ON THE SPACE OF p-SUMMABLE SEQUENCES

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Abstract. On the space ℓ^p of p-summable sequences (of real numbers), one can derive a norm from the 2-norm as indicated by Gunawan [H. Gunawan, The space of p-summable sequences and its natural n-norms, Bull. Austral. Math. Soc. 64 (2001), 137–147]. The purpose of this note is to establish the equivalence between such a norm and the usual norm on ℓ^p. We show that our result is useful in understanding the topology of ℓ^p as a 2-normed space.

1. Introduction

In 2001, Gunawan [9] showed that the space $\ell^p = \ell_{\mathbb{R}}^p(\mathbf{N})$ of p-summable sequences (of real numbers) can be equipped with a 2-norm. In general, if X is a (real) vector space, then a mapping $\|\cdot, \cdot\| : X \times X \to \mathbb{R}$ which satisfies the following properties:

(1.1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(1.2) $\|x, y\| = \|y, x\|$ for every $x, y \in X$,

(1.3) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for every $x, y \in X$ and $\alpha \in \mathbb{R}$,

(1.4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for every $x, y, z \in X$,

is called a *2-norm* on X, and the pair $(X, \|\cdot, \cdot\|)$ is called a *2-normed space*. The concept of 2-normed spaces was first introduced by Gähler [3] and its generalization can be found in [5–7]. Related works may be found in [2, 4, 8, 12, 14]. See also [1, 10, 11, 13] for some results on this subject in the last decade.

For example, on the space ℓ^p $(1 \leq p \leq \infty)$ that we are interested in now, the following mapping

$$\|x, y\|_p := \left[\frac{1}{2} \sum_k \sum_l \left| \begin{vmatrix} x_k & x_l \\ y_k & y_l \end{vmatrix} \right|^p \right]^{1/p}, \tag{1}$$

where $x = (x_k)$ and $y = (y_k)$, defines a 2-norm. For $p = \infty$, the formula reduces to

$$\|x, y\|_\infty := \sup_k \sup_l \left| \begin{vmatrix} x_k & x_l \\ y_k & y_l \end{vmatrix} \right|.$$

We also note that, for $p = 2$, the 2-norm may be rewritten as

$$\|x, y\|_2 = \left| \begin{matrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{matrix} \right|^{1/2}, \tag{2}$$

where $\langle x, y \rangle$ denotes the usual inner product on ℓ^2. (Geometrically, in an inner product space $(X, \langle \cdot, \cdot \rangle)$, the entity $\|x, y\|_2$ given by (2) represents the area of the parallelogram spanned by x and y in X.)

Now, as shown in [9], we can derive a norm from the 2-norm in a certain way. Indeed, if $e_1 := (1, 0, 0, \dots)$ and $e_2 := (0, 1, 0, \dots)$, then the following function

$$\|x\| := \left[\|x, e_1\|_p^p + \|x, e_2\|_p^p \right]^{1/p} \tag{3}$$

defines a norm on ℓ^p. In general, if $\{a, b\}$ is a linearly independent set in ℓ^p, then one may observe that

$$\|x\|_\circ := \left[\|x, a\|_p^p + \|x, b\|_p^p \right]^{1/p} \tag{4}$$

defines a norm on ℓ^p. If the former can be easily seen to be equivalent to the usual norm $\| \cdot \|_p$ on ℓ^p, the latter is not so obvious.

The aim of this note is to prove the equivalence between the usual norm $\| \cdot \|_p$ and the norm $\| \cdot \|_\circ$ defined in (4), for any linearly independent set $\{a, b\}$ in ℓ^p. This result is important because in an infinite dimensional space there is no guarantee that any two given norms are equivalent. As we shall see, our result can also be used to understand the topology of ℓ^p as a 2-normed space.

2. Main results

Hereafter we shall consider only the case where $1 \le p < \infty$, leaving the case where $p = \infty$ to the reader. We assume that the reader has sufficient knowledge of ℓ^p space, especially its completeness.

Before we present our results, we shall first show by example that on ℓ^p there are norms which are not equivalent to the usual one. For every $x = (x_k)$ in ℓ^p, define

$$\|x\|_* := \left[\sum_k \left| \frac{x_k}{k} \right|^p \right]^{1/p}.$$

Clearly $\| \cdot \|_*$ defines a norm on ℓ^p. Here we have $\|x\|_* \le \|x\|_p$ for every $x \in \ell^p$. However, we cannot find a constant $A > 0$ such that $\|x\|_* \ge A\|x\|_p$ for every $x \in \ell^p$. If there were such a constant $A > 0$, we could take $x = e_k := (0, \cdots, 0, 1, 0, \cdots)$ where the only nonzero term is the k-th term, so that $\frac{1}{k} \ge A$. But this cannot be true for any $k \in \mathbf{N}$, for otherwise $\mathbf{N}$ would be bounded by $\frac{1}{A}$.

Let us now examine the norm $\| \cdot \|$ derived from the 2-norm $\| \cdot, \cdot \|_p$ through the formula (3). For every $x \in \ell^p$, the formula (1) gives

$$\|x, e_1\|_p^p = |x_2|^p + |x_3|^p + |x_4|^p + \cdots,$$
$$\|x, e_2\|_p^p = |x_1|^p + |x_3|^p + |x_4|^p + \cdots,$$

whence
$$\|x\|^p = |x_1|^p + |x_2|^p + 2(|x_3|^p + |x_4|^p + \cdots).$$

Thus we have $\|x\|_p \leq \|x\| \leq 2^{1/p}\|x\|_p$, for every $x \in \ell^p$. This tells us that $\|\cdot\|$ is equivalent to the usual norm $\|\cdot\|_p$.

The norm $\|\cdot\|$ defined by (3) is just a special case of the norm $\|\cdot\|_\circ$ defined by (4) for any linearly independent set $\{a, b\}$ in ℓ^p. To prove that $\|\cdot\|_\circ$ is equivalent to the usual norm $\|\cdot\|_p$, we need the following lemmas. (Lemma 2.1 is taken from [9].)

LEMMA 2.1. *For every $x, y \in \ell^p$, we have $\|x, y\|_p \leq 2^{1-1/p}\|x\|_p\|y\|_p$.*

Proof. By the triangle inequality for real numbers and Minkowski's inequality for double series, we have

$$\begin{aligned}
\|x, y\|_p &= \left[\frac{1}{2} \sum_k \sum_l |x_k y_l - x_l y_k|^p \right]^{1/p} \\
&\leq \left[\frac{1}{2} \sum_k \sum_l (|x_k||y_l| + |x_l||y_k|)^p \right]^{1/p} \\
&\leq 2^{-1/p} \left[\left[\sum_k \sum_l |x_k|^p |y_l|^p \right]^{1/p} + \left[\sum_k \sum_l |x_l|^p |y_k|^p \right]^{1/p} \right] \\
&= 2^{1-1/p}\|x\|_p\|y\|_p,
\end{aligned}$$

for every $x, y \in \ell^p$, as claimed. ■

LEMMA 2.2. *For every $x, y, z \in \ell^p$, we have $\|x\|_p \|y, z\|_p \leq 2 \|y\|_p \|x, z\|_p + \|z\|_p \|x, y\|_p$.*

Proof. For every $x, y, z \in \ell^p$, we have $\|y, z\|_p^p = \frac{1}{2} \sum_m \sum_n |y_m z_n - z_m y_n|^p$ and $\|x\|_p^p = \sum_k |x_k|^p$, so that

$$\|x\|_p^p \|y, z\|_p^p = \frac{1}{2} \sum_k \sum_m \sum_n |x_k y_m z_n - x_k y_n z_m|^p.$$

Now observe that

$$\frac{1}{2} \sum_k \sum_m \sum_n |x_k y_m z_n - x_k y_n z_m|^p$$
$$= \frac{1}{2} \sum_k \sum_m \sum_n |(x_k y_m z_n - x_n y_m z_k) + (x_n y_m z_k - x_m y_n z_k) + (x_m y_n z_k - x_k y_n z_m)|^p$$
$$\leq \frac{1}{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (|x_k y_m z_n - x_n y_m z_k| + |x_n y_m z_k - x_m y_n z_k| + |x_m y_n z_k - x_k y_n z_m|)^p$$

It follows from Minkowski's inequality that

$$\left(\frac{1}{2} \sum_k \sum_m \sum_n |x_k y_m z_n - x_k y_n z_m|^p \right)^{\frac{1}{p}} \leq \left(\frac{1}{2} \sum_k \sum_m \sum_n |x_k y_m z_n - x_n y_m z_k|^p \right)^{\frac{1}{p}} +$$

$$+ \left(\frac{1}{2} \sum_k \sum_m \sum_n |x_n y_m z_k - x_m y_n z_k|^p \right)^{\frac{1}{p}} + \left(\frac{1}{2} \sum_k \sum_m \sum_n |x_m y_n z_k - x_k y_n z_m|^p \right)^{\frac{1}{p}}$$
$$= \|y\|_p \, \|x, z\|_p + \|z\|_p \, \|x, y\|_p + \|y\|_p \, \|x, z\|_p$$

Thus we obtain

$$\|x\|_p \, \|y, z\|_p \leq 2 \, \|y\|_p \, \|x, z\|_p + \|z\|_p \, \|x, y\|_p \, ,$$

as desired. ∎

Now we come to the main result.

THEOREM 2.3. *For any linearly independent set $\{a, b\}$ in ℓ^p, the norm $\| \cdot \|_\circ$ defined by (4) is equivalent to the usual norm $\| \cdot \|_p$.*

Proof. For every $x \in \ell^p$, we have $\|x, a\|_p \leq 2^{1 - \frac{1}{p}} \|x\|_p \, \|a\|_p$ and $\|x, b\|_p \leq 2^{1 - \frac{1}{p}} \|x\|_p \, \|b\|_p$ by Lemma 2.1. Hence we obtain

$$\|x\|_\circ = \left[\|x, a\|_p^p + \|x, b\|_p^p \right]^{\frac{1}{p}} \leq 2^{1 - 1/p} \left[\|a\|_p^p + \|b\|_p^p \right]^{\frac{1}{p}} \|x\|_p.$$

Meanwhile, by Lemma 2.2, we have $\|x\|_p \, \|a, b\|_p \leq 2 \, \|a\|_p \, \|x, b\|_p + \|b\|_p \, \|x, a\|_p$ and by swapping a and b we get $\|x\|_p \, \|b, a\|_p \leq 2 \, \|b\|_p \, \|x, a\|_p + \|a\|_p \, \|x, b\|_p$. Hence

$$2 \, \|x\|_p \, \|a, b\|_p \leq 3 \, \|a\|_p \, \|x, b\|_p + 3 \, \|b\|_p \, \|x, a\|_p \, .$$

Next, we know that $\|x, a\|_p \leq \left[\|x, a\|_p^p + \|x, b\|_p^p \right]^{\frac{1}{p}} = \|x\|_\circ$ and similarly $\|x, b\|_p \leq \|x\|_\circ$. It thus follows that

$$2 \, \|x\|_p \|a, b\|_p \leq 3 \, (\|a\|_p + \|b\|_p) \|x\|_\circ,$$

whence

$$\|x\|_p \leq \frac{3(\|a\|_p + \|b\|_p)}{2\|a, b\|_p} \, \|x\|_\circ.$$

Combining this and the previous inequality, we obtain

$$\frac{2\|a, b\|_p}{3(\|a\|_p + \|b\|_p)} \|x\|_p \leq \|x\|_\circ \leq 2^{1 - 1/p} \left[\|a\|_p^p + \|b\|_p^p \right]^{1/p} \|x\|_p.$$

This shows that $\| \cdot \|_\circ$ and $\| \cdot \|_p$ are equivalent. ∎

COROLLARY 2.4. *The space $(\ell^p, \| \cdot \|_\circ)$ is complete. In other words, it is a Banach space.*

3. Applications

We recall that a sequence $(x(n))$ in a 2-normed space $(X, \|\cdot, \cdot\|)$ is said to *converge* to an element $x \in X$ (with respect to $\|\cdot, \cdot\|$) if for every $y \in X$, we have

$$\|x(n) - x, y\| \to 0, \quad \text{as } n \to \infty.$$

Also, a sequence $(x(n))$ in a 2-normed space $(X, \|\cdot, \cdot\|)$ is called a *Cauchy* sequence (with respect to $\|\cdot, \cdot\|$) if for every $y \in X$, we have

$$\|x(m) - x(n), y\| \to 0, \quad \text{as } m, n \to \infty.$$

Clearly if $(x(n))$ converges to an element in $(X, \|\cdot, \cdot\|)$, then it is a Cauchy sequence. If the converse is also true for every sequence in $(X, \|\cdot, \cdot\|)$, then X is said to be *complete*. A complete 2-normed space is called a 2-Banach space.

By Lemma 2.1, we see that, in ℓ^p, if a sequence $(x(n))$ converges to x with respect to the usual norm $\|\cdot\|_p$, then it also converges to x with respect to the 2-norm $\|\cdot, \cdot\|_p$. Similarly, if $(x(n))$ is a Cauchy sequence in ℓ^p with respect to $\|\cdot\|_p$, then it is also a Cauchy sequence with respect to $\|\cdot, \cdot\|_p$. As a consequence of Theorem 2.3, we have the following results.

THEOREM 3.1. *In ℓ^p, if a sequence $(x(n))$ converges to x with respect to $\|\cdot, \cdot\|_p$, then it also converges to x with respect to $\|\cdot\|_p$. Also, if $(x(n))$ is a Cauchy sequence with respect to $\|\cdot, \cdot\|_p$, then it is a Cauchy sequence with respect to $\|\cdot\|_p$.*

Proof. Let $\{a, b\}$ be a linearly independent set in ℓ^p, and $\|\cdot\|_\circ$ be defined by (4). Now, if $(x(n))$ converges to x with respect to $\|\cdot, \cdot\|_p$, then we have

$$\|x(n) - x, a\| \to 0, \quad \text{as } n \to \infty$$

and

$$\|x(n) - x, b\| \to 0, \quad \text{as } n \to \infty.$$

It follows that

$$\|x(n) - x\|_\circ \to 0, \quad \text{as } n \to \infty,$$

that is, $(x(n))$ converges to x with respect to $\|\cdot\|_\circ$. By Theorem 2.3, we conclude that $(x(n))$ also converges to x with respect to $\|\cdot\|_p$. The second part of the theorem is proved in a similar way. ∎

COROLALRY 3.2. *$(\ell^p, \|\cdot, \cdot\|_p)$ is a 2-Banach space.*

Proof. Let $(x(n))$ be a Cauchy sequence in ℓ^p with respect to $\|\cdot, \cdot\|_p$. Then, by Theorem 3.1, $(x(n))$ is a Cauchy sequence with respect to $\|\cdot\|_p$. We know that $(\ell^p, \|\cdot\|_p)$ is a Banach space, and so $(x(n))$ must converge to an element $x \in \ell^p$ with respect to $\|\cdot\|_p$. By Lemma 2.1, $(x(n))$ must also converge to x with respect to $\|\cdot, \cdot\|_p$. Therefore, $(\ell^p, \|\cdot, \cdot\|_p)$ is a 2-Banach space. ∎

Our result can also be applied to prove a fixed point theorem in $(\ell^p, \|\cdot, \cdot\|_p)$, which is a refinement of the one in [9].

THEOREM 3.3. (Fixed Point Theorem) *Let $\{a, b\}$ be a linearly independent set in ℓ^p, and T be a self-mapping of ℓ^p. If there exists a constant $K \in (0, 1)$ such that, for $z \in \{a, b\}$, we have*

$$\|Tx - Ty, z\| \le K\|x - y, z\|, \quad \text{for every } x, y \in \ell^p,$$

then T has a unique fixed point (that is, there exists an element $x \in \ell^p$ such that $Tx = x$).

REMARK. In [9], the inequality is assumed to hold for every $z \in \ell^p$. Here we only require that it holds for $z = a$ and $z = b$ only.

Proof. For every $x, y \in \ell^p$, we observe that

$$\|Tx - Ty\|_\circ = \left[\|Tx - Ty, a\|_p^p + \|Tx - Ty, b\|_p^p\right]^{1/p}$$
$$\leq K \left[\|x - y, a\|_p^p + \|x - y, b\|_p^p\right]^{1/p} = K \|x - y\|_\circ.$$

This tells us that T is a contractive mapping on $(\ell^p, \|\cdot\|_\circ)$, which is a Banach space (by Corollary 2.4). Thus T must have a unique fixed point in ℓ^p. $\blacksquare$

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