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PROPERTY (gz) FOR BOUNDED LINEAR OPERATORS

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Abstract. A bounded linear operator T acting on a Banach space possesses property (gaw) if $\sigma(T) \setminus E_a(T) = \sigma_{BW}(T)$, where $\sigma_{BW}(T)$ is the B-Weyl spectrum of T, $\sigma(T)$ is the usual spectrum of T and $E_a(T)$ is the set of all eigenvalues of T which are isolated in the approximate point spectrum of T. In this paper we introduce and study the new spectral properties (z), (gz), (az) and (gaz) as a continuation of [M. Berkani, H. Zariouh, *New extended Weyl type theorems*, Mat. Vesnik **62** (2010), 145–154], which are related to Weyl type theorems. Among other results, we prove that T possesses property (gz) if and only if T possesses property (gaw) and $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$; where $\sigma_{SBF_+^-}(T)$ is the essential semi-B-Fredholm spectrum of T.

1. Introduction

Throughout this paper, let $L(X)$ denote the Banach algebra of all bounded linear operators acting on a complex infinite-dimensional Banach space X. For $T \in L(X)$, let $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_a(T)$ denote respectively the null space, the range, the spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} R(T)$. Recall that an operator $T \in L(X)$ is called an *upper semi-Fredholm* if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in L(X)$ is called a *lower semi-Fredholm* if $\beta(T) < \infty$. Let $SF_+(X)$ denote the class of all upper semi-Fredholm operators. If $T \in L(X)$ is an upper or lower semi-Fredholm operator, then T is called a *semi-Fredholm* operator, and the *index* of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a *Fredholm* operator. An operator $T \in L(X)$ is called a *Weyl* operator if it is a Fredholm operator of index 0. Define $SF_+^-(X) = \{T \in SF_+(X) : \operatorname{ind}(T) \leq 0\}$. The classes of operators defined above generate the following spectra : the *Weyl spectrum* $\sigma_W(T)$ of $T \in L(X)$ is defined by $\sigma_W(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not a Weyl operator}\}$, while the *Weyl essential approximate spectrum* $\sigma_{SF_+^-}(T)$ of T is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin SF_+^-(X)\}$.

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For $T \in L(X)$, let $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$ and $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Following Coburn [12], we say that *Weyl's theorem* holds for $T \in L(X)$ if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$. Here and elsewhere in this paper, for $A \subset \mathbf{C}$, iso A is the set of all isolated points of A. According to Rakočević [17], an operator $T \in L(X)$ is said to satisfy *a-Weyl's theorem* if $\Delta_a(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$. It is known [17] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

Recall that the *ascent* $a(T)$, of an operator T, is defined by $a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and the *descent* $\delta(T)$ of T, is defined by $\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf \emptyset = \infty$. An operator $T \in L(X)$ is called *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* $\sigma_D(T)$ of an operator T is defined by $\sigma_D(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Drazin invertible}\}$.

Define also the set $LD(X)$ by $LD(X) = \{T \in L(X) : a(T) < \infty$ and $R(T^{a(T)+1})$ is closed$\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin LD(X)\}$. Following [7], an operator $T \in L(X)$ is said to be *left Drazin invertible* if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. Let $\Pi_a(T)$ denote the set of all left poles of T and let $\Pi_a^0(T)$ denote the set of all left poles of T of finite rank, that is $\Pi_a^0(T) = \{\lambda \in \Pi_a(T) : \alpha(T - \lambda I) < \infty\}$. We say that *a-Browder's theorem* holds for $T \in L(X)$ if $\Delta_a(T) = \Pi_a^0(T)$.

Let $\Pi(T)$ be the set of all poles of the resolvent of T and let $\Pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\Pi^0(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda I) < \infty\}$. According to [14], a complex number λ is a *pole* of the resolvent of T if and only if $0 < \max\,(a(T - \lambda I), \delta(T - \lambda I)) < \infty$. Moreover, if this is true then $a(T - \lambda I) = \delta(T - \lambda I)$. According also to [14], the space $R((T - \lambda I)^{a(T-\lambda I)+1})$ is closed for each $\lambda \in \Pi(T)$. Hence we have always $\Pi(T) \subset \Pi_a(T)$ and $\Pi^0(T) \subset \Pi_a^0(T)$. We say that *Browder's theorem* holds for $T \in L(X)$ if $\Delta(T) = \Pi^0(T)$.

For $T \in L(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an *upper* (resp. a *lower*) *semi-B-Fredholm* operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [8]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm* operator, see [5]. A *semi-B-Fredholm* operator is an upper or a lower semi-B-Fredholm operator. An operator T is said to be a *B-Weyl* operator [4, Definition 1.1] if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not a B-Weyl operator}\}$. For $T \in L(X)$, let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$. Following [7], an operator $T \in L(X)$ is said to satisfy *generalized Weyl's theorem* if $\Delta^g(T) = E(T)$, where $E(T) = \{\lambda \in \text{iso}\sigma(T) : \alpha(T - \lambda I) > 0\}$ is the set of all isolated eigenvalues of T, and is said to satisfy *generalized Browder's theorem* if $\Delta^g(T) = \Pi(T)$. It is proved in [2, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem. In [7, Theorem 3.9], it is shown that an operator satisfying generalized

Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \Pi(T)$, it is proved in [6, Theorem 2.9] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, $SBF_+^-(X) = \{T \in SBF_+(X) : \text{ind}(T) \leq 0\}$. The *essential semi-B-Fredholm spectrum* of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \notin SBF_+^-(X)\}$. For $T \in L(X)$, let $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Following also [7], we say that *generalized a-Weyl's theorem* holds for $T \in L(X)$ if $\Delta_a^g(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I)\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that T obeys *generalized a-Browder's theorem* if $\Delta_a^g(T) = \Pi_a(T)$. It is proved in [2, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [7, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \Pi_a(T)$ it is proved in [6, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

Following [16], an operator $T \in L(X)$ is said to possess *property (w)* if $\Delta_a(T) = E^0(T)$. It is proved in [16, Corollary 2.3] that property (w) implies Weyl's theorem, but the converse is not true in general. According to [1], an operator $T \in L(X)$ is said to possess *property (gw)* if $\Delta_a^g(T) = E(T)$, which is an extension to the context of B-Fredholm theory of property (w), and it is proved in [1, Theorem 2.3] that an operator possessing property (gw) possesses property (w), but the converse is not true in general. According to [10], an operator $T \in L(X)$ is said to possess *property (b)* if $\Delta_a(T) = \Pi^0(T)$, and is said to possess *property (gb)* if $\Delta_a^g(T) = \Pi(T)$. It is shown [10, Theorem 2.3] that property (gb) implies property (b), but the converse does not hold in general. It is proved in [10, Theorem 2.13] that property (b) is a weak version of property (w), and it is proved also in [10, Theorem 2.15] that property (gb) is a weak version of property (gw).

According to [9], an operator $T \in L(X)$ is said to possess *property (ab)* if $\Delta(T) = \Pi_a^0(T)$, and is said to possess *property (gab)* if $\Delta^g(T) = \Pi_a(T)$. It is proved in [9, Theorem 2.2] that property (gab) implies property (ab), but the converse is not true in general. It is shown also [9] that property (gb) is a fortified version of property (gab), which is a fortified version of generalized Browder's theorem. According also to [9], we say that $T \in L(X)$ possesses *property (aw)* if $\Delta(T) = E_a^0(T)$, and that T possesses *property (gaw)* if $\Delta^g(T) = E_a(T)$. In [9, Theorem 3.3], it is proved that property (gaw) implies property (aw), but the converse is not true in general, and it is proved in [9, Theorem 3.5] that an operator possessing property (gaw) possesses property (gab), but the converse does not hold in general.

As a continuation of our previous article jointly with M. Berkani [9], we define and study in this paper the new spectral *properties (z), (gz), (az)* and *(gaz)* (see Definition 2.1 and Definition 3.1), which are analogous respectively to a-Weyl's theorem, generalized a-Weyl's theorem and a-Browder's theorem. We prove in

Theorem 2.2 that an operator $T \in L(X)$ possessing property (gz) possesses property (z) but the converse is not true in general as shown by Example 2.3, nonetheless and under the assumption that $E_a(T) = \Pi_a(T)$ we prove in Theorem 2.6 that the two properties are equivalent. We also prove in Theorem 2.4 that an operator possessing property (gz) satisfies generalized a-Weyl's theorem, and an operator possessing property (z) satisfies a-Weyl's theorem, but the converses do not hold in general. We also show in Theorem 2.7 that an operator possessing property (gz) possesses property (gaw) and in Theorem 2.8 we show that an operator possessing property (z) possesses property (aw), but the converses of these theorems are not true in general. Conditions for the equivalence of properties (gz) and (gaw), and properties (z) and (aw), are given in Theorem 2.7 and Theorem 2.8, respectively. Precisely, we prove that an operator $T \in L(X)$ possesses property (gz) if and only if T possesses property (gaw) and $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$, and that T possesses property (z) if and only if T possesses property (aw) and $\sigma_W(T) = \sigma_{SF_+^-}(T)$. We prove in Corollary 3.5 that property (az) is equivalent to property (gaz) and in Theorem 3.6 we show that an operator $T \in L(X)$ possesses property (z) if and only if T possesses property (az) and $E_a^0(T) = \Pi_a^0(T)$.

In the last part, as a conclusion, we give a diagram summarizing the different relations between Weyl type theorems and properties, extending a similar diagram given in [9].

2. Properties (z) and (gz)

For $T \in L(X)$, let $\Delta_+(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T)$ and let $\Delta_+^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$.

Definition 2.1. A bounded linear operator $T \in L(X)$ is said to possess property (z) if $\Delta_+(T) = E_a^0(T)$ and is said to possess property (gz) if $\Delta_+^g(T) = E_a(T)$.

Theorem 2.2. *Let $T \in L(X)$. If T possesses property (gz), then T possesses property (z).*

Proof. Suppose that T possesses property (gz), then $\Delta_+^g(T) = E_a(T)$. If $\lambda \in \Delta_+(T)$, then $\lambda \in \Delta_+^g(T) = E_a(T)$. Since $T - \lambda I$ is an upper semi-Fredholm, then $\alpha(T - \lambda I) < \infty$. So $\lambda \in E_a^0(T)$ and $\Delta_+(T) \subset E_a^0(T)$. To show the opposite inclusion, let $\lambda \in E_a^0(T)$ be arbitrary. Then λ is an eigenvalue of T isolated in $\sigma_a(T)$. Since T possesses property (gz), it follows that $\lambda \in \Delta_+^g(T)$ and $T - \lambda I$ is an upper semi-B-Fredholm operator. As $\alpha(T - \lambda I)$ is finite, then from [10, Lemma 2.2], we conclude that $T - \lambda I$ is an upper semi-Fredholm with $\mathrm{ind}(T - \lambda I) \leq 0$. Hence $\lambda \in \Delta_+(T)$. Finally, we have $\Delta_+(T) = E_a^0(T)$, and T possesses property (z). ∎

The converse of Theorem 2.2 does not hold in general as shown by the following example.

Example 2.3. Let Q be defined for each $x = (\xi_i) \in \ell^1$ by

$$Q(\xi_1, \xi_2, \xi_3, \dots, \xi_k, \dots) = (0, \alpha_1\xi_1, \alpha_2\xi_2, \dots, \alpha_{k-1}\xi_{k-1}, \dots),$$

where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$. We observe that $\overline{R(Q^n)} \neq R(Q^n)$, $n = 1, 2, \ldots$ Indeed, for a given $n \in \mathbf{N}$ let $x_k^{(n)} = (1, \ldots, 1, 0, 0, \ldots)$ (with $n+k$ times 1). Then the limit $y^{(n)} = \lim_{k\to\infty} Q^n x_k^{(n)}$ exists and lies in $\overline{R(Q^n)}$. However, there is no element $x^{(n)} \in \ell^1$ satisfying the equation $Q^n x^{(n)} = y^{(n)}$ as the algebraic solution to this equation is $(1, 1, 1, \ldots) \notin \ell^1$. Define T on $X = \ell^1 \oplus \ell^1$ by $T = Q \oplus 0$. Then $\sigma(T) = \sigma_a(T) = \{0\}$, $E_a(T) = \{0\}$ and $E_a^0(T) = \emptyset$. Since $R(T^n) = R(Q^n) \oplus \{0\}$, $R(T^n)$ is not closed for any $n \in \mathbf{N}$; so T is not a semi-B-Fredholm operator, and $\sigma_{SBF_+^-}(T) = \{0\}$. Furthermore, $T \notin SF_+^-(X)$, and $\sigma_{SF_+^-}(T) = \{0\}$. We then have $\Delta_+^g(T) \neq E_a(T)$, $\Delta_+(T) = E_a^0(T)$. Hence T possesses property (z), but it does not possess property (gz).

Theorem 2.4. *Let $T \in L(X)$. Then the following assertions hold.*

(i) T possesses property (z) if and only if T satisfies a-Weyl's theorem and $\sigma(T) = \sigma_a(T)$.

(ii) T possesses property (gz) if and only if T satisfies generalized a-Weyl's theorem and $\sigma(T) = \sigma_a(T)$.

Proof. (i) Assume that T possesses property (z). If $\lambda \in \Delta_a(T)$, then $\lambda \in \Delta_+(T)$. Therefore $\lambda \in E_a^0(T)$ and $\Delta_a(T) \subset E_a^0(T)$. Now if $\lambda \in E_a^0(T)$, then $\lambda \in \sigma_a(T)$ and since T possesses property (z), it follows that $\lambda \in \Delta_a(T)$. Hence $\Delta_a(T) = E_a^0(T)$ and T satisfies a-Weyl's theorem. Consequently, $\Delta_+(T) = E_a^0(T)$ and $\Delta_a(T) = E_a^0(T)$. Hence $\sigma(T) = \mathrm{iso}\sigma_a(T) \cup \sigma_{SF_+^-}(T)$ and $\sigma_a(T) = \mathrm{iso}\sigma_a(T) \cup \sigma_{SF_+^-}(T)$. This implies that $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies a-Weyl's theorem and $\sigma(T) = \sigma_a(T)$. Then $\Delta_a(T) = E_a^0(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+(T) = E_a^0(T)$ and T possesses property (z).

(ii) Assume that T possesses property (gz). If $\lambda \in \Delta_a^g(T)$, then $\lambda \in \Delta_+^g(T)$. Thus $\lambda \in E_a(T)$ and $\Delta_a^g(T) \subset E_a(T)$. Now if $\lambda \in E_a(T)$, then $\lambda \in \sigma_a(T)$ and since T possesses property (gz), $\lambda \in \Delta_a^g(T)$. Therefore $\Delta_a^g(T) = E_a(T)$ and T satisfies generalized a-Weyl's theorem. We then have $\Delta_+^g(T) = E_a(T)$ and $\Delta_a^g(T) = E_a(T)$ which implies that $\sigma(T) = \mathrm{iso}\sigma_a(T) \cup \sigma_{SBF_+^-}(T)$ and $\sigma_a(T) = \mathrm{iso}\sigma_a(T) \cup \sigma_{SBF_+^-}(T)$, so that $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies generalized a-Weyl's theorem and $\sigma(T) = \sigma_a(T)$. Then $\Delta_a^g(T) = E_a(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+^g(T) = E_a(T)$ and T possesses property (gz). ∎

The following example shows that generalized a-Weyl's theorem and generalized Weyl's theorem do not imply property (gz). It shows also that a-Weyl's theorem and Weyl's theorem do not imply property (z).

Example 2.5. Let R be the unilateral right shift operator defined on the Hilbert space $\ell^2(\mathbb{N})$. It is known from [15, Theorem 3.1] that $\sigma(R) = D(0, 1)$ is the closed unit disc in $\mathbf{C}$, $\sigma_a(R) = C(0, 1)$ is the unit circle of $\mathbf{C}$. Define T on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = 0 \oplus R$. Then $\sigma(T) = D(0, 1)$, $\sigma_a(T) = C(0, 1) \cup \{0\}$, $\sigma_{SF_+^-}(T) = C(0, 1) \cup \{0\}$, $\sigma_{SBF_+^-}(T) = C(0, 1)$, $E_a^0(T) = \emptyset$ and $E_a(T) = \{0\}$. Hence $\Delta_a^g(T) = E_a(T)$ and $\Delta_a(T) = E_a^0(T)$, i.e. T satisfies generalized a-Weyl's theorem and a-Weyl's theorem. Moreover, we have

$\sigma_{BW}(T) = \sigma_W(T) = D(0,1)$ and $E(T) = E^0(T) = \emptyset$. So $\Delta^g(T) = E(T)$ and $\Delta(T) = E^0(T)$, i.e. T satisfies generalized Weyl's theorem and Weyl's theorem. But T does not possess property (gz) or property (z), since $\Delta_+^g(T) \neq E_a(T)$ and $\Delta_+(T) \neq E_a^0(T)$.

Theorem 2.6. *Let $T \in L(X)$. The following statements are equivalent.*

(i) T possesses property (gz)

(ii) T possesses property (z) and $E_a(T) = \Pi_a(T)$.

Proof. (i)$\Longrightarrow$ (ii) Assume that T possesses property (gz), then T possesses property (z). If $\lambda \in E_a(T)$, then $\lambda \in \mathrm{iso}\sigma_a(T)$ and since T possesses property (gz), $T - \lambda I$ is an upper semi-B-Fredholm operator such that $\mathrm{ind}(T - \lambda I) \leq 0$. From [7, Theorem 2.8] we deduce that $\lambda \in \Pi_a(T)$. Consequently, $E_a(T) \subset \Pi_a(T)$. As $E_a(T) \supset \Pi_a(T)$ is always true, then $E_a(T) = \Pi_a(T)$.

(ii)$\Longrightarrow$ (i) Assume that T possesses property (z) and $E_a(T) = \Pi_a(T)$. By Theorem 2.4, T satisfies a-Weyl's theorem, which implies from [7, Corollary 3.5] that T satisfies a-Browder's theorem. As we know from [2, Theorem 2.2] that a-Browder's theorem is equivalent to generalized a-Browder's theorem, then T satisfies generalized a-Browder's theorem. Hence we have $\Delta_a^g(T) = \Pi_a(T)$. Since T possesses property (z), $\sigma(T) = \sigma_a(T)$ and by the hypothesis $\Pi_a(T) = E_a(T)$, it then follows that $\Delta_+^g(T) = E_a(T)$ and T possesses property (gz). ■

Theorem 2.7. *Let $T \in L(X)$. Then T possesses property (gz) if and only if T possesses property (gaw) and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.*

Proof. Assume that T possesses property (gz). Let $\lambda \in \Delta^g(T)$ be arbitrary, then $T - \lambda I$ is a B-Weyl operator, thus $\lambda \notin \sigma_{SBF_+^-}(T)$. As T possesses property (gz) then $\lambda \in E_a(T)$ and hence $\Delta^g(T) \subset E_a(T)$. Now if $\lambda \in E_a(T)$, then $\lambda \in \Delta_+^g(T)$. So $T - \lambda I$ is a semi-B-Fredholm operator such that $\mathrm{ind}(T - \lambda I) \leq 0$. Since λ is isolated in $\sigma_a(T)$ and T possesses property (gz), it follows from Theorem 2.4 that λ is isolated in $\sigma(T)$. Using the punctured neighborhood theorem [8, Corollary 3.2], we deduce that $\mathrm{ind}(T - \lambda I) = 0$. Hence $T - \lambda I$ is a B-Weyl operator. As a conclusion, we see that T possesses property (gaw). We then have $\Delta_+^g(T) = E_a(T)$ and $\Delta^g(T) = E_a(T)$. Hence $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Conversely, if T possesses property (gaw) and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$, then $\Delta_+^g(T) = E_a(T)$ and T possesses property (gz). ■

Similarly we have the following result in the case of property (z), which we give without proof.

Theorem 2.8. *Let $T \in L(X)$. Then T possesses property (z) if and only if T possesses property (aw) and $\sigma_{SF_+^-}(T) = \sigma_W(T)$.*

Remark 2.9. Generally, property (aw) and property (gaw) do not imply property (z) and property (gz) respectively. Indeed, let R be the unilateral right shift operator defined on $\ell^2(\mathbf{N})$. Then $\sigma(R) = D(0,1)$, $\sigma_a(R) = C(0,1)$, $\sigma_{BW}(R) = D(0,1)$, $\sigma_W(R) = D(0,1)$, $E_a^0(R) = \emptyset$ and $E_a(R) = \emptyset$. Hence $\Delta^g(R) = E_a(R)$ and $\Delta(R) = E_a^0(R)$, i.e. R possesses property (gaw) and property (aw). Moreover, we have $\sigma_{SBF_+^-}(R) = C(0,1)$, $\sigma_{SF_+^-}(R) = C(0,1)$. This implies that $\Delta_+^g(R) \neq E_a(R)$ and $\Delta_+(R) \neq E_a^0(R)$. So R does not possess property (z) or property (gz).

3. Properties (az) and (gaz)

Definition 3.1. A bounded linear operator $T \in L(X)$ is said to possess property (az) if $\Delta_+(T) = \Pi_a^0(T)$ and is said to possess property (gaz) if $\Delta_+^g(T) = \Pi_a(T)$.

Theorem 3.2. *Let $T \in L(X)$. Then T possesses property (az) if and only if T satisfies a-Browder's theorem and $\sigma(T) = \sigma_a(T)$.*

Proof. Suppose that T possesses property (az), then $\Delta_+(T) = \Pi_a^0(T)$. If $\lambda \in \Delta_a(T)$ be arbitrary, then $\lambda \in \Delta_+(T) = \Pi_a^0(T)$ and so $\Delta_a(T) \subset \Pi_a^0(T)$. If $\lambda \in \Pi_a^0(T)$ then $T - \lambda I$ is an upper semi-Fredholm operator of index less or equal than 0, see [7, Theorem 2.8] and $\lambda \in \sigma_a(T)$. Hence $\Delta_a(T) = \Pi_a^0(T)$, i.e. T satisfies a-Browder's theorem. Consequently, $\Delta_a(T) = \Pi_a^0(T)$ and $\Delta_+(T) = \Pi_a^0(T)$. Hence $\sigma_a(T) = \text{iso}\sigma_a(T) \cup \sigma_{SF_+^-}(T)$ and $\sigma(T) = \text{iso}\sigma_a(T) \cup \sigma_{SF_+^-}(T)$, so that $\sigma(T) = \sigma_a(T)$. Conversely, suppose that T satisfies a-Browder's theorem and $\sigma(T) = \sigma_a(T)$. Then $\Delta_a(T) = \Pi_a^0(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+(T) = \Pi_a^0(T)$ and T possesses property (az). ■

The following example shows that in general, a-Browder's theorem and Browder's theorem do not imply property (az).

Example 3.3. Let R be the unilateral right shift operator defined on $\ell^2(\mathbf{N})$ and let S defined on $\ell^2(\mathbf{N})$ by $S(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$. Define T on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus S$. Then $\sigma(T) = D(0,1)$, $\sigma_a(T) = C(0,1) \cup \{0\}$, $\sigma_{SF_+^-}(T) = C(0,1)$, $\sigma_W(T) = D(0,1)$, $\Pi_a^0(T) = \{0\}$ and $\Pi^0(T) = \emptyset$. Hence $\Delta_a(T) = \Pi_a^0(T)$, $\Delta(T) = \Pi^0(T)$. So T satisfies a-Browder's theorem and Browder's theorem. But T does not possess property (az), since $\Delta_+(T) \neq \Pi_a^0(T)$.

Theorem 3.4. *Let $T \in L(X)$. Then T possesses property (gaz) if and only if T satisfies generalized a-Browder's theorem and $\sigma(T) = \sigma_a(T)$.*

Proof. Assume that T possesses property (gaz). If $\lambda \in \Delta_a^g(T)$, then $\lambda \in \Delta_+^g(T) = \Pi_a(T)$. So $\Delta_a^g(T) \subset \Pi_a(T)$. If $\lambda \in \Pi_a(T)$, then $T - \lambda I$ is an upper semi-B-Fredholm operator such that $\text{ind}(T - \lambda I) \leq 0$ and $\lambda \in \sigma_a(T)$. Hence $\Delta_a^g(T) = \Pi_a(T)$ and T satisfies generalized a-Browder's theorem. Consequently, $\Delta_a^g(T) = \Pi_a(T)$ and $\Delta_+^g(T) = \Pi_a(T)$. This implies that $\sigma_a(T) = \text{iso}\sigma_a(T) \cup \sigma_{SBF_+^-}(T)$ and $\sigma(T) = \text{iso}\sigma_a(T) \cup \sigma_{SBF_+^-}(T)$, which implies that $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies generalized a-Browder's theorem and $\sigma(T) = \sigma_a(T)$. Then $\Delta_a^g(T) = \Pi_a(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+^g(T) = \Pi_a(T)$ and T possesses property (gaz). ■

The operator T defined as in Example 3.3 shows that in general, generalized a-Browder's theorem and generalized Browder's theorem do not imply property (gaz). Indeed, $\sigma(T) = D(0,1)$ and $\sigma_a(T) = C(0,1) \cup \{0\}$. Moreover, $\sigma_{SBF_+^-}(T) = C(0,1)$, $\Pi_a(T) = \{0\}$, $\Pi(T) = \emptyset$ and $\sigma_{BW}(T) = D(0,1)$. Hence $\Delta_a^g(T) = \Pi_a(T)$ and $\Delta^g(T) = \Pi(T)$, i.e. T satisfies generalized a-Browder's theorem and generalized Browder's theorem. But T does not possess property (gaz), since $\Delta_+^g(T) \neq \Pi_a(T)$.

Corollary 3.5. *Let $T \in L(X)$. Then T possesses property (gaz) if and only if T possesses property (az).*

Proof. If T possesses property (gaz) then by Theorem 3.4, T satisfies generalized a-Browder's theorem and $\sigma(T) = \sigma_a(T)$. Since a-Browder's theorem is equivalent to generalized a-Browder's theorem, then T satisfies a-Browder's theorem and $\sigma(T) = \sigma_a(T)$. By virtue of Theorem 3.2, T possesses property (az). Conversely, if T possesses property (az), again by Theorem 3.2, T satisfies a-Browder's theorem and $\sigma(T) = \sigma_a(T)$. Hence T satisfies generalized a-Browder's theorem that's $\Delta_a^g(T) = \Pi_a(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+^g(T) = \Pi_a(T)$ and T possesses property (gaz). ∎

Theorem 3.6. *Let $T \in L(X)$. Then T possesses property (z) if and only if T possesses property (az) and $E_a^0(T) = \Pi_a^0(T)$.*

Proof. Assume that T possesses property (z), then $\Delta_+(T) = E_a^0(T)$. If $\lambda \in \Delta_+(T)$, then $\lambda \in \mathrm{iso}\sigma_a(T)$ and $T - \lambda I$ is an upper semi-Fredholm operator such that $\mathrm{ind}(T-\lambda I) \leq 0$. Hence $\lambda \in \Pi_a^0(T)$. Therefore $\Delta_+(T) \subset \Pi_a^0(T)$. Now if $\lambda \in \Pi_a^0(T)$, then $T - \lambda I$ is an upper semi-Fredholm operator such that $\mathrm{ind}(T - \lambda I) \leq 0$ and $\lambda \in \sigma(T)$. Hence $\Delta_+(T) = \Pi_a^0(T)$, i.e. T possesses property (az) and $\Pi_a^0(T) = E_a^0(T)$. Conversely, assume that T possesses property (az) and $\Pi_a^0(T) = E_a^0(T)$. Then $\Delta_+(T) = \Pi_a^0(T)$ and $\Pi_a^0(T) = E_a^0(T)$. So $\Delta_+(T) = E_a^0(T)$ and T possesses property (z). ∎

Corollary 3.7. *Let $T \in L(X)$. Then T possesses property (gz) if and only if T possesses property (gaz) and $E_a(T) = \Pi_a(T)$.*

Proof. If T possesses property (gz), then T possesses property (z). From Theorem 3.6 we have T possesses property (az). Since property (az) is equivalent to property (gaz), it follows that T possesses property (gaz). Hence we have $\Delta_+^g(T) = E_a(T)$ and $\Delta_+^g(T) = \Pi_a(T)$, and this implies that $E_a(T) = \Pi_a(T)$. Conversely, if T possesses property (gaz) and $E_a(T) = \Pi_a(T)$ then $\Delta_+^g(T) = \Pi_a(T)$ and $E_a(T) = \Pi_a(T)$. So $\Delta_+^g(T) = E_a(T)$ and T possesses property (gz). ∎

Remark 3.8. In general, property (az) or property (gaz) does not imply property (z) or property (gz), respectively. Indeed, let $T \in L(\ell^2(\mathbf{N}))$ be defined by $T(x_1, x_2, x_3, \dots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$. Then $\sigma(T) = \{0\}$, $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T) = \{0\}$, $\Pi_a^0(T) = \Pi_a(T) = \emptyset$. We then have $\Delta_+(T) = \Pi_a^0(T)$, $\Delta_+^g(T) = \Pi_a(T)$. So T possesses properties (az) and (gaz). Moreover, $E_a^0(T) = E_a(T) = \{0\}$. Consequently, $\Delta_+(T) \neq E_a^0(T)$, $\Delta_+^g(T) \neq E_a(T)$. Thus T does not possess property (z) or property (gz).

4. Conclusion

In this last part, we give a summary of the known Weyl type theorems as in [7], including the properties introduced in [1, 9, 10, 16] and in this paper. We use the abbreviations gaW, aW, gW, W, (gw), (w), (gaw), (aw), (gz) and (z) to signify that an operator $T \in L(X)$ obeys generalized a-Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem, Weyl's theorem, property (gw), property (w), property (gaw), property (aw), property (gz) and property (z), respectively. Similarly, the abbreviations gaB, aB, gB, B, (gb), (b), (gab), (ab), (gaz) and (az) have analogous meaning with respect to Browder's theorem or the properties introduced in [9, 10] or the new properties introduced in this paper.

The following table summarizes the meaning of various theorems and properties.

gaW	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$	gaB	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$
aW	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$	aB	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$
gW	$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$	gB	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$
W	$\sigma(T) \setminus \sigma_W(T) = E^0(T)$	B	$\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$
(gw)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$	(gb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$
(w)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$	(b)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$
(gaw)	$\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$	(gab)	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$
(aw)	$\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$	(ab)	$\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$
(gz)	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$	(gaz)	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$
(z)	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$	(az)	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$

In the diagram on the next page, which extends the similar diagram presented in [9], arrows signify implications between various Weyl type theorems, Browder type theorems, property (gw), property (gb), property (gab), property (gaw), property (gz) and property (gaz). The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

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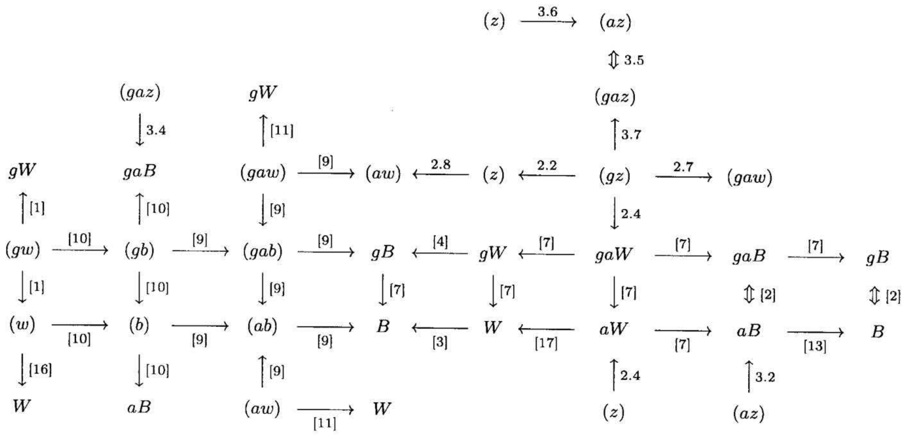

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