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FIXED POINTS OF MULTIVALUED SUZUKI-ZAMFIRESCU-(f, g) CONTRACTION MAPPINGS

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Abstract. Coincidence point theorems for hybrid pairs of single valued and multivalued mappings on an arbitrary nonempty set have been proved. As an application of our main result, the existence of common solutions of functional equations arising in dynamic programming are discussed.

1. Introduction and preliminaries

Fixed point theory has provided very useful results applicable in other related disciplines to solve functional equations [3]. In 1968, Banach contraction principle was generalized by Markin [14] for multivalued mappings on complete bounded metric spaces. In 1969, Nadler [17] obtained a multivalued analogue of the Banach contraction principle in complete metric spaces, which was subsequently generalized by several authors (see [5–8, 15, 20, 22]). Different contraction conditions have been introduced and compared in this context (see [10, 11, 21]). Hybrid contractive conditions involving single valued and multivalued mappings are the further additions to metric fixed point theory and its applications (see for details [18, 23, 24, 26, 27]).

Suzuki [28, Theorem 2.1] obtained a generalization of the classical Banach contraction principle which led to a number of results in metric fixed point theory by Kikkawa and Suzuki [12, 13], Moţ and Petruşel [16], Dhompongsa and Yingtaweesittikul [9], and Singh and Mishra [26], among others. It is interesting to note that in all the above results contractivity condition is assumed to hold not for all elements from a domain of a mapping, but only for elements satisfying an additional condition. In this paper we obtain some coincidence point theorems for hybrid pairs of single valued and multivalued mappings on an arbitrary nonempty set with values in a metric space and derive fixed point theorems. Our results extend, unify and generalize several known results in the existing literature ([16, 26, 28, 29]). As an application, we discuss the existence of a common solution for Suzuki-Zamfirescu

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class of functional equations under much weaker contractive conditions than those given in [1–4] and [19]).

The following definitions and results will be needed in the sequel.

Let (X,d) be a metric space and let $CB(X)$, $CL(X)$ and $B(X)$ denote respectively the collection of all nonempty closed and bounded subsets, nonempty closed subsets, and nonempty bounded subsets of X. For $A,B \in CL(X)$, set

$$E_{A,B} = \{\varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A)\},$$

where $N_\varepsilon(A) = \{x \in X : d(x,A) < \varepsilon\}$.

We define a generalized Hausdorff metric H on $CL(X)$ by

$$H(A,B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \emptyset \\ \infty & \text{if } E_{A,B} = \emptyset. \end{cases}$$

Further, for any $A,\ B \in CL(X)$, let $d(a,B) = \inf\{d(a,b) : b \in B\}$, and $\rho(A,B) = \sup\{d(a,b) :\ a \in A, b \in B\}$. Throughout this work, the mapping $\eta : [0,1) \to (\frac{1}{2},1]$ is defined by

$$\eta(r) = \frac{1}{1+r} \text{ for all } r \in [0,1).$$

Definition 1.1. [26] Let (X,d) be a metric space, $f : X \to X$ and $T : X \to CL(X)$. The hybrid pair (f,T) is said to satisfy *Suzuki-Zamfirescu hybrid contraction condition* if there exists $r \in [0,1)$ such that $\eta(r)d(fx,Tx) \leq d(fx,fy)$ implies that

$$H(Tx,Ty) \leq r \max\left\{d(fx,fy), \frac{d(fx,Tx)+d(fy,Ty)}{2}, \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}$$

for all $x,y \in X$.

Definition 1.2. Let $f : X \to X$ and $T : X \to CL(X)$. A point $x \in X$ is said to be: (i) a *fixed point of* f if $f(x) = x$; (ii) a *fixed point of* T if $x \in T(x)$; (iii) a *coincidence point of the pair* (f,T) if $fx \in Tx$; (iv) a *common fixed point of the pair* (f,T) if $x = fx \in Tx$.

We denote the set of all fixed points of f, the set of all coincidence points of the pair (f,T) and the set of all common fixed points of the pair (f,T) by $F(f)$, $C(f,T)$ and $F(f,T)$, respectively. Motivated by the work of [5] and [26], we give the following definition.

Definition 1.3. Let (X,d) be a metric space, and Y be any nonempty set. Let $f,g : Y \to X$ and $T : Y \to CL(X)$. Suppose that $x_0 \in Y$. Then the set

$$O\left(f,g,T;x_0\right) = \left\{\begin{array}{l} y_i : y_i = fx_i \in Tx_{i-1} \text{ for } i = 2k+2, \\ y_i = gx_i \in Tx_{i-1} \text{ for } i = 2k+1, \text{ where } k \geq 0 \end{array}\right\}$$

is called an orbit for the triplet (T,f,g) at x_0. A metric space X is called (T,f,g)-orbitally complete if and only if every Cauchy sequence in the orbit for (T,f,g) at x_0 is convergent in X.

Definition 1.4. Let (X,d) be a metric space, Y be any nonempty set and $f,g : Y \to X$. A mapping $T : Y \to CL(X)$ is called a *multivalued Suzuki-Zamfirescu-(f,g) contraction* if there exists $r \in [0,1)$ such that $\eta(r)d(fx,Tx) \leq d(fx,gy)$ or $\eta(r)d(gx,Tx) \leq d(gx,fy)$ implies that

$$H(Tx,Ty) \leq r \min\{M(fx,gy;T),\ M(fy,gx;T)\}$$

for all $x,y \in X$, where

$$M(fx,gy;T) = \max\left\{d(fx,gy), \frac{d(fx,Tx)+d(gy,Ty)}{2}, \frac{d(fx,Ty)+d(gy,Tx)}{2}\right\},$$

$$M(fy,gx;T) = \max\left\{d(fy,gx), \frac{d(fy,Ty)+d(gx,Tx)}{2}, \frac{d(fy,Tx)+d(gx,Ty)}{2}\right\}$$

Definition 1.5. Let $f : X \to X$ and $T : X \to CL(X)$. The pair (f,T) is called: (i) *commuting* if $Tfx = fTx$ for all $x \in X$; (ii) *weakly compatible* if they commute at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f,T)$; (iii) IT-commuting [23] at $x \in X$ if $fTx \subseteq Tfx$.

2. Coincidence and fixed point theorems

The following theorem is our main result on a multivalued Suzuki-Zamfirescu-(f,g) contraction.

Theorem 2.1. *Let (X,d) be a metric space, and Y be any nonempty set. Let $f,g : Y \to X$ and $T : Y \to CL(X)$ be a multivalued Suzuki-Zamfirescu-(f,g) contraction with $T(Y) \subset f(Y) \cap g(Y)$. If there exists $u_0 \in Y$ such that $f(Y) \cap g(Y)$ is (T,f,g)-orbitally complete at u_0, then the pairs (f,T) and (g,T) have a coincidence point. If $Y = X$ and (f,T) and (g,T) are IT-commuting at coincidence points of (f,T) and (g,T) respectively, then (f,T) has a common fixed point provided that fw is a fixed point of f for some $w \in C(f,T)$ and (g,T) has a common fixed point provided that gz is a fixed point of g for some $z \in C(g,T)$. Moreover, if $C(f,T) \cap C(g,T) \neq \phi$, then f,g and T have a common fixed point.*

Proof. Suppose that $q = 1/\sqrt{r} > 0$, f and g are non-constant mappings and $y_0 = fu_0$. By our assumption, we have $Tu_0 \subseteq g(Y)$. So, there exists a point $u_1 \in Y$ such that $y_1 = gu_1 \in Tu_0$. We choose a point $y_2 \in Tu_1$ such that

$$d(gu_1, y_2) \leq qH(Tu_0, Tu_1).$$

Using the fact $Tu_1 \subseteq f(Y)$, we obtain a point $u_2 \in Y$ such that $y_2 = fu_2 \in Tu_1$. Therefore

$$d(gu_1, fu_2) \leq qH(Tu_0, Tu_1).$$

Since

$$\eta(r)d(fu_0, Tu_0) \leq \eta(r)d(fu_0, gu_1) \leq d(fu_0, gu_1),$$

we have

$$d(gu_1, fu_2) \leq qH(Tu_0, Tu_1)$$

$$
\begin{aligned}
&\le q\min\{M(fu_0,gu_1;T),\ M(fu_1,gu_0;T)\}\\
&\le qM(fu_0,gu_1;T)\\
&\le qr\max\{d(fu_0,gu_1),\frac{d(gu_1,Tu_1)+d(fu_0,Tu_0)}{2},\frac{d(gu_1,Tu_0)+d(fu_0,Tu_1)}{2}\}\\
&\le \frac{1}{\sqrt{r}}r\max\{d(y_0,y_1),\frac{d(y_1,y_2)+d(y_0,y_1)}{2},\frac{d(y_0,y_2)}{2}\}\\
&\le \sqrt{r}\max\{d(y_0,y_1),\frac{d(y_0,y_1)+d(y_1,y_2)}{2}\}.
\end{aligned}
$$

This yields

$$d(y_1,y_2)\le qH(Tu_0,Tu_1)\le \sqrt{r}d(y_0,y_1).$$

As $fu_2\in Tu_1$, we choose $y_3\in Tu_2$ such that $d(fu_2,y_3)\le qH(Tu_1,Tu_2)$. Using the fact $Tu_2\subseteq g(Y)$, we obtain a point $u_3\in Y$ such that $y_3=gu_3\in Tu_2$, and

$$d(fu_2,gu_3)\le qH(Tu_1,Tu_2).$$

Since

$$\eta(r)d(gu_1,Tu_1)\le\eta(r)d(gu_1,fu_2)\le d(gu_1,fu_2),$$

we have

$$
\begin{aligned}
d(fu_2,gu_3)&\le qH(Tu_1,Tu_2)\\
&\le q\min\{M(fu_1,gu_2;T),M(fu_2,gu_1;T)\}\\
&\le qM(fu_2,gu_1;T)\\
&\le qr\max\{d(fu_2,gu_1),\frac{d(gu_1,Tu_1)+d(fu_2,Tu_2)}{2},\frac{d(gu_1,Tu_2)+d(fu_2,Tu_1)}{2}\}\\
&\le \frac{1}{\sqrt{r}}r\max\{d(y_2,y_1),\frac{d(y_1,y_2)+d(y_2,y_3)}{2},\frac{d(y_1,y_3)}{2}\}\\
&\le \sqrt{r}\max\{d(y_2,y_1),\frac{d(y_1,y_2)+d(y_2,y_3)}{2}\}.
\end{aligned}
$$

This implies

$$d(y_2,y_3)\le qH(Tu_1,Tu_2)\le\sqrt{r}d(y_1,y_2)\le(\sqrt{r})^2d(y_0,y_1).$$

As $y_3=gu_3\in Tu_2$, there exists $y_4\in Tu_3$ such that

$$d(gu_3,y_4)\le qH(Tu_2,Tu_3).$$

Since, $Tu_3\subseteq f(Y)$, we obtain a point $u_4\in Y$ such that $y_4=fu_4\in Tu_3$, and

$$d(gu_3,fu_4)\le qH(Tu_2,Tu_3).$$

Now

$$\eta(r)d(fu_2,Tu_2)\le\eta(r)d(fu_2,gu_3)\le d(fu_2,gu_3).$$

This implies

$$
\begin{aligned}
d(gu_3,fu_4)&\le qH(Tu_2,Tu_3)\\
&\le q\min\{M(fu_2,gu_3;T),M(fu_3,gu_2;T)\}
\end{aligned}
$$

$$\begin{aligned}
&\leq qM(fu_2, gu_3; T) \\
&\leq qr \max\{d(fu_2, gu_3), \frac{d(gu_3, Tu_3) + d(fu_2, Tu_2)}{2}, \frac{d(gu_3, Tu_2) + d(fu_2, Tu_3)}{2}\} \\
&\leq \frac{1}{\sqrt{r}} r \max\{d(y_2, y_3), \frac{d(y_3, y_4) + d(y_2, y_3)}{2}, \frac{d(y_2, y_4)}{2}\} \\
&\leq \sqrt{r} \max\{d(y_2, y_3), \frac{d(y_2, y_3) + d(y_3, y_4)}{2}\}.
\end{aligned}$$

Consequently,

$$d(y_3, y_4) \leq qH(Tu_2, Tu_3) \leq \sqrt{r} d(y_2, y_3) \leq (\sqrt{r})^3 d(y_0, y_1).$$

Continuing this process, we obtain a sequence $\{y_n\} \subset Y$ such that for any integer $k \geq 0$

$$y_{2k+1} = gu_{2k+1} \in Tu_{2k} \text{ and } y_{2k+2} = fu_{2k+2} \in Tu_{2k+1}$$

and

$$d(y_n, y_{n+1}) \leq (\sqrt{r})^n d(y_0, y_1).$$

Now for $m > n \geq 1$, we have

$$\begin{aligned}
d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\
&\leq \{(\sqrt{r})^n + (\sqrt{r})^{n+1} + \cdots + (\sqrt{r})^{m-1}\} d(y_0, y_1).
\end{aligned}$$

It follows that $\{y_n\}$ is a Cauchy sequence such that $\{y_n\} \subset O(f, g, T; u_0) \cap f(Y) \cap g(Y)$. Since $f(Y) \cap g(Y)$ is (T, f, g)-orbitally complete at u_0, there exists an element $u \in f(Y) \cap g(Y)$ such that $\lim_{n\to\infty} y_n = u$. Let $z \in g^{-1}u$ and $w \in f^{-1}u$. Then $z, w \in Y$ and $u = gz = fw$. Now we will show that

$$d(fw, Tx) \leq rd(fw, gx) \text{ for any } gx \in g(Y) \cap f(Y) - \{fw\}. \tag{2.1}$$

Since $y_{2k+1} \to fw$ and $y_{2k} \to fw$, therefore there exists a positive integer k_0 such that for all $k \geq k_0$

$$d(fw, gu_{2k+1}) \leq \frac{1}{3} d(fw, gx) \text{ and } (fw, fu_{2k}) \leq \frac{1}{3} d(fw, gx).$$

So, for any $k \geq k_0$, we have

$$\begin{aligned}
\eta(r) d(fu_{2k}, Tu_{2k}) &\leq d(fu_{2k}, Tu_{2k}) \leq d(fu_{2k}, gu_{2k+1}) \\
&\leq d(fu_{2k}, fw) + d(fw, gu_{2k+1}) \leq \frac{2}{3} d(fw, gx) \\
&\leq d(fw, gx) - \frac{1}{3} d(fw, gx) \leq d(fw, gx) - d(fw, fu_{2k}) \\
&\leq d(fu_{2k}, gx).
\end{aligned}$$

This implies

$$\begin{aligned}
&d(gu_{2k+1}, Tx) \leq H(Tu_{2k}, Tx) \\
&\quad \leq r \min\{M(fu_{2k}, gx; T), M(fx, gu_{2k}; T)\} \\
&\quad \leq r \min M(fu_{2k}, gx; T)
\end{aligned}$$

$$\leq r\max\{d(fu_{2k},gx),\frac{d(gx,Tx)+d(fu_{2k},Tu_{2k})}{2},\frac{d(gx,Tu_{2k})+d(fu_{2k},Tx)}{2}\}$$
$$\leq r\max\{d(y_{2k},gx),\frac{d(gx,Tx)+d(y_{2k},y_{2k+1})}{2},\frac{d(gx,y_{2k+1})+d(y_{2k},Tx)}{2}\}.$$

Letting $k\to\infty$ we get

$$d(fw,Tx)\leq r\max\{d(fw,gx),\frac{d(gx,Tx)}{2},\frac{d(gx,fw)+d(fw,Tx)}{2}\}$$
$$\leq r\max\{d(fw,gx),\frac{d(gx,fw)+d(fw,Tx)}{2}\}.$$

If $\max\left\{d(fw,gx),\frac{d(gx,fw)+d(fw,Tx)}{2}\right\}=d(fw,gx)$, we are done.

If $\max\left\{d(fw,gx),\frac{d(gx,fw)+d(fw,Tx)}{2}\right\}=\frac{d(gx,fw)+d(fw,Tx)}{2}$, then we obtain

$$\frac{2-r}{2}d(fw,Tx)\leq\frac{r}{2}d(gx,fw).$$

So

$$d(fw,Tx)\leq\frac{r}{2-r}d(gx,fw)\leq rd(gx,fw)=rd(fw,gx),$$

and hence (2.1) holds.

Next, we show that

$$H(Tw,Tx)\leq r\max\{d(fw,gx),\frac{d(gx,Tx)+d(fw,Tw)}{2},\frac{d(gx,Tw)+d(fw,Tx)}{2}\} \tag{2.2}$$

for any $x\in Y$.

If $x=w$, then (2.2) holds trivially. Suppose that $x\neq w$, then $gx\neq gz\Rightarrow gx\neq fw$. Such a choice is permissible as g is not a constant map. We have

$$d(gx,Tx)\leq d(gx,fw)+d(fw,Tx)$$
$$\leq d(gx,fw)+rd(gx,fw)=(1+r)d(gx,fw)$$

and so $\frac{1}{1+r}d(gx,Tx)\leq d(gx,fw)$. Therefore

$$H(Tw,Tx)=H(Tx,Tw)\leq r\min\{M(fx,gw;T),M(fw,gx;T)\}$$
$$\leq rM(fw,gx;T)$$
$$\leq r\max\{d(fw,gx),\frac{d(gx,Tx)+d(fw,Tw)}{2},\frac{d(gx,Tw)+d(fw,Tx)}{2}\}.$$

Hence (2.2) holds for any $x\in Y$. Therefore

$$d(Tw,fu_{2k+2})\leq H(Tw,Tu_{2k+1})$$
$$\leq r\min\{M(fw,gu_{2k+1};T),M(fu_{2k+1},gw;T)\}$$
$$\leq rM(fw,gu_{2k+1};T)$$
$$\leq r\max\{d(fw,gu_{2k+1}),\frac{d(gu_{2k+1},Tu_{2k+1})+d(fw,Tw)}{2},$$

$$\frac{d(gu_{2k+1}, Tw) + d(fw, Tu_{2k+1})}{2}\}$$

$$\leq r \max\{d(fw, y_{2k+2}), \frac{d(y_{2k+2}, y_{2k+2}) + d(fw, Tw)}{2},$$
$$\frac{d(y_{2k+2}, Tw) + d(fw, y_{2k+2})}{2}\}.$$

On taking limit as $k \to \infty$, we obtain

$$d(fw, Tw) \leq \frac{r}{2} d(fw, Tw).$$

This implies $fw \in Tw$. Further, if $Y = X$ and $ffw = fw$, then due to IT-commutativity of the pair (f, T)

$$fw \in Tw \quad \Rightarrow \quad ffw \in fTw \subseteq Tfw \quad \Rightarrow$$
$$fw = ffw \in fTw \subseteq Tfw \Rightarrow fw = ffw \in Tfw.$$

This shows that fw is a common fixed point of (f, T).

Now we will show that

$$d(gz, Tx) \leq rd(gz, fx) \text{ for any } fx \in g(Y) \cap f(Y) - \{gz\}.$$

Since $y_{2k+1} \to gz$ and $y_{2k+2} \to gz$, there exists a positive integer k_0 such that for all $k \geq k_0$ we have

$$d(gz, gu_{2k+1}) \leq \frac{1}{3} d(gz, fx) \text{ and } d(gz, fu_{2k+2}) \leq \frac{1}{3} d(gz, fx).$$

So, for any $k \geq k_0$, we have,

$$\eta(r) d(gu_{2k+1}, Tu_{2k+1}) \leq d(gu_{2k+1}, Tu_{2k+1}) \leq d(gu_{2k+1}, fu_{2k+2})$$
$$\leq d(gu_{2k+1}, gz) + d(gz, fu_{2k+2}) \leq \frac{2}{3} d(gz, fx)$$
$$\leq d(gz, fx) - \frac{1}{3} d(gz, fx) \leq d(gz, fx) - d(gz, gu_{2k+1})$$
$$\leq d(gu_{2k+1}, fx).$$

This implies that

$$d(fu_{2k+2}, Tx) \leq H(Tu_{2k+1}, Tx)$$
$$\leq r \min\{M(fu_{2k+1}, gx; T), M(fx, gu_{2k+1}; T)\}$$
$$\leq rM(fx, gu_{2k+1}; T)$$
$$\leq r \max\{d(fx, gu_{2k+1}), \frac{d(gu_{2k+1}, Tu_{2k+1}) + d(fx, Tx)}{2},$$
$$\frac{d(gu_{2k+1}, Tx) + d(fx, Tu_{2k+1})}{2}\}$$
$$\leq r \max\{d(fx, y_{2k+1}), \frac{d(y_{2k+1}, y_{2k+2}) + d(fx, Tx)}{2},$$
$$\frac{d(y_{2k+1}, Tx) + d(fx, y_{2k+2})}{2}\}.$$

Letting $k \to \infty$ we get

$$d(gz, Tx) \le r \max\{d(fx, gz), \frac{d(fx, Tx)}{2}, \frac{d(gz,Tx) + d(fx, gz)}{2}\}$$
$$\le r \max\{d(gz, fx), \frac{d(fx,gz) + d(gz, Tx)}{2}\}.$$

If $\max\{d(gz, fx), \frac{d(fx,gz)+d(gz,Tx)}{2}\} = d(gz, fx)$, then we are done.

If $\max\{d(gz, fx), \frac{d(fx,gz)+d(gz,Tx)}{2}\} = \frac{d(fx,gz)+d(gz,Tx)}{2}$, then

$$d(gz, Tx) \le \frac{r}{2} d(gz, fx) + \frac{r}{2} d(gz, Tx).$$

Therefore

$$d(gz, Tx) - \frac{r}{2} d(gz, Tx) \le \frac{r}{2} d(gz, fx),$$
$$\frac{2-r}{2} d(gz, Tx) \le \frac{r}{2} d(gz, fx),$$

and

$$d(gz, Tx) \le \frac{r}{2-r} d(fx,gz) \le r d(gz, fx).$$

Hence

$$d(gz, Tx) \le r d(gz, fx) \text{ for any } fx \in g(Y) \cap f(Y) - \{gz\}.$$

Now we shall show that

$$H(Tx, Tz) \le r \max\left\{d(fx, gz), \frac{d(gz, Tz) + d(fx, Tx)}{2}, \frac{d(gz, Tx) + d(fx, Tz)}{2}\right\} \tag{2.3}$$

for any $x \in Y$.

If $x = z$ then it holds trivially. If $x \ne z$ such that $gz \ne fx$, then we have

$$d(fx, Tx) \le d(fx, gz) + d(gz, Tx) \le d(fx, gz) + r d(gz, fx)$$

and

$$\frac{1}{1+r} d(fx, Tx) \le d(fx, gz).$$

This implies that (2.3) holds for any $x \in Y$ and so

$$d(gu_{2k+1}, Tz) \le H(Tu_{2k}, Tz) \le r \min\{M(fu_{2k}, gz; T), M(fz, gu_{2k}; T)\}$$
$$\le r M(fu_{2k}, gz; T)$$
$$\le r \max\{d(fu_{2k}, gz), \frac{d(gz, Tz) + d(fu_{2k}, Tu_{2k})}{2}, \frac{d(gz, Tu_{2k}) + d(fu_{2k}, Tz)}{2}\}$$
$$\le r \max\{d(y_{2k}, gz), \frac{d(gz, Tz) + d(y_{2k}, y_{2k+1})}{2}, \frac{d(gz, y_{2k+1}) + d(y_{2k}, Tz)}{2}\}.$$

Letting $k \to \infty$ we get $d(gz, Tz) \le \frac{r}{2} d(gz, Tz)$ and $gz \in Tz$.

Further, if $Y = X$, $ggz = gz$, then by the $IT-$commutativity of the pair pair (g, T) we have

$$gz \in Tz \quad \Rightarrow \quad ggz \in gTz \subseteq Tgz \quad \Rightarrow$$
$$gz = ggz \in gTz \subseteq Tgz \Rightarrow gz = ggz \in Tgz.$$

This shows that gz is a common fixed point of (g, T). Let $a \in C(f, T) \cap C(g, T)$. Then

$$d(fa, ga) \leq d(fa, Ta) + d(Ta, ga) = 0.$$

This implies $fa = ga \in Ta$. This shows that f, g and T have a coincidence point. Since $fa = ffa$, we have $fa = ga = ffa = fga = ga = gga \in Tga$. That is, ga is a common fixed point of f, g and T. ■

The above theorem extends and improves Theorem 3.1 in [26].

EXAMPLE 2.2. Let $Y = \{1, 2, 3, 4\}$ and $X = \{2, 3, 4, 7\}$. Let d be the usual metric on X, f, g and T be defined as

$$Tx = \begin{cases} \{2,3,4\}, & \text{if } x = 1,2,3 \\ \{4\}, & \text{if } x = 4, \end{cases} \quad fx = \begin{cases} 4, & \text{if } x = 1 \\ 3, & \text{if } x = 2 \\ 2, & \text{if } x = 3 \\ 7, & \text{if } x = 4, \end{cases} \quad gx = \begin{cases} 2, & \text{if } x = 1 \\ 4, & \text{if } x = 2 \\ 3, & \text{if } x = 3 \\ 4, & \text{if } x = 4. \end{cases}$$

If $r = \frac{2}{3}$, then $\eta(r) = \frac{3}{5}$, and we have

$$d(gx, Tx) = 0 \text{ for all } x \in Y \text{ and } d(fx, Tx) = 0 \text{ for } x = 1, 2, 3.$$

Now for $x = 4$, we have $d(f4, T4) = 3$ and $\eta(r)d(f4, T4) = \frac{9}{5} \leq d(f4, gy)$ for all $y \in Y$. Hence

$$\eta(r)d(fx, Tx) \leq d(fx, gy) \text{ and } \eta(r)d(gx, Tx) \leq d(gx, fy)$$

hold. For all $x, y \in \{1, 2, 3\}$ and for $x = y$, we have $H(Tx, Ty) = 0$. For $x = 4$ and for any $y \in \{1, 2, 3\}$, $H(T4, Ty) = H(\{4\}, \{2, 3, 4\}) = 2$. On the other hand $d(f4, gy) \geq 3$ for any $y \in Y$. Hence

$$H(Tx, Ty) \leq r \min\{M(fx, gy; T), M(fy, gx; T)\},$$

is satisfied. Hence T is a multivalued Suzuki-Zamfirescu-(f, g) contraction with $T(Y) \subset f(Y) \cap g(Y)$. Let $u_0 = 1$, $y_0 = f(1) = 4$, $T(1) \subseteq g(Y)$. Hence, there exists a point $u_1 = 4$ in Y such that $y_1 = g(4) = 4 \in T(1)$. $T(4) = \{4\} \subseteq f(Y)$, we obtain a point $u_2 = 1$ in Y such that $y_2 = 4 = f(1) \in T(4)$. Continuing in this way we construct orbit $\{y_0 = y_1 = y_2 = \cdots = 4\}$ for (f, g, T) at $u_0 = 1$. Also, $f(Y) \cap g(Y)$ is (T, f, g)-orbitally complete at $u_0 = 1$. Moreover $C(f, T) = \{1, 2, 3\}$ and $C(g, T) = \{1, 2, 3, 4\}$.

REMARK 2.3. If we take $f = g$ in Theorem 2.1, we obtain Theorem 3.1 in [26]. Further, by choosing $f = g = I$ (identity map) in Theorem 2.1 we recover Corollary 3.2 in [26] as a special case.

COROLLARY 2.4. *Let $f, g, T : Y \to X$ such that $T(Y) \subset f(Y) \cap g(Y)$. Let there exist $u_0 \in Y$ such that $f(Y) \cap g(Y)$ is (T, f, g)-orbitally complete at u_0. Assume further that there exists an $r \in [0, 1)$ such that $\eta(r)d(fx, Tx) \leq d(fx, gy)$ or $\eta(r)d(gx, Tx) \leq d(gx, fy)$ implies that*

$$d(Tx, Ty) \leq r \min\{M(fx, gy; T), M(fy, gx; T)\}$$

for all $x, y \in X$. Then (f, T) and (g, T) have coincidence point. Further if $Y = X$ and (f, T), (g, T) are commuting pairs at x where $x \in C(f, T) \cap C(g, T)$ then (f, T)

and (g,T) have unique common fixed points. Moreover f,g and T have common fixed point.

Proof. It follows from Theorem 2.1 that $C(f,T)\neq\phi$ and $C(g,T)\neq\phi$. If $u\in C(f,T)\cap C(g,T)$ then $fu=Tu=gu$. Further if $Y=X$ and (f,T) and (g,T) are commuting at u then

$$ffu=fTu=Tfu=fgu \text{ and } gfu=gTu=Tgu=fgu=ffu.$$

Now

$$\eta(r)d(fu,Tu)=0\le d(fu,gfu)$$

implies that

$$\begin{aligned}
d(fu,ffu)&=d(Tu,Tfu)\le r\min\{M(fu,gfu;T),M(ffu,gu;T)\}\\
&\le rM(fu,gfu;T)\\
&\le r\max\{d(fu,gfu),\frac{d(gfu,Tfu)+d(fu,Tu)}{2},\frac{d(gfu,Tu)+d(fu,Tfu)}{2}\}\\
&\le r\max\{d(fu,gfu),\frac{d(gfu,ffu)}{2},\frac{d(gfu,fu)+d(fu,ffu)}{2}\}\\
&\le rd(fu,gfu)=rd(fu,ffu)
\end{aligned}$$

which further implies that fu is a common fixed point of f and T. Similarly we can show that gu is a common fixed point of g and T. The uniqueness of common fixed point is straightforward. Also it can be shown the set of common fixed point of f,g and T is nonempty. ■

REMARK 2.5. By setting $f=g$ in Corollary 2.4, we obtain Corollary 3.3 in [26]. And, if $f=g=I$ (identity map) then Corollary 2.4 recovers Corollary 3.4 in [26].

THEOREM 2.6. *Let $T:Y\to B(X)$ and $f,g:Y\to X$ be such that $T(Y)\subset f(Y)\cap g(Y)$. Suppose there exists $u_0\in Y$ such that $f(Y)\cap g(Y)$ is (T,f,g)-orbitally complete at u_0. Assume there exists $r\in[0,1)$ such that $\eta(r)\rho(fx,Tx)\le d(fx,gy)$ or $\eta(r)\rho(gx,Tx)\le d(gx,fy)$ implies*

$$\rho(Tx,Ty)\le r\min\{m(fx,gy;T),m(fy,gx;T)\}$$

for all $x,y\in X$, where

$$\begin{aligned}
m(fx,gy;T)&=\max\left\{d(fx,gy),\frac{\rho(fx,Ty)+\rho(gy,Tx)}{2},\frac{\rho(fx,Tx)+\rho(gy,Ty)}{2}\right\},\\
m(fy,gx;T)&=\max\left\{d(fy,gx),\frac{\rho(fy,Tx)+\rho(gx,Ty)}{2},\frac{\rho(fy,Ty)+\rho(gx,Tx)}{2}\right\}.
\end{aligned}$$

Then the pairs (f,T) and (g,T) have a coincidence point.

Proof. Chose $\lambda\in(0,1)$. Define $h:Y\to X$ such that for all $x\in Y$, $hx\in Tx$ and we have

$$\begin{aligned}
d(fx,hx)\ge r^{\lambda}\rho(fx,Tx) &\text{ and } d(gx,hx)\ge r^{\lambda}\rho(gx,Tx),\\
d(gy,hx)\ge r^{\lambda}\rho(gy,Tx) &\text{ and } d(fx,hy)\ge r^{\lambda}\rho(fx,Ty).
\end{aligned}$$

Since $hx \in Tx$, we obtain

$$d(fx, hx) \leq \rho(fx, Tx) \text{ and } d(gx, hx) \leq \rho(gx, Tx).$$

This implies

$$\eta(r)d(fx, hx) \leq \eta(r)\rho(fx, Tx) \leq d(fx, gy)$$
$$\eta(r)d(gx, hx) \leq \eta(r)\rho(gx, Tx) \leq d(gx, fy).$$

This further gives

$$\begin{aligned}
&d(hx, hy) \leq \rho(Tx, Ty) \leq r \min\{m(fx, gy; T), m(fy, gx; T)\} \\
&\leq rr^{-\lambda} \max\{r^{\lambda} d(fx, gy), \frac{r^{\lambda}\rho(gy, Ty) + r^{\lambda}\rho(fx, Tx)}{2}, \frac{r^{\lambda}\rho(gy, Tx) + r^{\lambda}\rho(fx, Ty)}{2}\} \\
&\leq r^{1-\lambda} \max\{d(fx, gy), \frac{d(gy, hy) + d(fx, hx)}{2}, \frac{d(gy, hx) + d(fx, hy)}{2}\}.
\end{aligned}$$

Thus Corollary 2.4 can be applied as $h(Y) = \cup\{hx \in Tx\} \subseteq T(Y) \subset f(Y) \cap g(Y)$. Hence (f, h) and (g, h) have a coincidence point. Clearly

$$hx \in Tx \text{ implies } fz \in Tz \text{ and } gz \in Tz. \quad \blacksquare$$

Remark 2.6. By taking $f = g$ in Theorem 2.6 we obtain Theorem 3.5 in [26]. Further, if $f = g = I$ (identity map) then Theorem 2.6 gives Theorem 3.6 in [26].

3. Applications

In this section we assume that U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$. Let $\mathbb{R}$ denote the set of real numbers and

$$\begin{aligned}
\tau &: W \times D \to W, \\
g, g', g'' &: W \times D \to \mathbb{R}, \\
G, F, E &: W \times D \times \mathbb{R} \to \mathbb{R}.
\end{aligned}$$

Considering W and D as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p(x) := \sup_{y \in D}\{g(x, y) + G(x, y, p(\tau(x, y)))\}, \text{ for } x \in W, \tag{3.1}$$

$$q(x) := \sup_{y \in D}\{g'(x, y) + F(x, y, q(\tau(x, y)))\}, \text{ for } x \in W, \tag{3.2}$$

$$s(x) := \sup_{y \in D}\{g''(x, y) + E(x, y, s(\tau(x, y)))\}, \text{ for } x \in W. \tag{3.3}$$

For more on multistage process involving such functional equations, we refer to [1–4, 19, 25]. In this section, we study the existence of the common solution of the functional equations (3.1), (3.2), (3.3) arising in dynamic programming.

Let $B(W)$ denote the set of all bounded real valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold:

$(C1)$ $G, F, E, g, g'$and g'' are bounded.

$(C2)$ Let η be defined as in section (1). There exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D, h, k \in B(W)$ and $t \in W$

$$\eta(r)\,|Kh(t) - Jh(t)| \leq |Jh(t) - Ik(t)|, \eta(r)\,|Kh(t) - Ih(t)| \leq |Ih(t) - Jk(t)| \quad (3.4)$$

implies

$$|G(x, y, h(t)) - G(x, y, k(t))| \leq r \min\{M(Jh(t), Ik(t); K), M(Jk(t), Ih(t); K\},$$

where

$$\begin{aligned} M(Jh(t), Ik(t); K) = \max\{&|Jh(t) - Ik(t)|\,, \frac{|Ik(t) - Kk(t)| + |Jh(t) - Kh(t)|}{2}, \\ &\frac{|Jh(t) - Kk(t)| + |Ik(t) - Kh(t)|}{2}\}, \\ M(Jk(t), Ik(t); K) = \max\{&|Jk(t) - Ih(t)|\,, \frac{|Ih(t) - Kh(t)| + |Jk(t) - Kk(t)|}{2}, \\ &\frac{|Jk(t) - Kh(t)| + |Ih(t) - Kk(t)|}{2}\}. \end{aligned}$$

For $x \in W$ and $h \in B(W)$, define

$$\begin{aligned} Kh(x) &= \sup_{y \in D}\{g(x, y) + G(x, y, h(\tau(x, y)))\}, \\ Jh(x) &= \sup_{y \in D}\{g'(x, y) + F(x, y, h(\tau(x, y)))\}, \\ Ih(x) &= \sup_{y \in D}\{g''(x, y) + E(x, y, h(\tau(x, y)))\}. \end{aligned}$$

$(C3)$ For any $h \in B(W)$, there exists $k \in B(W)$ such that for $x \in W$

$$Kh(x) = Jk(x) \text{ and } Kh(x) = Ik(x).$$

$(C4)$ There exists $h \in B(W)$ such that

$$\begin{aligned} Kh(x) = Jh(x) &\text{ implies } JKh(x) = KJh(x) \\ Kh(x) = Ih(x) &\text{ implies } IKh(x) = KIh(x). \end{aligned}$$

Theorem 3.1. *Assume that the conditions* $(C1)$–$(C4)$ *are satisfied. If* $J(B(W))$ *is a closed convex subspace of* $B(W)$*, then the functional equations* (3.1), (3.2) *and* (3.3) *have a unique common bounded solution.*

Proof. Notice that $(B(W), d)$ is a complete metric space, where d is the metric induced by the supremum norm on $B(W)$. By $(C1)$, J, K and I are self-maps of $B(W)$. The condition $(C3)$ implies that $K(B(W)) \subseteq J(B(W)) \cap I(B(W))$. It follows from $(C4)$ that (J, K) and (I, K) commute at their coincidence points. Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$ and choose $y_1, y_2 \in D$ such that

$$Kh_j < g(x, y_j) + G(x, y_j, h_j(x_j) + \lambda, \quad (3.5)$$

where $x_j = \tau(x, y_j)$, $j = 1, 2$. Further

$$Kh_1 \geq g(x, y_2) + G(x, y_2, h_1(x_2)), \tag{3.6}$$
$$Kh_2 \geq g(x, y_1) + G(x, y_1, h_2(x_1)). \tag{3.7}$$

Therefore (3.4) in $(C2)$ becomes

$$\begin{aligned}\eta(r)\left|Kh_1(x) - Jh_1(x)\right| &\leq \left|Jh_1(x) - Ih_2(x)\right|, \\ \eta(r)\left|Kh_1(x) - Ih_1(x)\right| &\leq \left|Ih_1(x) - Jh_2(x)\right|.\end{aligned} \tag{3.8}$$

Then (3.8) together with (3.5) and (3.7) imply

$$\begin{aligned}Kh_1(x) - Kh_2(x) &< G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_2)) + \lambda \\ &\leq \left|G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_2))\right| + \lambda \\ &\leq r \min\{M(Jh_1(x_1), Ih_2(x_2); K), M(Jh_2(x_2), Ih_1(x_1); K\} \\ &\leq r \max\{\left|Jh_1(x) - Ih_2(x)\right|, \frac{\left|Ih_2(x) - Kh_2(x)\right| + \left|Jh_1(x) - Kh_1(x)\right|}{2}, \\ &\quad \frac{\left|Ih_2(x) - Kh_1(x)\right| + \left|Jh_1(x) - Kh_2(x)\right|}{2}\} + \lambda.\end{aligned} \tag{3.9}$$

So (3.5), (3.6) and (3.8) imply that

$$\begin{aligned}Kh_2(x) - Kh_1(x) &\leq G(x, y_1, h_2(x_2)) - G(x, y_1, h_1(x_1)) \\ &\leq \left|G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_2))\right| \\ &\leq r \min\{M(Jh_1(x_1), Ih_2(x_2); K), M(Jh_2(x_2), Ih_1(x_1); K\} \\ &\leq r \max\{\left|Jh_1(x) - Ih_2(x)\right|, \frac{\left|Ih_2(x) - Kh_2(x)\right| + \left|Jh_1(x) - Kh_1(x)\right|}{2}, \\ &\quad \frac{\left|Ih_2(x) - Kh_1(x)\right| + \left|Jh_1(x) - Kh_2(x)\right|}{2}\} + \lambda.\end{aligned} \tag{3.10}$$

From (3.9) and (3.10) we have

$$\begin{aligned}\left|Kh_1(x) - Kh_2(x)\right| \leq r \max\{&\left|Jh_1(x_1) - Ih_2(x_2)\right|, \\ &\frac{\left|Ih_2(x_2) - Kh_2(x_2)\right| + \left|Jh_1(x_1) - Kh_1(x_2)\right|}{2}, \\ &\frac{\left|Ih_2(x) - Kh_1(x)\right| + \left|Jh_1(x) - Kh_2(x)\right|}{2}\} + \lambda.\end{aligned}$$

Since the above inequality is true for any $x \in W$, and $\lambda > 0$ is taken arbitrary, we find from (3.8) that

$$\eta(r)d(Kh_1, Jh_2) \leq d(Jh_1, Ih_2) \text{ and } \eta(r)d(Kh_1, Ih_2) \leq d(Ih_1, Jh_2).$$

This implies

$$\begin{aligned}d(Kh_1, Kh_2) \leq r \max\{&d(Jh_1, Ih_2), \frac{d(Jh_1, Kh_1) + d(Ih_2, Kh_2)}{2}, \\ &\frac{d(Jh_1, Kh_2) + d(Ih_2, Kh_1)}{2}\}.\end{aligned}$$

Therefore by Corollary 2.4, wherein K, J and I correspond, respectively to the maps T, f, and g and the pairs (K, J) and (K, I) have unique a common fixed point h^*, that is, $h^*(x)$ is a unique bounded common solution. ■

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REFERENCES

[**1**] R. Baskaran, P.V. Subrahmanyam, *A note on the solution of a class of functional equations*, Applicable Analysis, **22** (1986), 235–241.

[**2**] R. Bellman, *Methods of Nonlinear Analysis*, Vol. II, vol. 61 of Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 1973.

[**3**] R. Bellman, E.S. Lee, *Functional equations in dynamic programming*, Aequationes Math. **17** (1978), 1–18.

[**4**] P.C. Bhakta, S. Mitra, *Some existence theorems for functional equations arising in dynamic programming*, J. Math. Anal. Appl. **98** (1984), 348–362.

[**5**] Lj.B. Ćirić, *Fixed points for generalized multivalued contractions*, Mat. Vesnik **9(24)** (1972), 265–272.

[**6**] Lj.B. Ćirić, *Multivalued nonlinear contraction mappings*, Nonlinear Anal. **71** (2009), 2716–2723.

[**7**] H. Covitz, S.B. Nadler Jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11.

[**8**] P.Z. Daffer, H. Kaneko, *Fixed points of generalized contractive multivalued mappings*, J. Math Anal. Appl. **192** (1995), 655–666.

[**9**] S. Dhompongsa, H. Yingtaweesittikul, *Fixed points for multivalued mappings and the metric completeness*, Fixed Point Theory Appl. **2009** (2009), Article ID 972395, pages 1–15.

[**10**] K. Goebel, *A coincidence theorem*, Bull. Acad. Polon. Sci. Math. **16** (1968), 733–735.

[**11**] S. Itoh, W. Takahashi, *Single valued mappings, multivalued mappings and fixed point theorems*, J. Math. Anal. Appl. **59** (1977), 514–521.

[**12**] M. Kikkawa, T. Suzuki, *Three fixed point theorems for generalized contractions with constants in complete metric spaces*, Nonlinear Anal. **69** (2008), 2942–2949.

[**13**] M. Kikkawa, T. Suzuki, *Some similarity between contractions and Kannan mappings*, Fixed Point Theory Appl. **2008** (2008), Article ID 649749, pp. 1–8.

[**14**] J. Markin, *A fixed point theorem for set valued mappings*, Bull. Amer. Math. Soc. **74** (1968), 639–640.

[**15**] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. **141** (1989), 177–188.

[**16**] G. Moţ ̧, A. Petruşel, *Fixed point theory for a new type of contractive multivalued operators*, Nonlinear Analysis **70** (2009), 3371–3377.

[**17**] S.B. Nadler Jr., *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[**18**] S.A. Naimpally, S.L. Singh, J.H.M. Whitfield, *Coincidence theorems for hybrid contractions*, Math. Nachr. **127** (1986), 177–180.

[**19**] H.K. Pathak, Y.J. Cho, S.M. Kang, B.S. Lee, *Fixed point theorems for compatible mappings of type P and applications to dynamic programming*, Le Matematiche **50** (1995), 15–33.

[**20**] S. Reich, *Fixed points of contractive functions*, Boll. UMI **5** (1972), 26–42.

[**21**] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, Romania, 2001.

[**22**] P.V. Semenov, *Fixed points of multivalued contractions*, Funct. Anal. Appl. **36** (2002), 159–161.

[**23**] S.L. Singh, S.N. Mishra, *Nonlinear hybrid contractions*, J. Natur. Phy. Sci. **5-8** (1994), 191–206.

[**24**] S.L. Singh, S.N. Mishra, *Coincidences and fixed points of nonself hybrid contractions*, J. Math. Anal. Appl. **256** (2001), 486–497.

[**25**] S.L. Singh, S.N. Mishra, *On a Ljubomir Ćirić fixed point theorem for nonexpansive type maps with applications*, Indian J. Pure Appl. Math. **33** (2002), 531–542.

[**26**] S.L. Singh, S.N. Mishra, *Coincidence theorems for certain classes of hybrid contractions*, Fixed Point Theory Appl. **2010** (2010) Article ID 898109, 1–14.

[**27**] S.L. Singh, S.N. Mishra, *Remarks on recent fixed point theorems*, Fixed Point Theory Appl. **2010** (2010) Article ID 452905, 1–18.

[**28**] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc. **136** (2008), 1861–1869.

[**29**] T. Zamfirescu, *Fixed point theorems in metric spaces*, Arch. Math. **23** (1972), 292–298.

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