МАТЕМАТИЧКИ ВЕСНИК
66, 1 (2014), 101–108
March 2014

оригинални научни рад
research paper

ON THE INVERTIBILITY OF $AA^+ - A^+A$ IN A HILBERT SPACE

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Abstract. Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. In this paper, we study the class of operators $A \in B(H)$ with closed range such that $AA^+ - A^+A$ is invertible, where A^+ is the Moore-Penrose inverse of A. Also, we present new relations between $(AA^* + A^*A)^{-1}$ and $(A + A^*)^{-1}$. The present paper is an extension of results from [J. Benítez and V. Rakočević, Appl. Math. Comput. 217 (2010) 3493–3503] to infinite-dimensional Hilbert space.

1. Introduction

Let H be a Hilbert space and $B(H)$ be the set of all bounded linear operators on H. Throughout this paper, the range, the null space and the adjoint of $A \in B(H)$ are denoted by $N(A)$, $R(A)$ and A^*, respectively. An operator $A \in B(H)$ is said to be positive if $(Ax, x) \geq 0$. An operator $P \in B(H)$ is said to be idempotent if $P^2 = P$. An orthogonal projection is a self-adjoint idempotent. Clearly, any orthogonal projection is positive. For $A \in B(H)$, if there exists an operator $A^+ \in B(H)$ satisfying the following four operator equations:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad AA^+ = (AA^+)^*, \quad A^+A = (A^+A)^*,$$

then A^+ is called the Moore-Penrose inverse (for short, MP inverse) of A. It is well known that A has the MP inverse if and only if $R(A)$ is closed, the MP inverse of A is unique [5]. It is easy to see that $R(A^+) = R(A^*)$, AA^+ is the orthogonal projection of H onto $R(A)$ and that A^+A is the orthogonal projection of H onto $R(A^*)$. $A \in B(H)$ is said to be an EP operator, if $R(A)$ is closed and $AA^+ = A^+A$ (see [1,7]). If A is an EP operator, then $AA^+ - A^+A$ is not invertible.

In this paper we study the class of operators $A \in B(H)$ with closed range, such that $AA^+ - A^+A$ is invertible. Since AA^+ and A^+A are orthogonal projections, the question of invertibility of $AA^+ - A^+A$ is strongly related to the invertibility of the difference $P - Q$, where P, Q are orthogonal projections on a Hilbert space.

2010 Math. Subject Classification: 47A05

Keywords and phrases: Moore-Penrose inverse; idempotent; orthogonal projection; positive operator.

Buckholtz [3,4] has proved that the operator $P-Q$ is invertible if and only if H is the direct sum $H=R(P)\oplus R(Q)$ of the ranges of P and Q. In this case there exists a linear idempotent M with range $R(P)$, kernel $R(Q)$ and $(P-Q)^{-1}=M+M^*-I$ (see [11,12,13] for further references).

Recently, J. Benítez and V. Rakočević (see [2]) obtained interesting results concerning the nonsingularity of AA^+-A^+A, where A is a square matrix. Notice that in [2] the finite-dimensional methods are mostly based on the CS decomposition and on the rank of a complex matrix. In the present paper we extend results obtained in [2] to infinite-dimensional Hilbert space.

2. Preliminary results

In this section, we present some Lemmas, needed in the sequel.

Lemma 2.1. [9] *Let A and B be in $B(H)$. Then the following statements hold:*

(i) *$R(A)$ is closed if and only if $R(A)=R(AA^*)$,*

(ii) *$R(A)$ is closed if and only if $R(A^*)$ is closed,*

(iii) *$R(A)=R(AA^*)^{\frac{1}{2}}$,*

(iv) *$R(A)+R(B)=R((AA^*+BB^*)^{\frac{1}{2}})$.*

Lemma 2.1. [6,8] *Let $A\in B(H)$ be a positive operator. Then the following statements hold:*

(i) *$R(A)\subseteq R(A^{\frac{1}{2}})$ and $\overline{R(A)}=\overline{R(A^{\frac{1}{2}})}$, where $\overline{K}$ denotes the closure of K,*

(ii) *$R(A)$ is closed if and only if $R(A)=R(A^{\frac{1}{2}})$,*

(iii) *$R(A)=H$ if and only if A is invertible.*

Lemma 2.1. [10] *If $P\in B(H)$ is an idempotent and $\|P\|\ \leq 1$, then P is an orthogonal projection.*

3. Main results

In this section we find several equivalent conditions that ensure the invertibility of AA^+-A^+A, where $A\in B(H)$ has the closed range.

Theorem 3.1. *If $A\in B(H)$ have closed range, then the following statements are equivalent:*

(i) *AA^+-A^+A is invertible,*

(ii) *$R(A)\oplus R(A^*)=H$,*

(iii) *There exists a bounded linear idempotent P with range $N(A^*)$ and kernel $N(A)$,*

(iv) *AA^++A^+A is invertible and $\|A(A^+)^2A\|<1$,*

(v) *AA^*+A^*A is invertible and $R(A)\cap R(A^*)=\{0\}$,*

(vi) *AA^*-A^*A is invertible and $R(A)\cap R(A^*)=\{0\}$.*

Proof. Since AA^+ and A^+A are orthogonal projections onto $R(A)$ and $R(A^*)$ respectively, then the equivalence of (i), (ii) and (iv) follows from [4].

(ii)$\Leftrightarrow$(iii). Assume first that $R(A) \oplus R(A^*) = H$. Then, there exists a bounded linear idempotent M in $B(H)$ such that $R(M) = R(A)$ and $N(M) = R(A^*)$.

Let us define $P = I - M^*$. Then P is an idempotent with range $N(M^*)$ and Kernel $R(M^*)$. By using relations $N(B^*) = R(B)^\perp$ and $R(B^*) = N(B)^\perp$, which are valid for closed range operators $B \in B(H)$, we get $R(P) = N(A^*)$ and $N(P) = N(A)$.

Conversely, if P is an idempotent with range $N(A^*)$ and kernel $N(A)$, then $I - P^*$ is idempotent with range $R(A)$ and kernel $R(A^*)$. According to the space decomposition $H = R(I - P^*) \oplus N(I - P^*)$, we obtain (ii).

(ii)$\Leftrightarrow$(v). Using Lemma 2.1, we obtain $R((AA^* + A^*A)^{\frac{1}{2}}) = R(A) + R(A^*)$. Since $(AA^* + A^*A)^{\frac{1}{2}}$ is a positive operator, it follows from Lemma 2.2, that $R(A) + R(A^*) = H$ if and only if $(AA^* + A^*A)^{\frac{1}{2}}$ is invertible, so $AA^* + A^*A$ is invertible. Hence, (ii)$\Leftrightarrow$(v).

(v)$\Rightarrow$(vi). Assume that (v) holds. By the equivalence (v)$\Leftrightarrow$(iii), there exists an idempotent P such that $R(P) = N(A^*)$ and $N(P) = N(A)$. This implies $A^*P = 0$ and $A(I - P) = 0$. Hence, $AP = A$ and $P^*A = 0$.

Then we easily obtain $(AA^* + A^*A)(I - 2P) = (AA^* - A^*A)$. Since $I - 2P$ is invertible (because $(I - 2P)^2 = I$), we get that $AA^* - A^*A$ is invertible.

(vi)$\Rightarrow$(v). Suppose that (vi) holds. From the invertibility of $AA^* - A^*A$, we deduce $H = R(AA^* - A^*A) = R(A) + R(A^*)$. According to Lemmas 2.1 and 2.2, $(AA^* + A^*A)^{\frac{1}{2}}$ is invertible. Hence $AA^* + A^*A$ is invertible. ∎

Remark 3.2. If P is the idempotent given by Theorem 3.1, then from the proof of (ii)$\Leftrightarrow$(iii), we deduce

$$A^+AP = A^+A, \quad AA^+P = 0, \quad AA^+(I - P^*) = I - P^*, \quad A^+AP^* = P^*.$$

Using these results, we obtain

$$(AA^+ - A^+A)(I - P - P^*) = I - P^* + P^* = I.$$

Taking the adjoint, we get

$$(I - P - P^*)(AA^+ - A^+A) = I.$$

Hence, $(AA^+ - A^+A)^{-1} = I - P - P^*$.

From Theorem 3.1, we see that if $AA^+ - A^+A$ is invertible, then $AA^* + A^*A$ is invertible. In the following example we show that the converse is not true.

Example 3.3. Consider the real Hilbert space ℓ_2 and let A$\in B(\ell_2)$ be the left shift, ie. $A(x_1, x_2, \dots) = (x_2, x_3, \dots)$, then $A^*(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ and $A^+ = A^*$. In this case $AA^+ = I$ and $A^+A(x_1, x_2, \dots) = (0, x_2, x_3, \dots)$. Then, $AA^* + A^*A$ is invertible and $AA^+ - A^+A$ is not injective. Hence $AA^+ - A^+A$ is not invertible.

Theorem 3.4. *Let $A \in B(H)$ have closed range, then the following statements are equivalent:*

(i) $AA^+ - A^+A$ *is invertible,*

(ii) $A+A^*$ *is invertible and there exists an idempotent $P \in B(H)$ such that $AP = A$ and $P^*A = 0$,*

(iii) $A-A^*$ *is invertible and there exists an idempotent $P \in B(H)$ such that $AP = A$ and $P^*A = 0$,*

(iv) $A + A^*$ *is invertible,* $A(A + A^*)^{-1}A = A$ *and* $A^*(A + A^*)^{-1}A = 0$,

(v) $A - A^*$ *is invertible,* $A(A - A^*)^{-1}A = A$ *and* $A^*(A - A^*)^{-1}A = 0$.

Proof. (i)$\Rightarrow$(ii). Since $AA^+ - A^+A$ is invertible, by the proof of Theorem 3.1,(v)$\Leftrightarrow$(vi), there exist an idempotent $P \in B(H)$, such that $AP = A$ and $P^*A = 0$. Then it is easy to check that

$$(A + A^*)(I - P - P^*)(A + A^*) = A^*A - AA^*.$$

We conclude that $R(A^*A - AA^*) \subset R(A + A^*)$ and $N(A + A^*) \subset N(A^*A - AA^*)$. Since, by Theorem 3.1, $A^*A - AA^*$ is invertible. Hence, $A + A^*$ is invertible.

(ii) $\Rightarrow$(i). If $A + A^*$ is invertible, we easily obtain $R(A) + R(A^*) = H$. On the other hand, if $P \in B(H)$ is an idempotent, such that $AP = A$ and $P^*A = 0$, then $R(A^*) \subset R(P^*)$ and $R(A) \subset N(P^*)$. Since $R(P^*) \cap N(P^*) = \{0\}$, we obtain $R(A) \cap R(A^*) = \{0\}$. Consequently, $R(A) \oplus R(A^*) = H$. Thus, by Theorem 3.1, $AA^+ - A^+A$ is invertible.

(ii)$\Leftrightarrow$(iii). Suppose that $P \in B(H)$ is idempotent such that $AP = A$ and $P^*A = 0$. then $(A+A^*)(2P-I) = A-A^*$. Since $2P-I$ is invertible, then $A+A^*$ is invertible if and only if $A - A^*$ is invertible. Hence,(ii) $\Leftrightarrow$(iii).

(ii)$\Rightarrow$(iv). From $AP = A$ and $P^*A = 0$, we have $(A+A^*)P = A$. Since $A+A^*$ is invertible, then $P = (A + A^*)^{-1}A$. This implies $A(A + A^*)^{-1}A = AP = A$ and $A^*(A + A^*)^{-1}A = A^*P = 0$.

(iv)$\Rightarrow$(ii). Let us define $P = (A + A^*)^{-1}A$. From $A(A + A^*)^{-1}A = A$ and $A^*(A + A^*)^{-1}A = 0$, we easily obtain $P^2 = P, AP = A$ and $A^*P = 0$.

The proof of (iii)$\Leftrightarrow$(v) works in the same way as in (ii) $\Leftrightarrow$(iv). ■

Remark 3.5. The existence of the idempotent P, such that $AP = A$ and $P^*A = 0$ is necessary for the invertibility of $AA^+ - A^+A$; for example, let $A \in B(H)$ be self-adjoint invertible, then $A + A^*$ is invertible, but $AA^+ - A^+A = 0$.

Corollary 3.6. *Let $A \in B(H)$ have closed range. If $AA^+ - A^+A$ is invertible, then the idempotent P given by Theorem 3.4 is unique and $R(P) = N(A^*)$ and $N(P) = N(A)$.*

Proof. Let P be the idempotent given in Theorem 3.4. From the proof of Theorem 3.4, (ii)$\Rightarrow$(iv), we get $P = (A + A^*)^{-1}A$. This proves the uniqueness of the idempotent P and the equality $N(P) = N(A)$.

Now, we prove that $R(P) = N(A^*)$. From $A^*P = (P^*A)^* = 0$, we get the inclusion $R(P) \subset N(A^*)$. To prove the reverse inclusion we first, observe that

$$(A + A^*)(I - AA^+) = A(I - AA^+) + [(I - AA^+)A]^* = A(I - AA^+).$$

Hence, we get

$$I - AA^+ = (A + A^*)^{-1}A(I - AA^+) = P(I - AA^+).$$

From $I - AA^+ = P(I - AA^+)$, we obtain $R(I - AA^+) \subset R(P)$. Since $R(I - AA^+) = N(AA^+) = N(A^*)$, Then $N(A^*) \subset R(P)$. Consequently, $R(P) = N(A^*)$. ■

COROLLARY 3.7. *Let $A \in B(H)$ have closed range. If $AA^+ - A^+A$ is invertible. Then*

(i) $(AA^+ - A^+A)^{-1} = (A + A^*)^{-1}(A^*A - AA^*)(A + A^*)^{-1}$,

(ii) $(AA^+ - A^+A)^{-1} = (A - A^*)^{-1}(AA^* - A^*A)(A - A^*)^{-1}$.

Proof. Let P be the idempotent given by Theorem 2.1.

(i) From the proof of Theorem 3.4, (i)⇒(ii), we get

$$(A + A^*)(I - P - P^*)(A + A^*) = A^*A - AA^*.$$

Using the equality $I - P - P^* = (AA^+ - A^+A)^{-1}$ and the invertibility of $A + A^*$ (guaranteed by Theorem 3.4), we deduce the equality (i).

(ii) From $AP = A$ and $P^*A = 0$, we get

$$(A - A^*)(I - P - P^*)(A - A^*) = AA^* - A^*A.$$

The rest of the proof of (ii) is similar to the proof of (i). ■

THEOREM 3.8. *Let $A \in B(H)$ have closed range, then the following statements are equivalent:*

(i) *$AA^+ - A^+A$ is invertible,*

(ii) *$AA^* + A^*A$ is invertible and $A^*A(AA^* + A^*A)^{-1}A^*A = A^*A$,*

(iii) *$AA^* - A^*A$ is invertible and $A^*A(A^*A - AA^*)^{-1}A^*A = A^*A$.*

Proof. (i)⇒(ii). Assume that (i) holds. Using Theorems 3.1 and 3.4, we get $AA^* + A^*A$ is invertible and there exists an idempotent $P \in B(H)$, such that $AP = A$ and $P^*A = 0$. Then $(AA^* + A^*A)P = A^*A$, which implies $P = (AA^* + A^*A)^{-1}A^*A$. Hence $A^*A(AA^* + A^*A)^{-1}A^*A = A^*A$.

(ii)⇒(i). Assume that (ii) holds. Let $P = (AA^* + A^*A)^{-1}A^*A$. From hypotheses, it is easy to get $P^2 = P$ and $N(P) = N(A^*A)$. Since $N(A^*A) = N(A)$ (see Lemma 2.1), then $N(P) = N(A)$.

On the other hand, we have:

$$A^*A(AA^* + A^*A)^{-1}AA^* = A^*A - A^*A(AA^* + A^*A)^{-1}A^*A = 0.$$

Hence, $AA^*P = 0$. So $R(P) \subset N(A^*)$. Since $R(P) + N(P) = H$, we get $N(A) + N(A^*) = H$. This implies $N(A)^\perp \cap N(A^*)^\perp = \{0\}$. Therefore $R(A) \cap R(A^*) = \{0\}$. Using Theorem 3.1 (v), we obtain $AA^+ - A^+A$ is invertible.

(i)$\Leftrightarrow$ (iii). This is similar as (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i). ■

From the above proof and Theorem 3.4, we obtain the following corollary.

COROLLARY 3.9. *Let $A \in B(H)$ with closed range, such that $AA^+ - A^+A$ is invertible. If P is the idempotent given by Theorem 3.1, then*

(i) $P = (AA^* + A^*A)^{-1}A^*A = (A^*A - AA^*)^{-1}A^*A = (A + A^*)^{-1}A = (A - A^*)^{-1}A$,

(ii) $A(AA^* + A^*A)^{-1}A^*A = A(A^*A - AA^*)^{-1}A^*A = A$,

(iii) $A^*(AA^* + A^*A)^{-1}A^*A = A^*(A^*A - AA^*)^{-1}A^*A = 0$.

As we have seen in Theorem 3.1, $AA^+ - A^+A$ is invertible if and only if $R(A) \oplus R(A^*) = H$. But what happens if H is the orthogonal direct sum $R(A) \oplus^\perp R(A^*) = H$ of the ranges of A and A^*?

In the next result we study the class of operators A with closed range such that $R(A)^\perp = R(A^*)$.

THEOREM 3.10. *Let $A \in B(H)$ have closed range, then the following statements are equivalent:*

(i) $R(A) \oplus^\perp R(A^*) = H$,

(ii) $AA^+ + A^+A = I$

(iii) $(AA^+ - A^+A)^2 = I$,

(iv) $A + A^*$ *is invertible and there exists a unique orthogonal projection P such that $AP = A$ and $PA = 0$,*

(v) $A - A^*$ *is invertible and there exists a unique orthogonal projection P such that $AP = A$ and $PA = 0$.*

Proof. (i)$\Leftrightarrow$(ii). It is well know that $R(A) \oplus^\perp R(A^*) = H$ if and only if $R(A)^\perp = R(A^*)$. Since AA^+ and A^+A are orthogonal projections onto $R(A)$ and $R(A^*)$ respectively, then $R(A)^\perp = R(A^*)$ if and only if $A^+A = I - AA^+$. So that $AA^+ + A^+A = I$. Hence, (i)$\Leftrightarrow$(ii).

(ii)$\Leftrightarrow$(iii). Let us first define the orthogonal projections $P_1 = AA^+$ and $P_2 = A^+A$. If $P_1 + P_2 = I$, then $P_1P_2 = P_1(I - P_1) = 0$ and $P_2P_1 = P_2(I - P_2) = 0$. Hence $(P_1 - P_2)^2 = P_1 + P_2 = I$.

Conversely, if $(P_1 - P_2)^2 = I$, then $P_1 + P_2 - P_1P_2 - P_2P_1 = I$. Multiplying the previous equality by P_1 from the left side, we get $P_1P_2P_1 = 0$. So that $(P_2P_1)^*(P_2P_1) = 0$. This is equivalent to $P_1P_2 = P_2P_1 = 0$. Thus $P_1 + P_2 = I$.

(iii)$\Rightarrow$(iv). Assume that (iii) holds. Then $AA^+ - A^+A$ is invertible and $(AA^+ - A^+A)^{-1} = AA^+ - A^+A$. By theorem 3.4, $A + A^*$ is invertible and there exists a unique idempotent $P \in B(H)$ such that $AP = A$ and $P^*A = 0$. It follows from Remark 3.2, that $I - P - P^* = AA^+ - A^+A$. Multiplying the previous equality by P^* from the left side, we get $P^*P = P^*A^+A = (A^+AP)^* = A^+A$. Hence, $\|P\| = (\|P^*P\|)^{\frac{1}{2}} = 1$. According to Lemma 2.3, we conclude that P is an orthogonal projection ($P = P^*$), which satisfies $AP = A$ and $PA = 0$.

(iv)$\Rightarrow$(i). Suppose that (iv) holds. From the invertibility of $A + A^*$, we deduce that $R(A) + R(A^*) = H$.

Now, we prove that $R(A) \perp R(A^*)$. From $AP = A$ and $PA = 0$, we get $A^2 = 0$. So $R(A) \subset N(A)$. Since $N(A) = R(A^*)^\perp$, we conclude that $R(A) \perp R(A^*)$.

(iv)$\Leftrightarrow$(v). This equivalence can be proved in a similar way as (ii)$\Leftrightarrow$(iii), Theorem 3.4. ∎

COROLLARY 3.11. *Let $A \in B(H)$ have closed range. If any item in Theorem 3.10, is satisfied, then*

(i) $A^+ = (A + A^*)^{-1}A(A + A^*)^{-1}$,

(ii) $A^+ = (A - A^*)^{-1}A(A - A^*)^{-1}$,

(iii) $A^+ + (A^+)^* = (A + A^*)^{-1}$,

(iv) $A^+ - (A^+)^* = (A - A^*)^{-1}$,

(v) $A^+ = \frac{1}{2}[(A + A^*)^{-1} + (A - A^*)^{-1}]$.

Proof. (i). By the proof of Theorem 3.10, (iv) $\Rightarrow$(i), we get $A^2 = 0$. Then $(A + A^*)A^+A = A$.

Since $A + A^*$ is invertible, then it is easy to check that

$$A^+ = (A^+A)^+(A + A^*)^{-1} = A^+A(A + A^*)^{-1}$$

By using $A^+A = (A + A^*)^{-1}A$, we obtain (i).

The proof of (ii) is similar to that of (i).

The proof of the remaining statements follows immediately from (i) and (ii) of this corollary. ∎

ACKNOWLEDGEMENT. We are grateful to the referees for helpful comments concerning the paper.

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(received 14.04.2012; in revised form 30.10.2012; available online 01.03.2013)

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