, F(a - 0))$ are functions satisfying:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,

(ii) $F(d(Ax, By)) \le \psi(F(\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(By, Sx) + d(Ax, Ty)]\}))$ *for each $x, y \in X$.*

(iii) *If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S, T have a unique common fixed point in X.*

2.1. Solution of nonlinear integral equations.

Now, we give the following application to Theorem 2.1 in the line of Pathak et. al. [8, 9, 10]. Consider the following simultaneous Voltera-Hammerstein nonlinear integral equations:

$$x(t) = w(t, x(t)) + \mu \int_0^t m(t, s) g_i(s, x(s))\, ds + \lambda \int_0^\infty k(t, s) h_j(s, x(s))\, ds \qquad (2.1)$$

for all $t \in [0,\infty)$, where $w(t,x(t)) \in L[0,\infty)$ is known, $m(t,s)$, $k(t,s)$, $g_i(s,x(s))$ and $h_j(s,x(s))$, $i,j = 1,2$ and $i \neq j$ are real or complex valued functions that are measurable both in t and s on $[0,\infty)$ and λ, μ are real or complex numbers. These functions satisfy the following conditions:

Condition (C_0): The integral $\int_0^\infty |w(s,x(s))|\,ds$ is bounded for all $x(s) \in L[0,\infty)$ and there exists $K_0 > 0$ such that for each $s \in [0,\infty)$,

$$|w(s,x(s)) - w(s,y(s))| \leq K_0|x(s) - y(s)|, \quad \forall x,y \in L[0,\infty).$$

Condition (C_1):

$$\int_0^\infty \sup_{0\leq s<\infty} |m(t,s)|\,dt = M_1 < +\infty.$$

Condition (C_2):

$$\int_0^\infty \sup_{0\leq s<\infty} |k(t,s)|\,dt = M_2 < +\infty.$$

Condition (C_3): $g_i(s,x(s)) \in L[0,\infty)$, $\forall x \in L[0,\infty)$ and there exists $K_1 > 0$ such that for all $s \in [0,\infty)$,

$$|g_1(s,x(s)) - g_2(s,y(s))| \leq K_1|x(s) - y(s)|, \quad \forall x,y \in L[0,\infty).$$

Condition (C_4): $h_i(s,x(s)) \in L[0,\infty)$ for all $x \in L[0,\infty)$ and there exists $K_2 > 0$ such that for all $s \in [0,\infty)$,

$$|h_1(s,x(s)) - h_2(s,y(s))| \leq K_2|x(s) - y(s)|, \quad \forall x,y \in L[0,\infty).$$

The existence theorem can be formulated as follows:

THEOREM 2.2. *Let F and ψ be two functions as defined in Theorem 2.1. If in addition to assumptions $(C_0)--(C_4)$, the following conditions are also satisfied:*

(a) For $i,j = 1,2$ with $i \neq j$,

$$\lambda \int_0^\infty k(t,s)h_i(s,w(s,x(s)) + \mu \int_0^s m(s,\tau)g_j(\tau,x(\tau))\,d\tau)\,ds = 0.$$

(b) For some $x \in L[0,\infty)$,

$$\begin{aligned}\mu \int_0^t m(t,s)g_i(s,x(s))\,ds &= x(t) - w(t,x(t)) - \lambda \int_0^\infty k(t,s)h_i(s,x(s))\,ds \\ &= \Gamma_i(t) \in L[0,\infty).\end{aligned}$$

If for some $\Gamma_i(t) \in L[0,\infty)$, there exists $\theta_i(t) \in L[0,\infty)$ such that:

$$\begin{aligned}(c)\quad \mu \int_0^t m(t,s)g_i(s,x(s) - \Gamma_i(s))\,ds &= w(t,x(t)) + \lambda \int_0^\infty k(t,s)h_i(s,x(s) - \Gamma_i(s))\,ds \\ &= \theta_i(t), \quad i = 1,2,\end{aligned}$$

then the system of simultaneous Voltera-Hammerstein nonlinear integral equation (2.1) *has a unique solution in* $L[0,\infty)$ *for each pair of real or complex numbers* λ, μ *with*

$$K_0 + |\mu| M_1 K_1 + |\lambda| M_2 K_2 < 1 \quad \text{and}$$
$$F(|\mu| M_1 K_1 p) \le \psi(1 - (K_0 + |\lambda| M_2 K_2) p), \quad p \ge 0. \tag{2.2}$$

Proof. Comparing the notation with Theorem 2.1, here $X = L[0,\infty)$. For every $x(s) \in L[0,\infty)$, we define the mappings A, B, S, T by:

$$Ax(t) = \mu \int_0^t m(t,s) g_1(s, x(s))\, ds, \quad Bx(t) = \mu \int_0^t m(t,s) g_2(s, x(s))\, ds,$$
$$Sx(t) = (I - C)x(t) \quad \text{and} \quad Tx(t) = (I - D)x(t),$$

where

$$Cx(t) = w(t, x(t)) + \lambda \int_0^\infty k(t,s) h_1(s, x(s))\, ds,$$
$$Dx(t) = w(t, x(t)) + \lambda \int_0^\infty k(t,s) h_2(s, x(s))\, ds.$$

Here $w(t, x(t)) \in L[0,\infty)$ is known and I is the identity operator on $L[0,\infty)$. First, let us show that each A, B, C, D, S, T are operators from $L[0,\infty)$ into itself.

Indeed, we have
$|Ax(t)| \le |\mu| \int_0^\infty |m(t,s)|.|g_1(s, x(s))|\, ds \le |\mu| \sup_{0\le s<\infty} |m(t,s)| \int_0^\infty |g_1(s, x(s))|\, ds$
applying conditions (C_1) and (C_3) and thus, we have
$\int_0^\infty |Ax(t)| dt \le |\mu| \int_0^\infty \sup_{0\le s<\infty} |m(t,s)|\, dt \int_0^\infty |g_1(s, x(s))|\, ds < +\infty$
and hence $Ax \in L[0,\infty)$. Similarly $Bx \in L[0,\infty)$.

For mapping C, we apply conditions (C_2) and (C_4) in the following way:
$\int_0^\infty |Cx(t) dt| \le \int_0^\infty |w(t, x(t))|\, dt + |\lambda| \int_0^\infty \sup\limits_{0\le s<\infty} |k(t,s)|\, dt \int_0^\infty |h_1(s, x(s))|\, ds < +\infty$, as $\int_0^\infty |w(t, x(t))|\, dt$ is bounded and hence C is a self operator on $L[0,\infty)$.

A similar argument is valid for D. Similarly S and $T \in L[0,\infty)$. Hence A, B, C, D, S, T are operators from $L[0,\infty)$ into itself.

Let us show the condition (i) of Theorem 2.1. First, to prove $A(X) \subseteq T(X)$, i.e., $A(L[0,\infty)) \subseteq T(L[0,\infty))$, let $x(t) \in L[0,\infty)$ be arbitrary, then we have

$$\begin{aligned} T(Ax(t) + w(t, x(t))) &= (I - D)(Ax(t) + w(t, x(t))) \\ &= Ax(t) - \lambda \int_0^\infty k(t,s) h_2(s, Ax(s) + w(s, x(s)))\, ds \\ &= Ax(t) - \lambda \int_0^\infty k(t,s) h_2[s, \mu \int_0^s m(s,\tau) g_1(\tau, x(\tau))\, d\tau + w(s, x(s))]\, ds \\ &= Ax(t), \quad \text{by assumption (a).} \end{aligned}$$

Thus $A(L[0,\infty)) \subseteq T(L[0,\infty))$. Similarly $B(L[0,\infty)) \subseteq S(L[0,\infty))$.

Further, we check (ii) of Theorem 2.1. Suppose $x, y \in L[0,\infty)$. Then LHS is:

$$\begin{aligned}
F(\|Ax - By\|) &= F(\int_0^\infty |Ax(t) - By(t)|\,dt), \quad \text{by the definition of } \|\cdot\| \\
&= F(\int_0^\infty |\mu \int_0^t m(t,s)[g_1(s,x(s)) - g_2(s,x(s))]\,ds|\,dt) \\
&\le F(\int_0^\infty |\mu| \sup_{0\le s<\infty} |m(t,s)|\,dt \int_0^\infty |g_1(s,x(s)) - g_2(s,y(s))|\,ds) \\
&\le F(|\mu| M_1 \int_0^\infty K_1 |x(s) - y(s)|\,ds), \quad \text{by } (C_1) \text{ and } (C_3) \\
&= F(|\mu| M_1 K_1 \|x - y\|).
\end{aligned}$$

Thus we obtain

$$F(\|Ax - By\|) \le F(|\mu| M_1 K_1 \|x - y\|). \tag{2.3}$$

Similarly, using assumptions (C_0), (C_2) and (C_4), we get

$$\begin{aligned}
\|Cx - Dy\| = &\int_0^\infty |w(t,x(t)) - w(t,y(t)) \\
&+ \lambda \int_0^\infty k(t,s)[h_1(s,x(s)) - h_2(s,y(s))]\,ds|\,dt \\
\le &\int_0^\infty |w(t,x(t)) - w(t,y(t))|\,dt \\
&+ |\lambda| \int_0^\infty \sup_{0\le s<\infty} |k(t,s)|\,dt. \int_0^\infty |h_1(s,x(s)) - h_2(s,y(s))\,ds| \\
\le &(K_0 + |\lambda| M_2 K_2)\|x - y\|.
\end{aligned}$$

Thus we obtain

$$\|Cx - Dy\| \le (K_0 + |\lambda| M_2 K_2)\|x - y\|. \tag{2.4}$$

Hence, for the RHS of (ii), we have

$$\begin{aligned}
&F(M(x,y)) \\
&\quad = F(\max\{\|Sx - Ty\|, \|Ax - Sx\|, \|By - Ty\|, \frac{1}{2}[\|By - Sx\| + \|Ax - Ty\|]\}) \\
&\quad \ge F(\|Sx - Ty\|), \quad \text{as } F \text{ is non-decreasing} \\
&\quad = F(\|(I - C)x - (I - D)y\|) \\
&\quad = F(\|x - y\| - \|Cx - Dy\|), \quad \text{by the triangle property of } \|\cdot\| \\
&\quad \ge F(\|x - y\| - (K_0 + |\lambda| M_2 K_2)\|x - y\|), \quad \text{by (2.4)} \\
&\quad = F(\{1 - K_0 - |\lambda| M_2 K_2\}\|x - y\|).
\end{aligned}$$

Thus we obtain

$$F(M(x,y)) \ge F(\{1 - K_0 - |\lambda| M_2 K_2\}\|x - y\|). \tag{2.5}$$

Next, since the function ψ is non-decreasing, so that

$$\begin{aligned}\psi(F(M(x,y))) &\geq \psi(F(\{1-K_0-|\lambda|M_2K_2\}.\|x-y\|)), \quad \text{by } (2.5)\\ &\geq F(|\mu|M_1K_1\|x-y\|), \quad \text{by } (2.2)\\ &\geq F(\|Ax-By\|), \quad \text{by } (2.3).\end{aligned}$$

Thus the generalized contractive condition (ii) of Theorem 2.1 is satisfied.

Now we prove that the pair (A,S) is weakly compatible. For this we have

$$\begin{aligned}\|SAx(t)-ASx(t)\| &= \|(I-C)Ax(t)-A(I-C)x(t)\|\\ &= \|Ax(t)-CAx(t)-Ax(t)+ACx(t)\| = \|ACx(t)-CAx(t)\| \end{aligned}\tag{2.6}$$

Now whenever $Ax(t)=Sx(t)$, we have

$$\mu\int_0^t m(t,s)g_1(s,x(s))\,ds = x(t)-w(t,x(t))-\lambda\int_0^\infty k(t,s)h_1(s,x(s))\,ds. \tag{2.7}$$

Using eq. (2.7) in (2.6), we get

$$\begin{aligned}\|SAx(t)-ASx(t)\| &= \|ACx(t)-CAx(t)\|\\ &= \|AC[w(t,x(t))+\lambda\int_0^\infty k(t,s)h_1(s,x(s))\,ds+\mu\int_0^t m(t,s)g_1(s,x(s))\,ds]\\ &\quad - CA[w(t,x(t))+\lambda\int_0^\infty k(t,s)h_1(s,x(s))\,ds+\mu\int_0^t m(t,s)g_1(s,x(s))\,ds]\|\\ &= \|A[w(t,x(t))+\lambda\int_0^\infty k(t,s)h_1(s,x(s)-\Gamma_1(s))\,ds]\\ &\quad - C[\mu\int_0^t m(t,s)g_1(s,x(s)-\Gamma_1(s))\,ds]\|\\ &= \|\mu\int_0^t m(t,s)g_1[s,w(s,x(s))+\lambda\int_0^\infty k(s,\tau)h_1(\tau,x(\tau)-\Gamma_1(\tau))\,d\tau]\,ds\\ &\quad - w(t,x(t))-\lambda\int_0^\infty k(t,s)h_1[s,\mu\int_0^s m(s,\tau)g_1(\tau,x(\tau)-\Gamma_1(\tau))\,d\tau]\,ds\|\\ &= 0, \quad \text{from } (2.1).\end{aligned}$$

This shows that the pair (A,S) is weakly compatible. Similarly (B,T) is also weakly compatible. Hence all the conditions of our Theorem 2.1 are satisfied and the solution of eq. (2.1) exists.

Finally, let us show the uniqueness of solution, let $v(t)\in L[0,\infty)$ be another solution of (2.1), then by (C_0)–(C_4), we have

$$\begin{aligned}\|u(t)-v(t)\| \leq \int_0^\infty &|w(t,u(t))-w(t,v(t))+\mu\int_0^t m(t,s)(g_1(s,u(s))\\ &- g_2(s,v(s)))\,ds+\lambda\int k(t,s)(h_1(s,u(s))-h_2(s,v(s)))\,ds|\,dt\end{aligned}$$

$$\le (K_0 + |\mu| M_1 K_1 + |\lambda| M_2 K_2)\|u(t) - v(t)\| < \|u(t) - v(t)\|$$

a contradiction. Thus the solution is unique. This completes the proof. ■

We have shown in the proof of Theorem 2.2 that, all the conditions (i)–(iii) of Theorem 2.1 are satisfied, and also that the nonlinear V-H integral equation (2.1) of Theorem 2.2 has a unique solution in $L[0,\infty)$.

Below we put $A = B$, $S = T$, $L[0,\infty) = BC[0,\infty)$, $g_1 = g_2 = g$, $h_1 = h_2 = h$, $\lambda = \mu = 1$ in eq. (2.1) of Theorem 2.2, to get the following Example 2.3, as a reduced nonlinear integral equation (2.8). Obviously, for these special values in this example, all the conditions (i)–(iii) of Theorem 2.1 are satisfied.

Now, by another method, we will show below that, the nonlinear integral equation have a unique solution in $BC[0,\infty)$. This will completely validate our Theorem 2.2.

EXAMPLE 2.3. Consider the following nonlinear integral equation in $BC[0,\infty)$:

$$x(t) = w(t, x(t)) + \int_0^t m(t,s) g(s, x(s))\, ds + \int_0^\infty k(t,s) h(s, x(s))\, ds \tag{2.8}$$

Let P and Q be two operators from $BC[0,\infty]$ into itself as defined below:

$$(Px)(t) = \int_0^t m(t,s).x(s)\, ds \quad \text{and} \quad (Qx)(t) = \int_0^\infty k(t,s).x(s)\, ds$$

Let the following conditions hold:

(C_0): $|w(t,x(t)) - w(t,y(t))| \le r|x(t) - y(t)|, \forall x(t), y(t) \in B_r$ and $r \ge 0$ a constant.

(C_1): $m(t,s)$ is such that $Px(t)$ is continuous operator from $BC[0,\infty]$ into itself.

(C_2): $k(t,s)$ is such that $Qx(t)$ is continuous operator from $BC[0,\infty]$ into itself.

(C_3): $|g(t,x(t)) - g(t,y(t))| \le K_1|x(t) - y(t)|$, $\forall x(t), y(t) \in B_r$ and $K_1 \ge 0$ a constant.

(C_4): $|h(t,x(t)) - h(t,y(t))| \le K_2|x(t) - y(t)|$, $\forall x(t), y(t) \in B_r$ and $K_2 \ge 0$ a constant.

Then there exists a unique solution of (2.8) provided $M_1K_1 + M_2K_2 + r < 1$ and $|w(t,x(t))| + M_1|g(t,0)| + M_2|h(t,0)| \le r(1 - M_1K_1 - M_2K_2)$, where M_1, M_2 are norms of P and Q, respectively.

Proof. Suppose the mappings A, B, S, T are defined below:

$(Ux)(t) = w(t,x(t)) + \int_0^s m(t,s) g(s,x(s))\, ds = w(t,x(t)) + (Ax)(t)$,

$(Uy)(t) = w(t,y(t)) + \int_0^s m(t,s) g(s,y(s))\, ds = w(t,y(t)) + (By)(t)$,

$(Vx)(t) = \int_0^\infty k(t,s) h(s,x(s))\, ds = (Cx)(t) - w(t,x(t)) = x(t) - (Sx)(t) - w(t,x(t))$,

$(Vy)(t) = \int_0^\infty k(t,s) h(s,y(s))\, ds = (Dx)(t) - w(t,y(t)) = y(t) - (Ty)(t) - w(t,y(t))$.

The operators U and V from B_r into $BC[0,\infty)$ defined above are Banach spaces. We show that $(U+V) : B_r \to B_r$ is a contraction. For, let $x \in B_r$ then

$$|U(x)(t) + V(x)(t)| \le |w(t,x(t))| + M_1|g(t,x(t))| + M_2|h(t,x(t))|$$

We have the following inequalities

$$|g(t,x(t))| \leq |g(t,x(t)) - g(t,0)| + |g(t,0)| \leq K_1|x(t)| + |g(t,0)| \tag{2.9}$$

Similarly,

$$|h(t,x(t))| \leq |h(t,x(t)) - h(t,0)| + |h(t,0)| \leq K_2|x(t)| + |h(t,0)| \tag{2.10}$$

From (2.9) and (2.10) we have

$$|U(x)(t)+V(x)(t)| \leq |w(t,x(t))|+K_1M_1|x(t)|+M_1|g(t,0)|+M_2|x(t)|+M_2|h(t,0)|.$$

Since by assumption

$$|w(t,x(t))| + M_1|g(t,0)| + M_2|h(t,0)| \leq 1 - K_1M_1 - K_2M_2$$

we have

$$|U(x)(t) + V(x)(t)| \leq r(1 - K_1M_1 - K_2M_2) + K_1M_1r + K_2M_2r = r.$$

Thus $U(x)(t) + V(x)(t) \in B_r$. Also

$$\begin{aligned} &|U(x)(t) + V(x)(t) - U(y)(t) - V(y)(t)| \\ &\qquad \leq r|x(t) - y(t)| + K_1M_1|x(t) - y(t)| + K_2M_2|x(t) - y(t)| \\ &\qquad = (r + K_1M_1 + K_2M_2)|x(t) - y(t)|. \end{aligned}$$

Since $(r + K_1M_1 + K_2M_2) < 1$, so $U + V$ is a contraction on B_r; therefore by Banach contraction theorem there exists a unique solution of (2.8). This completely validates Theorem 2.2.

Acknowledgements. The authors are highly indebted to the referee for his valuable comments and helpful suggestions.

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(received 02.12.2010; in revised form 01.02.2011; available online 20.04.2011)

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