CHARACTERIZATION OF *GCR*-LIGHTLIKE WARPED PRODUCT OF INDEFINITE KENMOTSU MANIFOLDS

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Abstract. In this paper we prove that there do not exist warped product GCR-lightlike submanifolds in the form $M = N_\perp \times_\lambda N_\top$ such that $N_\perp$ is an anti-invariant submanifold tangent to V and $N_\top$ an invariant submanifold of $\overline{M}$, other than GCR-lightlike product in an indefinite Kenmotsu manifold. We also obtain some characterizations for a GCR-lightlike submanifold to be locally a GCR-lightlike warped product.

1. Introduction

Cauchy-Riemann (CR)-submanifolds of Kaehler manifolds were introduced by Bejancu [2] as a generalization of holomorphic and totally real submanifolds of Kaehler manifolds and further investigated by [3–5, 8, 9] and others. Contact CR-submanifolds of Sasakian manifolds were introduced by Yano and Kon [24]. They all studied the geometry of CR-submanifolds with positive definite metric. Therefore this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily definite. Thus the geometry of CR-submanifolds with indefinite metric motivated many geometers to do research on this subject matter and Duggal and Bejancu [12] played a very crucial role in this study by introducing the notion of CR-lightlike submanifolds of indefinite Kaehler manifolds. Since there is a significant use of the contact geometry in differential equations, optics, and phase spaces of a dynamical system (see Arnol'd [1], MacLane [19], Nazaikinskii et al. [20]). Therefore Duggal and Sahin [14] introduced contact CR-lightlike submanifolds and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds. But there do not exist any inclusion relations between invariant and screen real submanifolds so new class of submanifolds called, Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Sasakian manifolds (which is an umbrella of invariant, screen real, contact CR-lightlike submanifolds) were derived by Duggal and Sahin [15]. In [7], the notion of warped product manifolds was introduced by Bishop and O'Neill in 1969 and it was further

studied by many mathematicians and physicists. These manifolds are generaliza-
tion of Riemannian product manifolds. This generalized product metric appears in
differential geometric studies in a natural way. For instance a surface of revolution
is a warped product manifold. Moreover, many important submanifolds in real and
complex space forms are expressed as warped product submanifolds. In view of its
physical applications many research articles have recently appeared exploring exis-
tence (or non existence) of warped product submanifolds in known spaces [21]. Chen
[10] introduced warped product CR-submanifolds and showed that there does not
exist a warped product CR-submanifold in the form $M = N_\perp \times_\lambda N_\top$ in a Kaehler
manifold where $N_\perp$ is a totally real submanifold and $N_\top$ is a holomorphic subman-
ifold of $\overline{M}$. He proved if $M = N_\perp \times_\lambda N_\top$ is a warped product CR-submanifold of a
Kaehler manifold $\overline{M}$, then M is a CR-product, that is, there do not exist warped
product CR-submanifolds of the form $M = N_\perp \times_\lambda N_\top$ other than CR-product.
Therefore he called a warped product CR-submanifold in the form $M = N_\top \times_\lambda N_\perp$
a CR-warped product. Chen also obtained a characterization for CR-submanifold
of a Kaehler manifold to be locally a warped product submanifold. He showed that
a CR-submanifold M of a Kaehler manifold $\overline{M}$ is a CR-warped product if and only
if $A_{JZ}X = JX(\mu)Z$ for each $X \in \Gamma(D)$, $Z \in \Gamma(D')$, μ a C^∞-function on M such
that $Z\mu = 0$ for all $Z \in \Gamma(D')$.

Warped product lightlike submanifolds of a semi-Riemann manifolds are ini-
tially studided by Sahin [22]. In this paper we prove that there do not exist warped
product GCR-lightlike submanifolds in the form $M = N_\perp \times_\lambda N_\top$ such that $N_\perp$
is an anti-invariant submanifold tangent to V and $N_\top$ an invariant submanifold of
$\overline{M}$, other than GCR-lightlike product in an indefinite Cosymplectic manifold. We
also obtain some characterizations for a GCR-lightlike submanifold to be locally a
GCR-lightlike warped product.

2. Lightlike submanifolds

We recall notations and fundamental equations for lightlike submanifolds,
which are due to the book [12] by Duggal and Bejancu.

Let $(\overline{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant
index q such that $m, n \geq 1$, $1 \leq q \leq m + n - 1$ and (M, g) be an m-dimensional
submanifold of $\overline{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the
tangent bundle TM of M then M is called a lightlike submanifold of $\overline{M}$. For a
degenerate metric g on M

$$TM^\perp = \bigcup\{u \in T_x\overline{M} : \bar{g}(u,v) = 0, \forall v \in T_xM, x \in M\},$$

is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus, both T_xM and $T_xM^\perp$
are degenerate orthogonal subspaces but no longer complementary. In this case,
there exists a subspace $Rad\, T_xM = T_xM \cap T_xM^\perp$ which is known as radical (null)
subspace. If the mapping

$$Rad\, TM : x \in M \longrightarrow Rad\, T_xM,$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\overline{M}$ is called r-lightlike submanifold and $Rad\,TM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\,(TM)$ in TM, that is

$$TM = Rad\,TM \perp S(TM),$$

$S(TM^{\perp})$ is a complementary vector subbundle to $Rad\,TM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}\mid_M$ and to $Rad\,TM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.1}$$
$$T\overline{M}\mid_M = TM \oplus tr(TM) = (Rad\,TM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.2}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\overline{M}$ along M, on u as $\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n,$ $N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}$, where $\{\xi_1, \ldots, \xi_r\}$, $\{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(Rad\,TM\mid_u)$, $\Gamma(ltr(TM)\mid_u)$ and $\{W_{r+1}, \ldots, W_n\}$, $\{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have

THEOREM 2.1. *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be an r-lightlike submanifold of a semi-Riemannian manifold* $(\overline{M}, \bar{g})$. *Then, there exists a complementary vector bundle* $ltr(TM)$ *of* $Rad\,TM$ *in* $S(TM^{\perp})^{\perp}$ *and a basis of* $\Gamma(ltr(TM)\mid_u)$ *consisting of smooth section* $\{N_i\}$ *of* $S(TM^{\perp})^{\perp}\mid_u$, *where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ *is a lightlike basis of* $\Gamma(Rad\,(TM))$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then, according to the decomposition (2.2), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM), \tag{2.3}$$
$$\overline{\nabla}_X U = -A_U X + \nabla_X^{\perp} U, \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{2.4}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla_X^{\perp} U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M and known as shape operator.

According to (2.1), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (2.3) and (2.4) give

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y),$$
$$\overline{\nabla}_X U = -A_U X + D_X^l U + D_X^s U, \tag{2.5}$$

where we put $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l_X U = L(\nabla^\perp_X U)$, $D^s_X U = S(\nabla^\perp_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N),$$
$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W),$$
(2.6)

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Using (2.1)–(2.2) and (2.5)–(2.6), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y,D^l(X,W)) = g(A_W X,Y),$$
(2.7)
$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y,h^l(X,\xi)) + g(Y,\nabla_X \xi) = 0,$$

for any $\xi \in \Gamma(Rad\,TM)$ and $W \in \Gamma(S(TM^\perp))$.

Next, we recall some basic definitions and results of indefinite Kenmotsu manifolds (see [6]). An odd-dimensional semi-Riemannian manifold $(\overline{M},\bar{g})$ is called an ϵ-contact metric manifold, if there is a $(1,1)$ tensor field ϕ, a vector field V, called characteristic vector field and a 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \epsilon\eta(X)\eta(Y), \quad \bar{g}(V,V) = \epsilon,$$
(2.8)
$$\phi^2(X) = -X + \eta(X)V, \quad \bar{g}(X,V) = \epsilon\eta(X),$$
$$d\eta(X,Y) = \bar{g}(X,\phi Y),$$

for any $X,Y \in \Gamma(TM)$, where $\epsilon = \pm 1$ then it follows that

$$\phi V = 0, \quad \eta o\phi = 0, \quad \eta(V) = \epsilon.$$

Then $(\phi,V,\eta,\bar{g})$ is called an ϵ-contact metric structure of $\overline{M}$. We say that $\overline{M}$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is Nijenhuis tensor field of ϕ. A normal ϵ-contact metric manifold is called indefinite Kenmotsu manifold and for this we have

$$(\overline{\nabla}_X \phi)Y = g(\phi X,Y)V - \eta(Y)\phi X.$$
(2.9)

3. Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Kenmotsu manifolds

Calin [11] proved that if the characteristic vector field V is tangent to $(M,g,S(TM))$ then it belongs to $S(TM)$. We assume characteristic vector V is tangent to M throughout this paper.

DEFINITION 3.1. Let $(M,g,S(TM))$ be a real lightlike submanifold of an indefinite Kenmotsu manifold $(\overline{M},\bar{g})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad\,(TM)$ such that

$$Rad\,(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM).$$

(B) There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V, \quad \phi(\bar{D}) = L \perp S,$$

where D_0 is invariant non-degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V and L, S are vector subbundles of $ltr(TM)$ and $S(TM)^\perp$, respectively.

Then tangent bundle TM of M is decomposed as

$$TM = D \oplus \bar{D} \oplus \{V\}, \quad \text{where} \quad D = Rad\,(TM) \oplus D_0 \oplus \phi(D_2).$$

Let Q, P_1 and P_2 be the projection morphisms on D, $\phi S = M_2$ and $\phi L = M_1$ respectively, therefore any $X \in \Gamma(TM)$ can be written as

$$X = QX + V + P_1X + P_2X,$$

or

$$X = QX + V + PX, \tag{3.1}$$

where P is projection morphism on $\bar{D}$. Applying ϕ to (3.1), we obtain

$$\phi X = fX + \omega P_1 X + \omega P_2 X, \tag{3.2}$$

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(S)$ and $\omega P_2 X \in \Gamma(L)$, or, we can write (3.2), as

$$\phi X = fX + \omega X,$$

where fX and ωX are the tangential and transversal components of ϕX, respectively.

Similarly, for any $U \in \Gamma(tr(TM))$, we have

$$\phi U = BU + CU, \tag{3.3}$$

where BU and CU are the sections of TM and $tr(TM)$, respectively.

Differentiating (3.2) and using (2.5)–(2.6) and (3.3), we have

$$D^l(X, \omega P_1 Y) = -\nabla_X^l \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X,Y) - \eta(Y)wP_2X, \tag{3.4}$$

$$D^s(X, \omega P_2 Y) = -\nabla_X^s \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X,Y) - \eta(Y)wP_1X, \tag{3.5}$$

for all $X, Y \in \Gamma(TM)$. By using Kenmotsu property of $\overline{\nabla}$ with (2.3) and (2.4), we have the following lemmas.

LEMMA 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ then we have*

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X,Y) + g(\phi X,Y)V - \eta(Y)fX, \tag{3.6}$$

and

$$(\nabla_X^t \omega)Y = Ch(X,Y) - h(X, fY) - \eta(Y)wPX, \tag{3.7}$$

where $X, Y \in \Gamma(TM)$ *and*

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \tag{3.8}$$

$$(\nabla_X^t \omega)Y = \nabla_X^t \omega Y - \omega \nabla_X Y. \tag{3.9}$$

LEMMA 3.3. *Let* M *be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$ *then we have*

$$(\nabla_X B)U = A_{CU}X - fA_U X + g(\phi X, U)V,$$

and

$$(\nabla_X^t C)U = -\omega A_U X - h(X, BU),$$

where $X \in \Gamma(TM)$ *and* $U \in \Gamma(tr(TM))$ *and*

$$(\nabla_X B)U = \nabla_X BU - B\nabla_X^t U,$$

$$(\nabla_X^t C)U = \nabla_X^t CU - C\nabla_X^t U.$$

THEOREM 3.4. *Let* M *be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold* $\overline{M}$ *then*

(A) *The distribution* $D \oplus \{V\}$ *is integrable, if and only if,*

$$h(X, \phi Y) = h(Y, \phi X), \quad \forall \quad X, Y \in \Gamma(D \oplus \{V\}).$$

(B) *The distribution* $\bar{D}$ *is integrable, if and only if,*

$$A_{\phi Z}U = A_{\phi U}Z, \quad \forall \quad Z, U \in \Gamma(\bar{D}).$$

Proof. Using (3.4) and (3.5), we have

$$wP\nabla_X Y = h(X, fY) - Ch(X,Y),$$

for any $X, Y \in \Gamma(D \oplus \{V\})$. Hence $wP[X,Y] = h(X, fY) - h(Y, fX)$, which proves (A). Next, using (3.6) and (3.8), we have

$$f\nabla_Z U = -A_{wU}Z - Bh(Z,U),$$

for any $Z, U \in \Gamma(\bar{D})$. Then we obtain $f[Z,U] = A_{wZ}U - A_{wU}Z$, which completes the proof. ∎

THEOREM 3.5. *Let* M *be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold* $\bar{M}$. *Then the distribution* $D \oplus \{V\}$ *defines a totally geodesic foliation in* M, *if and only if,* $Bh(X, \phi Y) = 0$, *for any* $X, Y \in D \oplus \{V\}$.

Proof. From the definition of GCR-lightlike submanifolds of an indefinite Kenmotsu manifold, it is clear that $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $g(\nabla_X Y, \phi\xi) = g(\nabla_X Y, \phi W) = 0$, for any $X, Y \in \Gamma(D \oplus \{V\})$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(S)$. Using (2.9) and (2.5), we have

$$g(\nabla_X Y, \phi\xi) = -g(\phi\overline{\nabla}_X Y, \xi) = -g(h^l(X, \phi Y), \xi),$$

similarly

$$g(\nabla_X Y, \phi W) = -g(\phi\overline{\nabla}_X Y, W) = -g(h^s(X, \phi Y), W).$$

Therefore from above equations, it is clear that the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $h^l(X, \phi Y)$ and $h^s(X, \phi Y)$ have no components in L and S, respectively, that is, if and only if, $Bh^l(X, \phi Y) = 0$ and $Bh^s(X, \phi Y) = 0$. Hence the assertion follows. ∎

4. GCR-lightlike warped product

Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and $\lambda > 0$ a differentiable function on B. Assume the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_\lambda F$ is the manifold $B \times F$ equipped with the Riemannian metric g where

$$g = g_B + \lambda^2 g_F. \tag{4.1}$$

If X is tangent to $M = B \times_\lambda F$ at (p, q) then using (4.1), we have

$$\|X\|^2 = \|\pi_* X\|^2 + \lambda^2(\pi(X))\|\eta_* X\|^2.$$

The function λ is called the warping function of the warped product. For differentiable function λ on M, the gradient $\nabla\lambda$ is defined by $g(\nabla\lambda, X) = X\lambda$, for all $X \in T(M)$.

LEMMA 4.1. [7] *Let $M = B \times_\lambda F$ be a warped product manifold. If $X, Y \in T(B)$ and $U, Z \in T(F)$ then $\nabla_X Y \in T(B)$,*

$$\nabla_X U = \nabla_U X = \frac{X\lambda}{\lambda} U, \tag{4.2}$$

$\nabla_U Z = -\frac{g(U,Z)}{\lambda}\nabla\lambda.$

COROLLARY 4.2. *On a warped product manifold $M = B \times_\lambda F$ we have*

(i) *B is totally geodesic in M.*

(ii) *F is totally umbilical in M.*

DEFINITION 4.3. [13] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that $h(X, Y) = Hg(X, Y)$, for all $X, Y \in \Gamma(TM)$, it is easy to see that M

is a totally umbilical if and only if on each coordinate neighborhood u, there exists smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$, such that

$$h^l(X,Y) = H^l g(X,Y), \quad h^s(X,Y) = H^s g(X,Y) \quad D^l(X,W) = 0,$$

for any $W \in \Gamma(S(TM^\perp))$.

LEMMA 4.4. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ then the distribution $\bar{D}$ defines a totally geodesic foliation in M.*

Proof. Let $X, Y \in \Gamma(\bar{D})$ then using (3.6) and (3.8) we have $f\nabla_X Y = -A_{wY}X - Bh(X,Y)$. Let $Z \in \Gamma(D_0)$ then using (2.9) we obtain

$$g(f\nabla_X Y, Z) = -g(A_{wY}X, Z) = \bar{g}(\overline{\nabla}_X \phi Y, Z)$$
$$= -\bar{g}(\overline{\nabla}_X Y, \phi Z) = -\bar{g}(\overline{\nabla}_X Y, Z') = g(Y, \nabla_X Z'), \qquad (4.3)$$

where $Z' = \phi Z \in \Gamma(D_0)$. Since $X \in \Gamma(\bar{D})$ and $Z \in \Gamma(D_0)$ then using (3.7), (3.9) and the hypothesis that M is a totally umbilical GCR-lightlike submanifold, we get $w\nabla_X Z = h(X, fZ) - Ch(X,Z) = Hg(X, fZ) - CHg(X,Z) = 0$, this implies that $\nabla_X Z \in \Gamma(D)$, then (4.3) implies that $g(f\nabla_X Y, Z) = 0$ then the non degeneracy of the distribution D_0 implies that $f\nabla_X Y = 0$, this gives $\nabla_X Y \in \Gamma(\bar{D})$ for any $X, Y \in \Gamma(\bar{D})$. Hence the proof is complete. ∎

THEOREM 4.5. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold then the totally real distribution $\bar{D}$ is integrable.*

Proof. Using (3.6) and (3.8) with above lemma, we get

$$A_{wY}X = -Bh(X,Y),$$

for any $X, Y \in \Gamma(\bar{D})$. Then using the symmetric property of h, we get $A_{wY}X = A_{wX}Y$, for any $X, Y \in \Gamma(\bar{D})$. This implies that the distribution $\bar{D}$ is integrable. ∎

DEFINITION 4.6. A GCR-lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ is called a GCR-lightlike product if both the distribution $D \oplus \{V\}$ and $\bar{D}$ define totally geodesic foliations in M.

Let $M = N_\perp \times_\lambda N_\top$ be a warped product GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Such submanifolds are always tangent to the structure vector field V. We distinguish two cases

 (i) V is tangent to $N_\top$.

(ii) V is tangent to $N_\perp$.

In this paper we consider the case when V is tangent to $N_\top$.

THEOREM 4.7. *Let M be a totally umbilical GCR-lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$. If $M = N_\perp \times_\lambda N_\top$ be a warped product GCR-lightlike submanifold such that $N_\perp$ is an anti-invariant submanifold and $N_\top$ is an invariant submanifold of $\overline{M}$ tangent to V, then it is a GCR-lightlike product.*

Proof. Since M is a totally umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold then using Lemma 4.4, the distribution $\bar{D}$ defines a totally geodesic foliation in M.

Let h^T and A^T be the second fundamental form and the shape operator of $N_\top$ in M then for $X, Y \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(\phi S) \subset \Gamma(\bar{D})$, we have $g(h^T(X,Y), Z) = g(\nabla_X Y, Z) = -\bar{g}(Y, \overline{\nabla}_X Z) = -g(Y, \nabla_X Z) - g(Y, h^l(X, Z)) = -g(Y, \nabla_X Z) - g(Y, H^l)g(X, Z) = -g(Y, \nabla_X Z)$. Using (4.2) for $M = N_\perp \times_\lambda N_\top$, we get

$$g(h^T(X,Y), Z) = -(Z \ln \lambda)g(X, Y). \tag{4.4}$$

Now, let $\hat{h}$ be the second fundamental form of $N_\top$ in $\overline{M}$ then

$$\hat{h}(X, Y) = h^T(X, Y) + h^s(X, Y) + h^l(X, Y), \tag{4.5}$$

for any X, Y tangent to $N_\top$ then using (4.4), we get

$$g(\hat{h}(X, Y), Z) = g(h^T(X, Y), Z) = -(Z \ln \lambda)g(X, Y). \tag{4.6}$$

Since $N_\top$ is a holomorphic submanifold of $\overline{M}$ then we have $\hat{h}(X, \phi Y) = \hat{h}(\phi X, Y) = \phi \hat{h}(X, Y)$ therefore we have

$$g(\hat{h}(X, Y), Z) = -g(\hat{h}(\phi X, \phi Y), Z) = (Z \ln \lambda)g(X, Y). \tag{4.7}$$

Adding (4.6) and (4.7) we get

$$g(\hat{h}(X, Y), Z) = 0. \tag{4.8}$$

Using (4.5) and (4.8), we have $g(h(X, Y), \phi Z) = g(\hat{h}(X, Y), \phi Z) - g(h^T(X, Y), \phi Z) = g(\hat{h}(X, Y), \phi Z) = -g(\phi \hat{h}(X, Y), Z) = -g(\hat{h}(X, \phi Y), Z) = 0$. Thus $g(h(X, Y), \phi Z) = 0$ implies that $h(X, Y)$ has no components in $L_1 \perp L_2$ for any $X, Y \in \Gamma(D \oplus \{V\})$. In other words, we can say that $Bh(X, Y) = 0$, for any $X, Y \in \Gamma(D \oplus \{V\})$. Therefore using Theorem 3.5, the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M. Hence M is a GCR-lightlike product. Hence the proof is complete. ∎

After the proof of Theorem 4.7, it is important to mention the theorems by Hasegawa and Mihai [16], Khan et al. [18] and Siraj Uddin and Khan [23] respectively.

THEOREM 4.8. [16] *Let $\overline{M}$ be a $(2m+1)$-dimensional Sasakian manifold. Then there do not exist warped product submanifolds $M = M_1 \times_\lambda M_2$ such that M_1 is an anti-invariant submanifold tangent to V and M_2 an invariant submanifold of $\overline{M}$.*

THEOREM 4.9. [18] *Let $\overline{M}$ be a $(2m + 1)$-dimensional Kenmotsu manifold. Then there do not exist warped product submanifolds $M = N_\perp \times_\lambda N_\top$ such that $N_\top$ is an invariant submanifold tangent to V and $N_\perp$ is anti-invariant submanifold of $\overline{M}$.*

THEOREM 4.10. [23] *There does not exist a proper warped product CR-submanifold $N_\perp \times_\lambda N_\top$ of a Cosymplectic manifold $\overline{M}$ such that V is tangent to*

$N_\perp$, where $N_\perp$ is an anti-invariant submanifold and $N_\top$ is an invariant submanifold of $\overline{M}$.

In this paper, Theorem 4.7 also shows that there do not exist warped product GCR-lightlike submanifolds of indefinite Kenmotsu manifolds the form $M = N_\perp \times_\lambda N_\top$ such that $N_\perp$ is an anti-invariant submanifold and $N_\top$ an invariant submanifold tangent to V of $\overline{M}$, other than GCR-lightlike product. Thus for simplicity we call a warped product GCR-lightlike submanifold of indefinite Kenmotsu manifolds in the form $M = N_\top \times_\lambda N_\perp$ such that $N_\perp$ is an anti-invariant submanifold and $N_\top$ is an invariant submanifold of $\overline{M}$ tangent to V, a GCR-lightlike warped product.

LEMMA 4.11. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Let $M = N_\top \times_\lambda N_\perp$ be a proper GCR-lightlike warped product of an indefinite Kenmotsu manifold $\overline{M}$ such that such that $N_\top$ is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$ then $N_\top$ is totally geodesic in M.*

Proof. Let $X, Y \in N_\top$ and $Z \in N_\perp$ then we have $g(\nabla_X Y, Z) = \bar{g}(\overline{\nabla}_X Y, Z) = -g(Y, \nabla_X Z) - g(Y, h^l(X, Z))$, using (4.2) we get $g(\nabla_X Y, Z) = -g(Y, h^l(X, Z))$. Since M is a totally umbilical GCR-lightlike submanifold therefore $h^l(X, Z) = h^s(X, Z) = 0$. Hence $g(\nabla_X Y, Z) = 0$ implies that $N_\top$ is totally geodesic in M. ∎

THEOREM 4.12. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. If the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M then it is integrable.*

Proof. Let $X, Y \in \Gamma(D \oplus \{V\})$ then using (3.7) and (3.9), we have $h(X, fY) = Ch(X, Y) + \omega \nabla_X Y$. Since the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M therefore $\omega \nabla_X Y = 0$ and we get $h(X, fY) = Ch(X, Y)$, then taking into account that h is symmetric therefore we obtain $h(X, fY) = h(fX, Y)$, for all $X, Y \in \Gamma(D \oplus \{V\})$. This proves the assertion. ∎

THEOREM 4.13. *Let M be a totally umbilical proper GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ then $H^l = 0$.*

Proof. Let M be a totally umbilical proper GCR-lightlike submanifold then using (3.6) and (3.8), we have $A_{wZ} Z = -f\nabla_Z Z - Bh^l(Z, Z) - Bh^s(Z, Z)$, for $Z \in \Gamma(\phi S)$. Taking inner product with $\phi\xi$, for any $\xi \in \Gamma(D_2)$ the we obtain $g(A_{wZ} Z, \phi\xi) = -g(Bh^l(Z, Z), \phi\xi)$. Using (2.7) and the hypothesis we obtain $g(Z, Z)g(H^l, \xi) = 0$, then using the non degeneracy of M_2, the result follows. ∎

5. A characterization of GCR-lightlike warped products of indefinite Kenmotsu manifolds

For a GCR-lightlike warped product in indefinite Kenmotsu manifolds, we have

LEMMA 5.1. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ then for a GCR-lightlike warped product $M = N_\top \times_\lambda N_\perp$ such that $N_\top$ is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$, we have*

$$\bar{g}(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0.$$

Proof. Since $\overline{M}$ is a Kenmotsu manifold therefore for $X \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(M_2)$ using (2.9), we have $\phi\overline{\nabla}_X Z = \overline{\nabla}_X \phi Z$. Since M is a totally umbilical therefore we have $\phi(\nabla_X Z) = -A_{wZ}X + \nabla^s_X wZ$, then taking inner product with ϕY where $Y \in \Gamma(D \oplus \{V\})$, we get $g(\phi\nabla_X Z, \phi Y) = -g(A_{wZ}X, \phi Y)$. Using (2.8) and (4.2), we obtain $g(A_{wZ}X, \phi Y) = 0$ then using (2.7) we get $\bar{g}(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0$. Hence the proof is complete. ∎

COROLLARY 5.2. *Let $Z \in \Gamma(M_1) \subset \Gamma(\bar{D})$ then clearly $g(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi Z) = 0$ and also $g(h^l(D \oplus \{V\}, D \oplus \{V\}), \phi Z) = 0$ for any $Z \in \Gamma(\bar{D})$. Thus $g(h(D \oplus \{V\}, D \oplus \{V\}), \phi\bar{D}) = 0$, this implies that $h(D \oplus \{V\}, D \oplus \{V\})$ has no component in $L_1 \perp L_2$, that is, $Bh(D \oplus \{V\}, D \oplus \{V\}) = 0$ therefore using Theorem 3.5 the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M.*

Next, we have the following characterizations of *GCR*-lightlike warped products.

THEOREM 5.3. *A proper totally umbilical GCR-lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ is locally a GCR-lightlike warped product $M = N_\top \times_\lambda N_\perp$ such that $N_\top$ is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$ if and only if*

$$A_{\phi Z}X = ((\phi X)\mu)Z, \tag{5.1}$$

for $X \in \Gamma(D \oplus \{V\})$, $Z \in \Gamma(\bar{D})$ and for some function μ on M satisfying $U\mu = 0, U \in \Gamma(\bar{D})$.

Proof. Assume that M is a proper *GCR*-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ satisfying (5.1). Let $X, Y \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(M_2) \subset \Gamma(\bar{D})$, we have $g(A_{\phi Z}X, \phi Y) = g(((\phi X)\mu)Z, \phi Y) = ((\phi X)\mu)g(Z, \phi Y) = 0$, then using (2.7) we get $g(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0$. Then as done in above corollary, the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M and consequently it is totally geodesic in M and using Theorem 4.12 the distribution $D \oplus \{V\}$ is integrable.

Now, taking inner product of (5.1) with $U \in \Gamma(M_2) \subset \bar{D}$ and using (2.8), (2.9), (4.2) and that M is a totally umbilical submanifold, we get $g(((\phi X)\mu)Z, U) = g(A_{\phi Z}X, U) = g(\phi Z, \nabla_X U) = g(\phi Z, \nabla_U X) = -\bar{g}(\overline{\nabla}_U \phi Z, X) = g(\nabla_U Z, \phi X) + \bar{g}(h^l(U, Z), \phi X)$, then using the definition of gradient $g(\nabla\mu, X) = X\mu$ we get

$$g(\nabla_U Z, \phi X) = g(\nabla\mu, \phi X)g(Z, U) - \bar{g}(h^l(U, Z), \phi X). \tag{5.2}$$

Let h' and ∇' be the second fundamental form and the metric connection of $\bar{D}$, respectively in M then we have

$$g(h'(U, Z), \phi X) = g(\nabla_U Z - \nabla'_U Z, \phi X). \qquad (5.3)$$

Therefore from (5.2) and (5.3), particularly for $X \in \Gamma(D_0)$, we get $g(h'(U, Z), \phi X)$ $= g(\nabla_U Z, \phi X) = g(\nabla\mu, \phi X)g(Z, U)$ this further implies that

$$h'(U, Z) = \nabla\mu g(Z, U), \qquad (5.4)$$

this implies that the distribution $\bar{D}$ is totally umbilical in M. Using Theorem 4.5, the totally real distribution $\bar{D}$ is also integrable. Hence using (5.4) and the condition $U\mu = 0$ for $U \in \bar{D}$ we obtain that each leaf of $\bar{D}$ is an extrinsic sphere in M. Hence by a result of [17] which says that "If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non trivial vector subbundles such that E_1 is spherical and its orthogonal complement E_0 is autoparallel, then the manifold M is locally isometric to a warped product $M_0 \times_\lambda M_1$", therefore we can conclude that M is locally a GCR-lightlike warped product $N_\top \times_\lambda N_\perp$ of $\overline{M}$ where $\lambda = e^\mu$.

Conversely, let $X \in \Gamma(N_\top)$ and $Z \in \Gamma(N_\perp)$, since $\overline{M}$ is a Kenmotsu manifold so we have $\overline{\nabla}_X \phi Z = \phi \overline{\nabla}_X Z$, which further becomes $-A_{\phi Z} X + \nabla^t_X \phi Z = ((\phi X) \ln \lambda) Z$, comparing tangential components, wet $A_{\phi Z} X = -((\phi X) \ln \lambda) Z$ for each $X \in \Gamma(D \oplus \{V\})$ and $Z \in (\bar{D})$. Since λ is a function on $N_\top$ so we also have $U(\ln \lambda) = 0$ for all $U \in \Gamma(\bar{D})$. Hence the proof is complete. ■

LEMMA 5.4. *Let $M = N_\top \times_\lambda N_\perp$ be a GCR-lightlike warped product of an indefinite Kenmotsu manifold such that $N_\top$ is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$ then*

$$(\nabla_Z f) X = f X (\ln \lambda) Z,$$
$$(\nabla_U f) Z = g(Z, U) f(\nabla \ln \lambda),$$

for any $U \in \Gamma(TM), X \in \Gamma(N_\top)$ and $Z \in \Gamma(N_\perp)$.

Proof. For any $X \in \Gamma(N_\top)$ and $Z \in \Gamma(N_\perp)$, using (3.8) and (4.2), we have $(\nabla_Z f) X = \nabla_Z f X - f((\frac{Xf}{f}) Z) = \nabla_Z f X - \frac{Xf}{f} f Z = \nabla_Z f X = f X (\ln \lambda) Z$. Next, again using (3.8) we get $(\nabla_U f) Z = -f \nabla_U Z$ this implies that $(\nabla_U f) Z \in \Gamma(N_\top)$, therefore for any $X \in \Gamma(D_0)$ we have

$$g((\nabla_U f) Z, X) = -g(f \nabla_U Z, X) = g(\nabla_U Z, f X) = \bar{g}(\overline{\nabla}_U Z, f X)$$
$$= -g(Z, \nabla_U f X) = -f X (\ln \lambda) g(Z, U).$$

Hence using the definition of gradient of λ and the non degeneracy of the distribution D_0, the result follows. ■

THEOREM 5.5. *A proper totally umbilical GCR-lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ is locally a GCR-lightlike warped product $M =*

$N_\top \times_\lambda N_\perp$ *such that* $N_\top$ *is an invariant submanifold tangent to* V *and* $N_\perp$ *an anti-invariant submanifold of* $\overline{M}$ *if*

$$(\nabla_X f)Y = (fY(\mu))PX + g(PX, PY)\phi(\nabla\mu) + g(\phi X, Y)V - \eta(Y)fX, \qquad (5.5)$$

for any $X, Y \in \Gamma(TM)$ *and for some function* μ *on* M *satisfying* $Z\mu = 0, Z \in \Gamma(\bar{D})$.

Proof. Let M be a *GCR*-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ satisfying (5.5). Let $X, Y \in \Gamma(D \oplus \{V\})$ then (5.5) implies that $(\nabla_X f)Y = g(\phi X, Y)V - \eta(Y)fX$ then (3.6) gives $Bh(X, Y) = 0$. Thus $D \oplus \{V\}$ defines a totally geodesic foliation in M and consequently it is totally geodesic in M and integrable using Theorem 4.12.

Let $X, Y \in \Gamma(\bar{D})$ then (5.5) gives

$$(\nabla_X f)Y = g(PX, PY)\phi(\nabla\mu). \qquad (5.6)$$

Let $U \in \Gamma(D_0)$ then (5.6) implies that

$$g((\nabla_X f)Y, U) = g(PX, PY)g(\phi(\nabla\mu), U). \qquad (5.7)$$

Also using (2.9) with (3.6), we have

$$g((\nabla_X f)Y, U) = g(A_{wY} X, U) = \bar{g}(\overline{\nabla}_X Y, \phi U) = g(\nabla_X Y, \phi U), \qquad (5.8)$$

therefore from (5.7) and (5.8) we get

$$g(\nabla_X Y, \phi U) = -g(\nabla\mu, \phi U)g(X, Y). \qquad (5.9)$$

Let h' and ∇' be the second fundamental form and the metric connection of $\bar{D}$, respectively in M then

$$g(h'(X, Y), \phi U) = g(\nabla_X Y - \nabla'_X Y, \phi U) = g(\nabla_X Y, \phi U), \qquad (5.10)$$

therefore from (5.9) and (5.10) we get $g(h'(X, Y), \phi U) = -g(\nabla\mu, \phi U)g(X, Y)$ then the non degeneracy of the distribution D_0 implies that

$$h'(X, Y) = -\nabla\mu g(X, Y),$$

this gives that the distribution $\bar{D}$ is totally umbilical in M and using Theorem 4.5, the distribution $\bar{D}$ is integrable. Also $Z\mu = 0$ for $Z \in \Gamma(\bar{D})$, hence as in Theorem 5.3 each leaf of $\bar{D}$ is an extrinsic sphere in M. Thus M is locally a *GCR*-lightlike warped product $N_\top \times_\lambda N_\perp$ of $\overline{M}$ where $\lambda = e^\mu$. ∎

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