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COMPOSITIONS OF SAIGO FRACTIONAL INTEGRAL OPERATORS WITH GENERALIZED VOIGT FUNCTION

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Abstract. The principal object of this paper is to provide the composition of Saigo fractional integral operators with different forms of Voigt functions. An alternative explicit representation of the generalized Voigt function in terms of Laplace integral transform is shown and the relations between the left-sided and the right-sided Saigo fractional integral operators are established with the ${}_1F_1$-transform and the Whittaker transform, respectively. Many interesting results are deduced in terms of some relatively more familiar hypergeometric functions in one and two variables.

1. Introduction

This paper deals with the generalized Voigt function defined for $\mu,\ y \in \mathbf{R}^+$, $x \in \mathbf{R}$, and $\Re(\mu+\nu) > -1$ by the integral

$$\Omega_{\mu,\alpha,\beta,\nu}(x,y) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu \mathrm{e}^{-yt-\frac{t^2}{4}} {}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right) \mathrm{d}t. \tag{1}$$

Here ${}_1F_2\,(\alpha;\beta,\gamma;x)$ is the the function defined as

$${}_1F_2\,(\alpha;\beta,\gamma;x) = \sum_{r=0}^\infty \frac{(\alpha)_r}{(\beta)_r(\gamma)_r}\ \frac{x^r}{r!},$$

where $\alpha,\ \beta,\ \gamma \in \mathbf{C}$, $\gamma \neq -p$ $(p = 0,1,2,\dots)$ and $(\alpha)_r$ is the Pochhammer symbol [1]. The function in (1) was introduced by Pathan and Shahwan [16]. The special case of the function $\Omega_{\mu,\alpha,\beta,\nu}(x,y)$ in the form

$$V_{\mu,\nu}(x,y) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu \exp(-yt-\frac{t^2}{4}) J_\nu(xt)\mathrm{d}t,\ \ x,y \in \mathbf{R},\ \Re(\mu+\nu) > -1, \tag{2}$$

with $x,y \in \mathbf{R}^+$ and $\Re(\mu+\nu) > -1$, known as generalized (unified) Voigt function, was introduced and studied by Srivastava and Miller [22] and Srivastava and Chen

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[20]. $J_\nu(x)$ appeared in (2) is the Bessel function [17] of the first kind of order ν and is defined as

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\nu+2n}}{\Gamma(n+1)\Gamma(\nu+n+1)}, \; |z| < \infty.$$

The functions defined in (1) and (2) are connected through the relation [16, p. 77]

$$\Omega_{\mu,\alpha,\alpha,\nu}(x,y) = \Gamma(\nu+1) \left(\frac{2}{x}\right)^{\nu} V_{\mu-\nu,\nu}(x,y).$$

The function $V_{\mu,\nu}(x,y)$ is connected to the Voigt functions $K(x,y)$ and $L(x,y)$ introduced by Voigt in 1899 through the following relations.

$$V_{\frac{1}{2},-\frac{1}{2}}(x,y) = K(x,y) \quad \text{and} \quad V_{\frac{1}{2},\frac{1}{2}}(x,y) = L(x,y),$$

where the functions $K(x,y)$ and $L(x,y)$ are respectively defined as follows:

$$K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-yt - \frac{t^2}{4}) \cos(xt) \mathrm{d}t$$

and

$$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-yt - \frac{t^2}{4}) \sin(xt) \mathrm{d}t,$$

with $x \in \mathbf{R}$, $y \in \mathbf{R}^+$. The function $K(x,y) + i\,L(x,y)$ is identical to the so called plasma dispersion function, except for a numerical factor, which is tabulated by Fried and Conte [5] and by Fettis et al. [4]. Further, (1) can be connected to the Struve function $H_\nu(x)$ [2] through the relations

$$\Omega_{\mu,1,\frac{3}{2},\beta-1}(x,y) = \frac{\sqrt{\pi}}{2} \left(\frac{x}{2}\right)^{1-\beta} \Gamma(\beta) \int_0^\infty t^{\mu-\beta+\frac{1}{2}} \mathrm{e}^{-yt-\frac{t^2}{4}} H_{\beta-\frac{3}{2}}(xt) \mathrm{d}t. \tag{3}$$

More special cases, representations and multivariate analogues of (1) can be seen in [14]. For a review of various mathematical properties and computational methods, see Haubold and John [6], Srivastava and Miller [22], Srivastava et al. [23], Yang [24], Khan et al. [7], Pathan et al. [15] etc. These functions have found applications in various fields like astrophysical spectroscopy, neutron physics, plasma physics, statistical communication theory and some problems of mathematical physics and engineering associated with multi-dimensional analysis of spectral harmonics. A further generalization of the Voigt function has been carried out in the literature starting from the fact that the original function is defined for physical reasons as the convolution between the Gaussian and Lorentz densities. Since these densities are both Levy stable densities, a probabilistic generalization can be done by convolution of two general Levy densities. Due to the strong relation between Levy densities and fractional calculus this probabilistic generalization is worth mentioning. The details may be seen from [12] and [13].

The present paper is devoted to the study of the composition of Saigo fractional integral operators with different forms of the generalized Voigt function. The fractional operators given by Saigo [18] for $\nu,\ \beta,\ \eta \in \mathbf{C}$, $x \in \mathbf{R}_+$ and for $\Re(\nu) > 0$, denoted by $I_{0,x}^{\nu,\beta,\eta}$ and $J_{x,\infty}^{\nu,\beta,\eta}$, are defined as

$$I_{0,x}^{\nu,\beta,\eta} f(x) = \frac{x^{-\nu-\beta}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1}\ {}_2F_1\left(\nu+\beta, -\eta; \nu; 1-\frac{t}{x}\right) f(t)\ \mathrm{d}t \tag{4}$$

and

$$J_{x,\infty}^{\nu,\beta,\eta} f(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1}\ t^{-\nu-\beta}\ {}_2F_1\left(\nu+\beta, -\eta; \nu; 1-\frac{x}{t}\right) f(t)\ \mathrm{d}t, \tag{5}$$

where ${}_2F_1(a, b; c; x)$ is the Gauss hypergeometric series. Formula (4) is the left-sided Saigo fractional integral operator and (5) is the right-sided Saigo fractional integral operator. Moreover

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \Re(c-a-b) > 0.$$

When $\beta = -\nu$ in (4) and (5) we get respectively, the classical Riemann-Liouville fractional integral and the Weyl integral [9, 10, 19] of order ν as

$${}_0D_x{}^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\mathrm{d}t,\ \Re(\nu) > 0$$

and

$${}_xW_\infty{}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} f(t)\mathrm{d}t,\ \Re(\nu) > 0$$

whereas if we set $\beta = 0$ in (4) and (5), we obtain the Erdélyi-Kober (E-K) fractional integration operators [9, 10, 19]

$$E_{0,x}^{\nu,\eta} f(x) = \frac{x^{-\nu-\eta}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^\eta f(t)\mathrm{d}t,\ x \in \mathbf{R}_+$$

and

$$K_{x,\infty}^{\nu,\eta} f(x) = \frac{x^\eta}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} t^{-\nu-\eta} f(t)\mathrm{d}t,\ x \in \mathbf{R}_+.$$

The paper is systemized into four sections. In the next section, we establish an alternative representation of the generalized Voigt function, which provide an extension for the result given by Pathan and Daman in 2010 (see reference [14]). Section 3 is devoted to the study of the composition of the left-sided Saigo fractional image of the generalized Voigt function and the results are expressed in closed forms via Kampé de Fériet's function. It is also shown that the image under Saigo operator is related to ${}_1F_1$ -transform. Many interesting results are deduced from so established results. Finally, in the last section, the effects of the right-sided Saigo operator on generalized Voigt functions and its relation with Whittaker transform are described.

2. An alternate representation of generalized Voigt function

The Laplace transform of the function $f(t)$ [2] having the transform parameter p is defined as

$$\mathcal{L}\left\{f(u);p\right\} = \int_0^\infty \mathrm{e}^{-pu} f(u)\mathrm{d}u, \ \Re(p) > 0. \tag{6}$$

Theorem 1. [14] *If* $\mathcal{L}\left\{h(t);p\right\} = g(p)$ *then*

$$\mathcal{L}\left\{\mathrm{e}^{-a\sqrt{t}}h(t);p\right\} = \frac{a}{2\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-\frac{a^2}{4t}}t^{-\frac{3}{2}}g(t+p)\mathrm{d}t, \ \Re(p) > 0, \ \Re(a) > 0.$$

Now we will discuss about an alternative representation of the generalized Voigt function with the aid of Theorem 1. By using the definition (6), we have

$$\mathcal{L}\left\{\mathrm{e}^{-bt}\ t^{\mu-1}\ {}_1F_2\left(\alpha;\beta,1+\nu;-\frac{xt}{2}\right);p\right\}$$
$$= \frac{\Gamma(\mu)}{(p+b)^\mu}{}_2F_2\left(\alpha,\mu;\beta,1+\nu;-\frac{x}{2(p+b)}\right) = G(p+b), \ \Re(p) > 0, \ b > 0.$$

Replacing $\frac{x}{2}$ by $(\frac{x}{2})^2$ and using Theorem 1, we find that

$$\mathcal{L}\left\{\mathrm{e}^{-a\sqrt{t}}\ t^{\mu-1}\ {}_1F_2\left(\alpha;\beta,1+\nu;-\frac{xt}{2}\right);b\right\} = \frac{a}{2\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-\frac{a^2}{4t}}t^{-\frac{3}{2}}G(t+b)\mathrm{d}t.$$

This can be simplified into the following form.

$$\int_0^\infty \mathrm{e}^{-at-bt^2}t^{2\mu-1}{}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right)\mathrm{d}t$$
$$= \frac{a}{2\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-\frac{a^2}{4t}}t^{-\frac{3}{2}}(t+b)^{-\mu}{}_2F_2\left(\alpha,\mu;\beta,1+\nu;-\frac{x^2}{4(t+b)}\right)\mathrm{d}t,$$
$$\Re(\mu) > 0, \ b > 0, \ x > 0, \ \Re(a) > 0.$$

On taking $b = \frac{1}{4}$, $\mu = \frac{\mu+1}{2}$, $a = y$, the expression on the L.H.S. of the above equation represents the generalized Voigt function $\Omega_{\mu,\alpha,\beta,\nu}$ introduced by Pathan and Shahwan [16]. If we assume that $\alpha = \beta$, this alternate representation will coincides with the result established by Pathan and Daman [14].

3. Composition of the left-sided Saigo fractional operator with generalized Voigt function

This section deals with the composition formulae of left-sided Saigo operator and Voigt functions.

Theorem 2. *If* $\alpha^*, \beta^*, \eta, \lambda \in \mathbf{C}$, $\mu \in \mathbf{R}$, $p > 0$, $x > 0$, $\Re(\alpha^*) > 0$, $\Re(\mu+\nu) > -1$, $\Re(1-\lambda) > 0$, *then the left-sided Saigo fractional image of the generalized Voigt* *function in* (1) *is given by*

$$I_{0,p}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)] = C\sum_{r=0}^{\infty}\frac{(\alpha^*+\beta^*)_r(-\eta)_r}{(1-\lambda+\alpha^*)_r r!}\left\{\frac{\Gamma(\frac{\mu+1}{2})}{\sqrt{2}}\right.$$
$$\times F_{0:3;2}^{1:2;1}\left[\begin{matrix}\frac{\mu+1}{2}:\frac{1-\lambda}{2},\frac{2-\lambda}{2};\alpha;\\ -:\frac{1}{2},\frac{1-\lambda+\alpha^*+r}{2},\frac{2-\lambda+\alpha^*+r}{2};\beta,\ 1+\nu;\end{matrix} -p^2,-x^2\right] + \frac{\sqrt{2}p\Gamma(\frac{\mu}{2}+1)(1-\lambda)}{(1-\lambda+\alpha^*+r)}$$
$$\left.\times F_{0:3;2}^{1:2;1}\left[\begin{matrix}\frac{\mu}{2}+1:\frac{2-\lambda}{2},\frac{3-\lambda}{2};\alpha;\\ -:\frac{3}{2},\frac{2-\lambda+\alpha^*+r}{2},\frac{3-\lambda+\alpha^*+r}{2};\beta,1+\nu;\end{matrix} -p^2,-x^2\right]\right\}. \quad (7)$$

Here $C = \frac{\sqrt{x}\ 2^{\mu}p^{-\beta^*-\lambda}B(1-\lambda,\alpha^*)}{\Gamma(\alpha^*)}$ *and* $F_{l:m;n}^{p:q;k}$ *denotes the generalized Kampé de Fériet's function* [21].

Proof. Employing the definitions given in (1), (4) and on interchanging the order of integrations which is permissible due to the uniform convergence, we have

$$I_{0,p}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)] = \sqrt{\frac{x}{2}}\ \frac{p^{-\beta^*-\alpha^*}}{\Gamma(\alpha^*)}\int_0^{\infty}u^{\mu}\mathrm{e}^{-\frac{u^2}{4}}{}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2u^2}{4}\right)$$
$$\times\left[\int_0^p(p-t)^{\alpha^*-1}\mathrm{e}^{-ut}t^{-\lambda}{}_2F_1\left(\alpha^*+\beta^*,-\eta;\alpha^*;1-\frac{t}{p}\right)\mathrm{d}t\right]\mathrm{d}u.$$

With the help of series representation of Gauss hypergeometric function and on taking $t = ps$, further simplification yields

$$I_{0,p}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)] = \sqrt{\frac{x}{2}}\ \frac{p^{-\beta^*-\lambda}}{\Gamma(\alpha^*)}\int_0^{\infty}u^{\mu}\mathrm{e}^{-\frac{u^2}{4}}{}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2u^2}{4}\right)$$
$$\times\sum_{r=0}^{\infty}\frac{(\alpha^*+\beta^*)_r(-\eta)_r}{(\alpha^*)_r\ r!}\left[\int_0^1(1-s)^{\alpha^*+r-1}\mathrm{e}^{-ups}s^{-\lambda}\mathrm{d}s\right]\mathrm{d}u. \quad (8)$$

By using the definition of type-1 beta function and series expansion for the exponential function, we get

$$I_{0,p}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)] = \sqrt{\frac{x}{2}}\ \frac{p^{-\beta^*-\lambda}\ B(1-\lambda,\alpha^*)}{\Gamma(\alpha^*)}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha^*+\beta^*)_r(-\eta)_r}{(1-\lambda+\alpha^*)_r r!}$$
$$\times\frac{(1-\lambda)_k}{(1-\lambda+\alpha^*+r)_k}\ \frac{(-p)^k}{k!}\int_0^{\infty}u^{\mu+k}\mathrm{e}^{-\frac{u^2}{4}}{}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2u^2}{4}\right)\mathrm{d}u.$$

$$I_{0,p}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)]$$
$$= \sqrt{\frac{x}{2}}\ \frac{2^{\mu}\ p^{-\beta^*-\lambda}\ B(1-\lambda,\alpha^*)}{\Gamma(\alpha^*)}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha^*+\beta^*)_r\ (-\eta)_r}{(1-\lambda+\alpha^*)_r r!}$$
$$\times\frac{\Gamma(\frac{\mu+k+1}{2})(1-\lambda)_k}{(1-\lambda+\alpha^*+r)_k}\frac{(-2p)^k}{k!}\ {}_2F_2\left(\alpha,\frac{\mu+k+1}{2};\beta,1+\nu;-x^2\right). \quad (9)$$

Now breaking the k-series into odd and even terms and using the formula $(a)_{2n} = 2^{2n}(\frac{a}{2})_n(\frac{a+1}{2})_n$ and the definition of generalized Kampé de Fériet's function

$$F^{p:q;k}_{l:m;n}\left[\begin{smallmatrix}(a_p):\ (b_q);\ (c_k);\\ (\alpha_l):\ (\beta_m);(\gamma_n)\end{smallmatrix}\ x, y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\ \frac{y^s}{s!},$$

the result in (7) follows. ∎

Corollary 2.1. *A relation also can be established among left-sided Saigo operator, Voigt function and* $_1\mathbf{F}_1$*-transform by evaluating* (4) *by using the series expansion of exponential function and definition of type-1 beta function as*

$$I_{0,p}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)] = \sqrt{\frac{x}{2}}\ p^{-\beta^*-\lambda-\mu}\ \sum_{r=0}^{\infty}\frac{(\alpha^*+\beta^*)_r\ (-\eta)_r}{r!}$$
$$\times\left[{_1\mathbf{F}_1}^{\mu}\left\{e^{-\frac{u^2}{4}}{_1F_2}\left(\alpha;\beta,1+\nu;-\frac{x^2u^2}{4}\right);p\right\}\right],$$

where ${_1\mathbf{F}_1}^{k}(.)$ *denotes an integral transform known as generalized* $_1\mathbf{F}_1$*-transform* [8], *firstly considered by Erdélyi* [3] *and is defined as*

$$({_1\mathbf{F}_1}^{k})(x) = \frac{\Gamma(a)}{\Gamma(c)}\int_0^{\infty}(xt)^k{_1F_1}(a;c;-xt)f(t)\mathrm{d}t,$$

with $a, c \in \mathbf{C}, \Re(a) > 0$ *and* $k \in \mathbf{R}$ *containing the confluent hypergeometric or Kummer function* $_1F_1(a;c;x)$ *in the kernel.*

Some special cases of the above theorem are explained in following corollaries.

Corollary 2.2. *When* $\alpha = \beta$, (9) *reduces to left-sided Saigo image of the function* $V_{\mu,\nu}$ *given as*

$$I_{0,p}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}V_{\mu,\nu}(x,t)] = C_1\ \sum_{r=0}^{\infty}\frac{(\alpha^*+\beta^*)_r(-\eta)_r}{(1-\lambda+\alpha^*)_r r!}$$
$$\times\left\{\frac{\Gamma(\frac{\mu+\nu+1}{2})}{\sqrt{2}}F^{1:2;0}_{0:3;1}\left[\begin{smallmatrix}\frac{\mu+\nu+1}{2}:\frac{1-\lambda}{2},\frac{2-\lambda}{2};-;\\ -:\frac{1}{2},\frac{1-\lambda+\alpha^*+r}{2},\frac{2-\lambda+\alpha^*+r}{2};\ 1+\nu;\end{smallmatrix}-p^2,-x^2\right]\right.$$
$$\left.+\frac{\sqrt{2}p\Gamma(\frac{\mu+\nu}{2}+1)(1-\lambda)}{(1-\lambda+\alpha^*+r)}F^{1:2;0}_{0:3;1}\left[\begin{smallmatrix}\frac{\mu+\nu}{2}+1:\frac{2-\lambda}{2},\frac{3-\lambda}{2};-;\\ -:\frac{3}{2},\frac{2-\lambda+\alpha^*+r}{2},\frac{3-\lambda+\alpha^*+r}{2};1+\nu;\end{smallmatrix}-p^2,-x^2\right]\right\},$$

$$C_1 = \frac{x^{\nu+\frac{1}{2}}\ 2^{\mu}p^{-\beta^*-\lambda}B(1-\lambda,\alpha^*)}{\Gamma(\nu+1)\Gamma(\alpha^*)}.$$

Corollary 2.3. *Let* $\alpha = \beta,\ \mu = \frac{1}{2}$*. For* $\nu = -\frac{1}{2}$*, we have the left-sided Saigo image of the function* $K(x,t)$ *and for* $\nu = \frac{1}{2}$*, the result for* $L(x,t)$ *can be obtained.*

Corollary 2.4. *By using the relation* (3)*, we can deduce a relation involving Struve function and Saigo operator by taking* $\alpha = 1,\ \beta = \frac{3}{2},\ \nu = \beta - 1$*. Similarly for* $\beta = \alpha + 1,\ \nu = \beta - 1$*, result involving Lommel function* [2] *can be obtained.*

Corollary 2.5. *Taking $\alpha = 0$, due to the relation* [14]

$$\Omega_{\mu,0,\beta,\nu}(x,t) = \sqrt{x}\ 2^{\frac{\mu}{2}}\ \Gamma(\mu+1)\mathrm{e}^{\frac{t^2}{4}} D_{-(\mu+1)}(\sqrt{2}t),$$

where $D_\mu(.)$ is the parabolic cylindrical function [2], *we can establish a relation between the Saigo operator and the parabolic cylindrical function from* (7) *as follows:*

$$I_{0,p}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\ \mathrm{e}^{\frac{t^2}{4}}\ D_{-(\mu+1)}(\sqrt{2}t)] = C_2 \sum_{r=0}^{\infty} \frac{(\alpha^*+\beta^*)_r\ (-\eta)_r}{(1-\lambda+\alpha^*)_r\ r!}$$
$$\times \left\{ \frac{\Gamma(\frac{\mu+1}{2})}{\sqrt{2}} {}_3F_3\left(\frac{\mu+1}{2}, \frac{1-\lambda}{2}, \frac{2-\lambda}{2}; \frac{1}{2}, \frac{1-\lambda+\alpha^*+r}{2}, \frac{2-\lambda+\alpha^*+r}{2}; -p^2\right)\right.$$
$$+ \frac{\sqrt{2}p\Gamma(\frac{\mu}{2}+1)(1-\lambda)}{(1-\lambda+\alpha^*+r)}$$
$$\left.\times {}_3F_3\left(\frac{\mu}{2}+1, \frac{2-\lambda}{2}, \frac{3-\lambda}{2}; \frac{3}{2}, \frac{2-\lambda+\alpha^*+r}{2}, \frac{3-\lambda+\alpha^*+r}{2}; -p^2\right)\right\},$$

where $C_2 = \frac{2^{\frac{\mu}{2}} p^{-\beta^*-\lambda} B(1-\lambda,\alpha^*)}{\Gamma(\alpha^*)\Gamma(\mu+1)}$.

Corollary 2.6. *Further, on taking $\beta^* = 0$, the results for Erdélyi-Kober operator can be deduced from above results.*

Corollary 2.7. *Instead of $\beta^* = 0$, if we take $\beta^* = -\alpha^*$ the corresponding results for Riemann-Liouville operator can be obtained.*

4. Right-sided Saigo fractional operator and generalized Voigt function

The function $\Omega_{\mu,\alpha,\beta,\nu}(x,y)$ can also be written as

$$\Omega_{\mu,\alpha,\beta,\nu}(x,s) = \sqrt{\frac{x}{2}}\ \mathcal{L}\left\{u^\mu \mathrm{e}^{-\frac{u^2}{4}} {}_1F_2\left(\alpha; \beta, 1+\nu; -\frac{x^2u^2}{4}\right); s\right\}, \qquad (10)$$

where $\mathcal{L}\{f(u); s\}$ represents the Laplace transform of the function f with the transform parameter s, see [2].

By using the above representation, we will next discuss about the compositions of right-sided Saigo operator with Voigt functions and hence some of their special cases are discussed.

Theorem 3. *If $\alpha^*, \beta^*, \eta, \lambda \in \mathbf{C}$, $\mu \in \mathbf{R}$, $p > 0$, $x > 0$, $\Re(\alpha^*) > 0$, $\Re(\mu+\nu) > -1$, then the Saigo fractional image of the the generalized Voigt function is given by*

$$J_{p,\infty}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)] = \sum_{r=0}^{\infty} \frac{(\alpha^*+\beta^*)_r\ (-\eta)_r}{r!}$$
$$\times \left\{ \frac{p^{2m}\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} \sum_{s=0}^{\infty} \frac{(\frac{1}{2}+m-k)_s}{(2m+1)_s}\ \frac{p^s}{s!}\Omega_{\mu+s,\alpha,\beta,\nu}(x,p)\right.$$
$$\left. + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} \sum_{q=0}^{\infty} \frac{(\frac{1}{2}-m-k)_q}{(1-2m)_q}\ \frac{p^q}{q!}\Omega_{\mu+q-2m,\alpha,\beta,\nu}(x,p)\right\}, \qquad (11)$$

where

$$k=\frac{1-2\alpha^*-2r-\beta^*-\lambda}{2};\ m=-\frac{(\beta^*+\lambda)}{2}. \tag{12}$$

Proof. Consider the right-sided Saigo fractional integral operator. By using (5) and (10), we have

$$J_{p,\infty}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)]$$
$$=\frac{1}{\Gamma(\alpha^*)}\int_p^\infty (t-p)^{\alpha^*-1}t^{-\alpha^*-\beta^*}{}_2F_1\left(\alpha^*+\beta^*,-\eta;\alpha^*;1-\frac{p}{t}\right)t^{-\lambda}$$
$$\times\left\{\sqrt{\frac{x}{2}}\ \mathcal{L}\left\{u^\mu \mathrm{e}^{-\frac{u^2}{4}}{}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2u^2}{4}\right);t\right\}\right\}\ \mathrm{d}t.$$

On taking $t-p=y$ and interchanging the order of integration, due to the uniform convergence of the integral,

$$J_{p,\infty}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)]=\sqrt{\frac{x}{2}}\frac{1}{\Gamma(\alpha^*)}\int_0^\infty u^\mu \mathrm{e}^{-pu-\frac{u^2}{4}}{}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2u^2}{4}\right)$$
$$\times\left[\mathcal{L}\left\{y^{\alpha^*-1}(p+y)^{-\alpha^*-\beta^*-\lambda}\ {}_2F_1\left(\alpha^*+\beta^*,-\eta;\alpha^*;\frac{y}{p+y}\right);u\right\}\right]\ \mathrm{d}t.$$

On simplification by using Gauss hypergeometric series, the Laplace integral on the right-side can be written as

$$\mathcal{L}\left\{y^{\alpha^*-1}(p+y)^{-\alpha^*-\beta^*-\lambda}\ {}_2F_1\left(\alpha^*+\beta^*,-\eta;\alpha^*;\frac{y}{p+y}\right);u\right\}$$
$$=\sum_{r=0}^\infty\frac{(\alpha^*+\beta^*)_r(-\eta)_r}{(\alpha^*)_r\ r!}\mathcal{L}\left\{y^{\alpha^*+r-1}(p+y)^{-\alpha^*-\beta^*-\lambda-r};u\right\}.$$

Employing the formula [2]

$$\mathcal{L}\left\{t^{m-k-\frac{1}{2}}(t+a)^{m+k-\frac{1}{2}};p\right\}=\frac{\Gamma(\frac{1}{2}-k+m)}{a^{\frac{1}{2}-m}}p^{m-\frac{1}{2}}\mathrm{e}^{\frac{ap}{2}}W_{k,m}(ap),$$

$|\arg a|<\pi$, $\Re(\frac{1}{2}-k+m)>0$, where $W_{k,m}(.)$ denotes the Whittaker function [2, 12] defined as

$$W_{k,m}(z)=\mathrm{e}^{-\frac{1}{2}z}\frac{\Gamma(-2m)z^{\frac{1}{2}+m}}{\Gamma(\frac{1}{2}-m-k)}{}_1F_1\left(\frac{1}{2}+m-k;2m+1;z\right)$$
$$+\mathrm{e}^{-\frac{1}{2}z}\frac{\Gamma(2m)z^{\frac{1}{2}-m}}{\Gamma(\frac{1}{2}+m-k)}{}_1F_1\left(\frac{1}{2}-m-k;1-2m;z\right), \tag{13}$$

and on further simplification, we get

$$J_{p,\infty}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)]=\sqrt{\frac{x}{2}}\ p^{m-\frac{1}{2}}\sum_{r=0}^\infty\frac{(\alpha^*+\beta^*)_r\ (-\eta)_r}{r!}$$
$$\times\int_0^\infty u^{\mu-m-\frac{1}{2}}\mathrm{e}^{-\frac{pu}{2}-\frac{u^2}{4}}{}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2u^2}{4}\right)W_{k,m}(pu)\mathrm{d}u. \tag{14}$$

By using (14) and rewriting the result using (1), we get the Theorem. ■

Corollary 3.1. *We also have a relation between* (11) *and generalized Whittaker transform* [8] *as follows:*

$$J_{p,\infty}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}\Omega_{\mu,\alpha,\beta,\nu}(x,t)] = \sqrt{\frac{x}{2}}\, p^{2m-\mu}\sum_{r=0}^{\infty}\frac{(\alpha^*+\beta^*)_r\ (-\eta)_r}{r!} \times W_{k,m}^{\mu-m}\left\{\mathrm{e}^{-\frac{u^2}{4}}{}_1F_2\left(\alpha;\beta,1+\nu;-\frac{x^2u^2}{4}\right);p\right\},$$

where $W_{k,m}^{\sigma}(.)$ *denotes the generalized Whittaker transform given by*

$$W_{k,m}^{\sigma}\{f(t);p\} = \int_0^{\infty}\mathrm{e}^{-\frac{1}{2}pt}(pt)^{\sigma-\frac{1}{2}}W_{k,m}(pt)f(t)\mathrm{d}t,$$

$W_{k,m}(.)$ *is the function defined in* (13). *It is known that the Meijer transform and Varma transform are particular case of generalized Whittaker transform.*

It is interesting to note that if we take $\alpha^* = -\beta^*$, the relation (14) reduces to the one established by Pathan and Daman [14] for the Weyl fractional operator.

Corollaty 3.2. *When* $\alpha = \beta$, *we have the following result:*

$$\begin{aligned} &J_{p,\infty}^{\alpha^*,\beta^*,\eta}[t^{-\lambda}V_{\mu-\nu,\nu}(x,t)] \\ &= \sum_{r=0}^{\infty}\frac{(\alpha^*+\beta^*)_r(-\eta)_r}{r!}\left\{\frac{p^{2m}\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}\sum_{s=0}^{\infty}\frac{(\frac{1}{2}+m-k)_s}{(2m+1)_s}\ \frac{p^s}{s!}V_{\mu-\nu+s,\nu}(x,p)\right. \\ &\qquad \left.+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\sum_{q=0}^{\infty}\frac{(\frac{1}{2}-m-k)_q}{(1-2m)_q}\ \frac{p^q}{q!}V_{\mu-\nu+q-2m,\nu}(x,p)\right\}, \end{aligned}$$

where m *and* k *are the same given in* (12).

Corollary 3.3. *Under the same restrictions of parameters given for Corollaries 2.3, 2.4 and 2.5, the corresponding results for the right-sided Saigo operator and the functions* $K(x,y)$, $L(x,y)$, *Struve function, Parabolic cylindrical function can be obtained from* (11).

Corollary 3.4. *Further, for* $\beta^* = 0$, *the images of Voigt functions can be obtained from above results under the right-sided Erdélyi-Kober fractional operator.*

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