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INTEGRAL AND COMPUTATIONAL REPRESENTATION OF SUMMATION WHICH EXTENDS A RAMANUJAN'S SUM

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Abstract. A generalized sum, which contains Ramanujan's summation formula recorded in Hardy's article [G.H. Hardy, *A chapter from Ramanujan's notebook*, Proc. Camb. Phil. Soc. **21** (1923), 492–503] as a special case, has been represented in the form of Mellin-Barnes type contour integral. A computational representation formula is derived for this summation in terms of the unified Hurwitz-Lerch Zeta function.

1. Introduction

The generalized hypergeometric function with p numerator and q denominator parameters is defined [6] as the power series

$$ {}_pF_q(z) = {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \Big| z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}\,, \tag{1}$$

where $(a)_n$ denotes the *Pochhammer or schifted factorial symbol* defined in terms of the familiar Gamma function

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (\ n=0,\ a \neq 0); \\ a(a+1)\cdots(a+n-1) & (\ n \in \mathbb{N},\ a \in \mathbb{C}). \end{cases} \tag{2}$$

When $p \leq q$ the series (1) converges for all finite values of z and defines an entire function. If one or more of the top parameters a_j is a nonpositive integer, the series terminates and the generalized hypergeometric function is a polynomial in z. If $p = q+1$, series converges in the open unit disc $|z| < 1$, while on the unit circle the generalized hypergeometric series is absolutely convergent if

$$\Delta := \Re\Big\{ \sum_{j=1}^{q} b_j - \sum_{j=1}^{n} a_j \Big\} > 0\,. \tag{3}$$

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One of the Ramanujan's curious summation recorded by Hardy [1, p. 495] is given by

$$\mathfrak{R}_1(x) := 1 + 3\Big\{\frac{x-1}{x+1}\Big\}^2 + 5\Big\{\frac{(x-1)(x-2)}{(x+1)(x+2)}\Big\}^2 + \cdots = \frac{x^2}{2x-1}\,. \tag{4}$$

Ten years ago Park and Seo [5, Eq. (1.1)] proved that summation (4) can be expressed *via* the generalized hypergeometric function ${}_4F_3$ as

$$\mathfrak{R}_1(x) = {}_4F_3\left[\begin{matrix} 1,\, 3/2,\, 1-x,\, 1-x \\ 1/2,\ 1+x,\ 1+x \end{matrix}\,\Big|\, 1\right] = \frac{x^2}{2x-1}\,. \tag{5}$$

Their rather long proving procedure includes the use of higher order generalized hypergeometric series. However, we remark that to show (5) it is enough to apply formula [3]

$${}_4F_3\left[\begin{matrix} a,\, a/2+1,\, b,\, c \\ a/2,\, a-b+1,\, a-c+1 \end{matrix}\,\Big|\, 1\right] = \frac{\Gamma\big(\frac{a+1}{2}\big)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma\big(\frac{a+1}{2}-b-c\big)}{\Gamma(a+1)\Gamma\big(\frac{a+1}{2}-b\big)\Gamma\big(\frac{a+1}{2}-c\big)\Gamma(a-b-c+1)}\,,$$

where $\Re\{a-2b-2c\} > -1$, specifying $a = 1, b = c = 1-x$ for some x such, that $\Re\{x\} > 1/2$.

The main goal of this short note is to derive a closed expression for a general summation formula $\mathfrak{R}^s_{p,q}(\alpha;x)$ in the form of a Mellin-Barnes type contour integral which contains $\mathfrak{R}_1(x)$ as a special case, see (6); second, to give a computational representation for series $\mathfrak{R}^s_{p,q}(\alpha;x)$, and a new formula achieved *via* (4).

2. Extension of $\mathfrak{R}_1(x)$

Let us denote $\mathbb{Q}^+$ the set of positive rationals, while $\mathbb{N}_0$ stands for the set of non-negative integers and $\mathbb{N}_2 = \{2, 3, \cdots\}$. Consider the sum

$$\mathfrak{R}^s_{p,q}(\alpha;x) = \sum_{n=0}^{\infty}(n+\alpha)^s\frac{[(x-1)\cdots(x-n)]^p}{[(x+1)\cdots(x+n)]^q} \qquad \alpha\in\mathbb{Q}^+,\, p,q,s\in\mathbb{N}_2\,. \tag{6}$$

Obviously $\mathfrak{R}_1(x) = 2\,\mathfrak{R}^1_{2,2}(1/2;x)$.

Theorem 1. *For all*

$$\max\Big\{0, \frac{1+s+p-q}{p+q}\Big\} < \Re\{x\} < 1$$

the following integral representation holds true

$$\mathfrak{R}^s_{p,q}(\alpha;x) = \frac{\Gamma^q(1+x)}{2\pi \mathrm{i}\Gamma^p(1-x)}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\frac{\Gamma(\xi)\Gamma(1-\xi)\Gamma^s(\alpha+1-\xi)\Gamma^p(1-x-\xi)}{\Gamma^s(\alpha-\xi)\Gamma^q(1+x-\xi)[-(-1)^p]^\xi}\,\mathrm{d}\xi\,, \tag{7}$$

where $\gamma\in\big(0, 1-\Re\{x\}\big)$.

Proof. It is obvious that

$$n+\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\cdot\frac{(\alpha+1)_n}{(\alpha)_n} = \alpha\cdot\frac{(\alpha+1)_n}{(\alpha)_n}\,,$$

therefore

$$\mathfrak{R}_{p,q}^{s}(\alpha;x) = \alpha^s \sum_{n=0}^{\infty} \frac{(1)_n[(\alpha+1)_n]^s[(1-x)_n]^p}{[(\alpha)_n]^s[(1+x)_n]^q}\, \frac{[(-1)^p]^n}{n!}\,.$$

Reading the last expression in the spirit of the definition (1) of generalized hypergeometric function ${}_pF_q$ we clearly conclude that

$$\mathfrak{R}_{p,q}^{s}(\alpha;x) = \alpha^s \cdot {}_{p+s+1}F_{q+s}\Big[\begin{matrix} 1, \overbrace{\alpha+1,\cdots,\alpha+1}^{s}, \overbrace{1-x,\cdots,1-x}^{p} \\ \underbrace{\alpha,\cdots,\alpha}_{s}, \underbrace{1+x,\cdots,1+x}_{q} \end{matrix} \Big| (-1)^p \Big]\,. \tag{8}$$

Because the argument of this special function is unimodal, we have to test the convergence of this series. However, the condition (3) gives us $\Delta = q-s-p-1+(p+q)\Re\{x\}>0$ if $p=q$, which is fulfilled by assumption of the Theorem.

Now, consider the following Mellin-Barnes type contour integral *viz.*

$$\mathfrak{I}(z) = \frac{\Gamma^q(1+x)}{2\mathrm{i}\pi\Gamma^p(1-x)} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{\Gamma(\xi)\Gamma(1-\xi)\Gamma^s(\alpha+1-\xi)\Gamma^p(1-x-\xi)}{\Gamma^s(\alpha-\xi)\Gamma^q(1+x-\xi)(-z)^\xi}\,\mathrm{d}\xi\,;$$

here the integration path is of Bromwich-Wagner type, that is, it consists from a straight line orthogonal to the real axis at $\gamma \in \big(0, 1-\Re\{x\}\big)$, which starts at $\gamma-\mathrm{i}\infty$, and terminates at the point $\gamma+\mathrm{i}\infty$. The simple poles of the integrand $\xi_n^{(1)} = -n+1, n\in\mathbb{N}$ have been separated by the integration path $\mathfrak{L}$ from other poles (because γ's definition). Calculating the residues of the function $\Gamma(\xi)$ at the values $\xi_n^{(1)}$ we easily find that

$$\begin{aligned}
\mathfrak{I}(z) &= \frac{\Gamma^q(1+x)}{\Gamma^p(1-x)} \sum_{n=0}^{\infty} \mathrm{Res}\big[\Gamma(\xi);-n\big]\cdot \frac{\Gamma(1+n)\Gamma^s(\alpha+1+n)\Gamma^p(1-x+n)(-z)^n}{\Gamma^s(\alpha+n)\Gamma^q(1+x+n)} \\
&= \frac{\alpha^s\Gamma^s(\alpha)\Gamma^q(1+x)}{\Gamma^s(\alpha+1)\Gamma^p(1-x)} \sum_{n=0}^{\infty} \frac{(1)_n\,\Gamma^s(\alpha+1+n)\Gamma^p(1-x+n)}{\Gamma^s(\alpha+n)\Gamma^q(1+x+n)}\,\frac{z^n}{n!} \\
&= \alpha^s \sum_{n=0}^{\infty} \frac{(1)_n[(\alpha+1)_n]^s[(1-x)_n]^p}{[(\alpha)_n]^s[(1+x)_n]^q}\,\frac{z^n}{n!} \\
&= \alpha^s \cdot {}_{p+s+1}F_{q+s}\Big[\begin{matrix} 1, \overbrace{\alpha+1,\cdots,\alpha+1}^{s}, \overbrace{1-x,\cdots,1-x}^{p} \\ \underbrace{\alpha,\cdots,\alpha}_{s}, \underbrace{1+x,\cdots,1+x}_{q} \end{matrix} \Big| z \Big]\,.
\end{aligned}$$

Thus, we deduce

$$\mathfrak{R}_{p,q}^{s}(\alpha;x) = \mathfrak{I}\big((-1)^p\big)\,.$$

This finishes the proof of Theorem 1. ■

Corollary 1. *For all x, $\Re\{x\} \in (1/2, 1)$ we have*

$$\frac{1}{\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{\Gamma(\xi)\Gamma(1-\xi)\Gamma(3/2-\xi)\Gamma^2(1-x-\xi)}{\Gamma(1/2-\xi)\Gamma^2(1+x-\xi)(-1)^\xi}\,\mathrm{d}\xi = \frac{\Gamma^2(1-x)}{(2x-1)\Gamma^2(x)}\,, \tag{9}$$

where $\gamma \in (0, 1/2)$.

Proof. Recalling equality $\mathfrak{R}_1(x) = 2\,\mathfrak{R}_{2,2}^1(1/2; x)$, by Theorem 1 and (7) we get

$$\mathfrak{R}_1(x) = \frac{\Gamma^2(1+x)}{\pi \mathrm{i}\Gamma^2(1-x)} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{\Gamma(\xi)\Gamma(1-\xi)\Gamma(3/2-\xi)\Gamma^2(1-x-\xi)}{\Gamma(1/2-\xi)\Gamma^2(1+x-\xi)(-1)^\xi}\,\mathrm{d}\xi\,.$$

Since $\mathfrak{R}_1(x) = x^2(2x-1)^{-1/2}$ by (4), obvious further transformations lead to the asserted formula (9). ■

3. Computational representation for $\mathfrak{R}_{p,q}^s(\boldsymbol{\alpha}; \boldsymbol{x})$

Next, we give a computational representation of extended Ramanujan's sum $\mathfrak{R}_{p,q}^s(\alpha; x)$. First we introduce the so-called *unified Hurwitz-Lerch Zeta function*, a new special function defined recently by Srivastava et al. [4]. Thus, for parameters $p, q \in \mathbb{N}_0$; $\lambda_j \in \mathbb{C}$, $\mu_k \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\sigma_j, \rho_k > 0$, $j = \overline{1,p}$, $k = \overline{1,q}$, the unified Hurwitz-Lerch Zeta function with $p+q$ both upper and lower, and two auxiliary parameters, reads as follows

$$\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z, w, a) := \Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z, w, a) = \sum_{n=0}^\infty \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{\prod_{j=1}^q (\mu_j)_{n\sigma_j}}\, \frac{z^n}{(n+a)^w\, n!}\,; \tag{10}$$

the auxiliary parameters $w \in \mathbb{C}, \Re\{a\} > 0$; the empty product is taken to be unity (if any). The series (10) converges

1. for all $z \in \mathbb{C} \setminus \{0\}$ if $\Omega > -1$;
2. in the open disc $|z| < \nabla'$ if $\Omega = -1$;
3. on the circle $|z| = \nabla'$, for $\Omega = -1$, $\Re\{\Xi\} > 1/2$,

   where

$$\nabla' := \prod_{j=1}^q \sigma_j^{\sigma_j} \prod_{j=1}^p \rho_j^{-\rho_j}, \quad \Omega := \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j + \Re\{w\}, \quad \Xi := \sum_{j=1}^q \mu_j - \sum_{j=1}^p \lambda_j + \frac{p-q}{2}. \tag{11}$$

Theorem 2. *Let the situation be the same as in the previous theorem. Then we have*

$$\begin{aligned}\mathfrak{R}_{p,q}^s(\alpha; x) &= \frac{p(-1)^{(p+1)(x-1)+s}\,\Gamma^q(1+x)}{\Gamma^q(2x)\Gamma^{p-1}(1-x)} \sum_{n=0}^\infty \frac{H_{n+1}(1-x)_n[(1-2x)_n]^q}{[(1)_n]^{p-1}(1-x-\alpha+n)^{-s}}\, \frac{(-1)^{(q+1)n}}{n!} \\ &\quad - p\frac{\boldsymbol{\gamma}(-1)^{(p+1)(x-1)+s}\,\Gamma^q(1+x)}{\Gamma^q(2x)\Gamma^{p-1}(1-x)}\, \Phi_{1-x,(\mathbf{1-2x})_q;(\mathbf{1})_{p-1}}^{((\mathbf{1})_{q+1};(\mathbf{1})_{p-1})}\big((-1)^{q+1}, -s, 1-x-\alpha\big) \\ &\quad + (-1)^s x^{q-p}\, \Phi_{1,(\mathbf{1-x})_q;(\mathbf{1+x})_p}^{((\mathbf{1})_{q+1};(\mathbf{1})_p)}\big((-1)^q, -s, 1-\alpha\big)\,,\end{aligned} \tag{12}$$

where H_n denotes the nth harmonic number, $\boldsymbol{\gamma}$ stands for the Euler-Mascheroni constant and $(\boldsymbol{a})_\nu := \underbrace{a, \cdots, a}_{\nu}$.

Proof. It is easy to see that integral $\mathfrak{I}(z)$ can be rewritten into equivalent form

$$\mathfrak{I}(z) = \frac{\Gamma^q(1+x)}{2\mathrm{i}\pi\Gamma^p(1-x)} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{\Gamma(\xi)\Gamma(1-\xi)\Gamma^p(1-x-\xi)(\alpha-\xi)^s}{\Gamma^q(1+x-\xi)\cdot(-z)^\xi}\,\mathrm{d}\xi\,.$$

Now, if we calculate the residues of the function $\Gamma(1-\xi)$ at the simple poles $\xi_n^{(2)} = n+1, n \in \mathbb{N}_0$ and the residues of $\Gamma^p(1-x-\xi)$ at the poles $\xi_n^{(3)} = 1-x+n, n \in \mathbb{N}_0$ of order p, then it yields exactly the asserted formula (12). Indeed, we have

$$\begin{aligned}\mathfrak{I}(z) = \frac{\Gamma^q(1+x)}{\Gamma^p(1-x)} \sum_{n=0}^\infty \Bigg\{ &\frac{\Gamma(1-x+n)(\alpha-1+x-n)^s}{\Gamma^q(2x-n)(-z)^{1-x-n}}\,\mathrm{Res}\big[\Gamma^p(1-x-\xi);\xi_n^{(3)}\big] \\ &+ \frac{\Gamma(n+1)\Gamma^p(-x-n)(\alpha-1-n)^s}{\Gamma^q(x-n)(-z)^{-n-1}}\,\mathrm{Res}\big[\Gamma(1-\xi);\xi_n^{(2)}\big]\Bigg\}\,. \qquad (13)\end{aligned}$$

First, it is well known that

$$\mathrm{Res}\Big[\Gamma(1-\xi);\xi_n^{(2)}\Big] = \frac{(-1)^{n+1}}{n!}\,.$$

On the other hand employing the power series representation formula [2]

$$\begin{aligned}\Gamma(z) = &\frac{(-1)^n}{(z+n)n!} + \frac{(-1)^n\psi(n+1)}{n!} \\ &+ \frac{1}{6}\big(3\psi^2(n+1)+\pi^2-3\psi^{(1)}(n+1)\big)(z+n) + \mathcal{O}\big((z+n)^2\big)\end{aligned}$$

we clearly conclude

$$\begin{aligned}\mathrm{Res}\big[\Gamma^p(1-x-\xi);\xi_n^{(3)}\big] &= \frac{1}{(p-1)!}\lim_{\xi\to\xi_n^{(3)}} \frac{\mathrm{d}^{p-1}}{\mathrm{d}\xi^{p-1}}\Big(\Gamma(1-x-\xi)\big(\xi-\xi_n^{(3)}\big)\Big)^p \\ &= \frac{(-1)^{pn}}{(p-1)!(n!)^p}\lim_{\xi\to\xi_n^{(3)}} \frac{\mathrm{d}^{p-1}}{\mathrm{d}\xi^{p-1}}\Big\{1+\psi(n+1)\big(\xi-\xi_n^{(3)}\big)+\mathcal{O}\Big(\big(\xi-\xi_n^{(3)}\big)^2\Big)\Big\}^p \\ &= \frac{(-1)^{pn}}{(p-1)!(n!)^p}\lim_{\xi\to\xi_n^{(3)}} \frac{\mathrm{d}^{p-1}}{\mathrm{d}\xi^{p-1}}\Big\{1+p\,\psi(n+1)\big(\xi-\xi_n^{(3)}\big)+\mathcal{O}\Big(\big(\xi-\xi_n^{(3)}\big)^2\Big)\Big\} \\ &= \frac{p(-1)^{pn}\psi(n+1)}{(n!)^p} = \frac{p(-1)^{pn}(H_n-\boldsymbol{\gamma})}{\Gamma^p(n+1)}\,;\end{aligned}$$

here $\psi(\cdot)$ denotes the *digamma function*, i.e. $\psi(z) = \Gamma'(z)/\Gamma(z)$. Hence $\mathfrak{I}(z)$ becomes

$$\begin{aligned}\mathfrak{I}(z) = &\frac{p\Gamma^q(1+x)(-1)^{x-1+s}z^{x-1}}{\Gamma^p(1-x)} \sum_{n=0}^\infty \frac{\Gamma(1-x+n)(H_n-\boldsymbol{\gamma})}{\Gamma^q(2x-n)(1-\alpha-x+n)^{-s}}\,\frac{[(-1)^{p+1}z]^n}{\Gamma^p(n+1)} \\ &+ \frac{\Gamma^q(1+x)z(-1)^s}{\Gamma^p(1-x)} \sum_{n=0}^\infty \frac{\Gamma(n+1)\Gamma^p(-x-n)}{\Gamma^q(x-n)(1-\alpha+n)^{-s}}\,\frac{z^n}{n!}\,. \qquad (14)\end{aligned}$$

Transforming in (14) the negative summation index Gamma-function terms into positive index Pochhammer symbols with the aid of the familiar formula

$$(a)_{-n} = \frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n} \qquad a \in \mathbb{C} \setminus \mathbb{Z}_0^-,\, n \in \mathbb{N}_0\,,$$

we arrive at

$$\begin{aligned} \mathfrak{I}(z) = \frac{p\Gamma^q(1+x)(-1)^{x-1+s}z^{x-1}}{\Gamma^q(2x)\Gamma^{p-1}(1-x)} \sum_{n=0}^\infty \frac{(1-x)_n[(1-2x)_n]^q(H_n-\gamma)}{[(1)_n]^{p-1}(1-\alpha-x+n)^{-s}} \frac{[(-1)^{p+q+1}z]^n}{n!} \\ + x^{q-p}z(-1)^{s+p} \sum_{n=0}^\infty \frac{(1)_n[(1-x)_n]^q}{[(1+x)_n]^p(1-\alpha+n)^{-s}} \frac{[(-1)^{p+q}z]^n}{n!}\,. \end{aligned} \tag{15}$$

Setting $z=(-1)^p$ in (15) routine calculations lead to asserted expression (12). ■

Finally, specifying $p=q=2, s=1$ in Theorem 2 we clearly conclude the following formula, far from being obvious by itself.

Corollary 2. *For all* x, $\Re\{x\} \in (1/2,1)$ *it holds true*

$$\begin{aligned} \sum_{n=0}^\infty (1-x)_n[(1-2x)_n]^2(1-x-\alpha+n)\,\frac{H_n}{(n!)^2} = \gamma\Phi^{(1,1,1;1)}_{1-x,1-2x,1-2x;1}\big(-1,-1,1/2\big) \\ + \frac{\Gamma^2(2x)\Gamma(1-x)(-1)^{3x+1}}{2\Gamma^2(1+x)}\Big\{\Phi^{(1,1,1;1,1)}_{1,1-x,1-x;1+x,1+x}\big(1,-1,1/2\big) + \frac{x^2}{2x-1}\Big\}\,. \end{aligned} \tag{16}$$

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