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AN ITERATIVE APPROXIMATION OF FIXED POINTS OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. We prove strong convergence of an iterative scheme for approximation of fixed point of λ-strict pseudocontractive mapping in a uniformly smooth real Banach space (which is not necessarily uniformly convex). We apply our result to approximation of common fixed point of a finite family of strictly pseudocontractive mappings. Our result extends the results of Li and Yao [M. Li, Y. Yao, Strong convergence of an iterative algorithm for λ-strictly pseudocontractive mappings in Hilbert spaces, An. St. Univ. Ovidius Constanta 18 (2010), 219-228] and complements other new interesting results in the literature.

1. Introduction

Let E be a real Banach space and E^* its dual space. We denote by J_q, $(q > 1)$ the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\},$$

where E^* denotes the dual space of E and $\langle .,.\rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. It is well known (see, for example, [8, 17]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$, and that if E^* is strictly convex then J_q is single valued. It is well known that if E is uniformly smooth then J_q is norm-to-norm uniformly continuous on bounded sets (see, e.g., [3, 19]). In the sequel we shall denote single-valued generalized duality mapping by j_q.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive in the terminology of Browder and Petryshy* [2] if there exists $\lambda > 0$

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2 \tag{1.1}$$

for all $x, y \in D(T)$ and for some $j(x - y) \in J(x - y)$. If I denotes the identity operator, then (1.1) can be written in the form

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \lambda\|(I - T)x - (I - T)y\|^2. \tag{1.2}$$

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In Hilbert spaces, (1.1) (and hence (1.2)), for $\lambda \in (0, \frac{1}{2})$, is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.3}$$

where $k = (1 - 2\lambda) < 1$. T is said to be *L-Lipschitzian or Lipschitz* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \tag{1.4}$$

for all $x, y \in D(T)$. If $L = 1$ then T is called *nonexpansive*. Clearly, in Hilbert spaces, every nonexpansive mapping is strictly pseudocontractive.

If E is a q-uniformly smooth Banach space with (single-valued) generalized duality mapping $j_q : E \to E^*$, we say that $T : C \to E$ is (q)-λ-strict pseudocontractive (briefly a (q)-strict pseudocontraction) if for all $x, y \in C$

$$\langle Tx - Ty, j_q(x - y)\rangle \leq \|x - y\|^q - \lambda\|x - y - (Tx - Ty)\|^q. \tag{1.5}$$

Remark 1.1. We note that for $q = 2$, the class of (q)-strict pseudocontractions coincides with that of strict pseudocontractions. For $q < 2$, (q)-strict pseudocontractions do represent a subclass of strict pseudocontractions (see Lemma 3 of [9]).

Browder and Petryshyn [2] introduced the class of λ-strict pseudocontractive mappings in 1967 and proved existence and convergence theorem in real Hilbert spaces. They proved the following theorem.

Theorem BP. [2] *Let H be a real Hilbert space and K a nonempty closed convex and bounded subset of H. Let $T : K \to K$ a λ-strict pseudocontractive mappings for some $0 \leq \lambda < 1$. Then for any fixed $\gamma \in (1 - \lambda, 1)$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by*

$$x_{n+1} = \gamma x_n + (1 - \gamma)Tx_n$$

converges weakly to a fixed point of T.

It is well known that for a nonexpansive mapping T with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$, the classical *Picard iteration sequence* $x_{n+1} = Tx_n$, $x_1 \in D(T)$ does not always converge. An iterative process commonly used for finding fixed points of nonexpansive mappings is the following: For a convex subset K of a Banach space E and $T : K \to K$, the sequence $\{x_n\}_{n=1}^{\infty}$ is defined iteratively by $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. The sequence of (1.6) is generally referred to as the *Mann sequence* in the light of [11].

Construction of fixed points for λ-strict pseudocontractive mappings using the Mann iteration (1.6) has been studied extensively by many authors (see, for example, [1, 4–7, 12–14, 23, 24] and the references contained therein). It is well known that in an infinite-dimensional Hilbert space, the Mann iteration (1.6) has only weak convergence, in general, even for nonexpansive mappings. In order to obtain strong convergence, one has to modify the Mann iteration (1.6).

In 2007, Marino and Xu [12] obtained weak convergence results using Mann iteration (1.6) for λ-strict pseudocontractive mappings in Hilbert spaces and used the "CQ" algorithm to obtain the strong convergence in Hilbert spaces. Furthermore, Acedo and Xu [1] used Mann iteration process to obtain weak convergence for finite family of λ-strict pseudocontractive mappings in Hilbert spaces and later used the "CQ" algorithm to obtain the strong convergence for the finite family of this class of mappings.

In 2008, Zhou [24] proved weak convergence theorem for approximation of λ-strict pseudocontractive mappings and later made a modification of the Mann iteration to obtain strong convergence results for λ-strict pseudocontractive mappings in a real 2-uniformly smooth Banach space. Thus, he extended the results of [12] from Hilbert spaces to 2-uniformly smooth Banach spaces. Zhang and Guo [21] furthermore obtained weak convergence result for λ-strict pseudocontractive mappings in a real q-uniformly smooth and uniformly convex Banach space which also improved on the result of Osilike and Udemene [13].

In 2009, Zhang and Su [23] extended the results of [24] and obtained weak convergence results using Mann iteration (1.6) for λ-strict pseudocontractive mappings in real q-uniformly smooth Banach space and further obtained strong convergence results for finite family of this same class of maps in q-uniformly smooth Banach space using a modification of normal Mann iteration (see [22]). For the strong convergence result, they proved the following theorem.

THEOREM 1.2. [22] *Let K be a nonempty closed convex subset of a q-uniformly smooth real Banach space E and let $T_i : K \to K$, $i = 1, 2, \dots, N$ be a finite family of λ_i-strict pseudocontractive mappings such that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\lambda := \min\{\lambda_i : 1 \le i \le N\}$. Assume for each n, $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \ge 1$ and $\inf_{n\ge 1} \eta_i^{(n)} > 0$, for all $1 \le i \le N$. For arbitrary fixed $u \in K$, define a sequence $\{x_n\}_{n=1}^\infty$ by $x_1 \in K$,*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n \sum\limits_{i=1}^N \eta_i^{(n)} T_i x_n \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \end{cases}$$

for all $n \ge 1$, where $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ are sequences in $(0,1)$ satisfying (i) $\lim_{n\to\infty} \beta_n = 0$, (ii) $\sum_{n=1}^\infty \beta_n = \infty$, (iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$, (iv) $\sum_{n=1}^\infty \sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}| < \infty$, (v) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$, (vi) $\beta_n + \gamma_n + \delta_n = 1$, (vii) $0 < a \le \alpha_n \le \mu$, $\mu = \min\left\{1, \left(\frac{q\lambda}{c_q}\right)^{\frac{1}{q-1}}\right\}$. Then $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point z of $\{T_i\}_{i=1}^N$, where $z = Q_F u$ and $Q_F : K \to F$ is the unique sunny nonexpansive retraction from K onto F.

Furthermore, Yao *et al.* [20] proved path convergence for a nonexpansive mapping in a real Hilbert space. In particular, they proved the following theorem.

THEOREM 1.3. [20] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For $t \in (0,1)$, let the net $\{x_t\}$ be generated by $x_t = TP_C[(1-t)x_t]$, then as $t \to 0$, the net $\{x_t\}$ converges strongly to a fixed point of T.*

Furthermore, they applied Theorem 1.3 to prove the following theorem.

THEOREM 1.4. [20] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0,1)$. For arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$\begin{cases} y_n = P_C[(1-\alpha_n)x_n] \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \ n \geq 1, \end{cases} \tag{1.7}$$

Suppose the following conditions are satisfied:

(a) $\lim \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*. Then the sequence* $\{x_n\}_{n=1}^{\infty}$ *generated by* (1.7) *converges strongly to a fixed point of* T*.*

In 2010, Chidume and Shahzad [5] obtained weak convergence results for λ-strict pseudocontractive mappings in some real uniformly smooth Banach space which is also uniformly convex. Thus, they extended the results of [12, 24, 23] and [21] to a real uniformly smooth Banach space which is also uniformly convex. However, Cholamjiak and Suantai [7] pointed out that the result of [5] (and hence the recent result of Sahu and Petrusel [15]) does not hold in real Hilbert spaces. Hence, Cholamjiak and Suantai improved and extended the results of [5] from a real uniformly smooth and uniformly convex Banach space to a real uniformly convex Banach space which has the Fréchet differentiable norm.

Motivated by the result of Yao *et al.* [20], Cholamjiak and Suantai [6] recently extended the result [20, Theorem 1.4] to countable family of λ-strict pseudocontractive mappings in q-uniformly smooth and uniformly convex real Banach space which also admits weakly sequentially continuous duality mapping j_q. We remark that the result of [6] does not hold in L_p, $3 < p < \infty$.

In [10], Li and Yao introduced the following iterative scheme

$$x_{n+1} = (1-\beta_n-\alpha_n)x_n + \beta_n T x_n, \ n \geq 1, \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfy some appropriate conditions. Furthermore, they proved that the sequence $\{x_n\}$ defined iteratively by (1.8) converges strongly to a fixed point of a λ-strictly pseudo-contractive mapping T in a real Hilbert space H, where $T : H \to H$ and $F(T) \neq \emptyset$.

Motivated by the results of [10], we prove strong convergence of the scheme for approximation of fixed point of λ-strict pseudocontractive mapping in a uniformly smooth real Banach space (which is not necessarily uniformly convex). Our results extend the results of [10] from real Hilbert spaces to uniformly smooth real Banach spaces and complements other new interesting results in the literature.

2. Preliminaries

In the sequel, we shall need the following.

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm (and E is called *smooth*) if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Further, E is said to be *uniformly smooth* if the limit exists uniformly for $(x, y) \in S \times S$. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| = \tau\right\}; \quad \tau > 0.$$

Equivalently, E is said to be *smooth* if $\rho_E(\tau) > 0, \quad \forall \tau > 0$. Let $q > 1$. E is said to be q-uniformly smooth (or to have a modulus of smoothness of power type $q > 1$) if there exists $c > 0$ such that $\rho_E(\tau) \le c\tau^q$. Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces, W_m^p, $1 < p < \infty$, are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p(\text{or } \ell_p) \text{ or } W_m^p \text{ is} \begin{cases} p\text{-uniformly smooth if } \ 1 < p \le 2 \\ 2\text{-uniformly smooth if } \ p \ge 2. \end{cases}$$

It is shown in [19] that there is no Banach space which is q-uniformly smooth with $q > 2$. It is also known that every uniformly smooth space (e.g., L_p space, $1 < p < \infty$) has uniformly Gâteaux differentiable norm.

We need the following lemmas in the sequel.

Lemma 2.1. [21] *Let E be a real Banach space and C a nonempty closed convex subset of E. For each $1 \le i \le N$, let $T_i : C \to C$ be a λ_i-strict pseudocontraction. Assume that $\{\eta_i\}_{i=1}^N$ is a sequence of positive numbers such that $\sum_{i=1}^N \eta_i = 1$. Then, $\sum_{i=1}^N \eta_i T_i$ is a λ-strict pseudocontraction with $\lambda := \min\{\lambda_i : 1 \le i \le N\}$. If in addition, $\{T_i\}_{i=1}^N$ has a common fixed point, then $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.2. *Let E be a real normed linear space. Then the following inequality holds*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \ \forall\ x, y \in E,\ \forall\ j(x+y) \in J(x+y).$$

Lemma 2.3. [18] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n,\ n \ge 1,$$

where $\{a_n\}_{n=1}^{\infty} \subset [0,1]$ *and* $\{\sigma_n\}_{n=1}^{\infty}$ *is a sequence in* $\mathbb{R}$ *satisfying:*

(*i*) $\sum \alpha_n = \infty$;

(*ii*) $\limsup \sigma_n \leq 0$ *or* $\sum |\alpha_n \sigma_n| < \infty$.

Then, $a_n \to 0$ *as* $n \to \infty$.

LEMMA 2.4. [3, p. 21] *Let* E *be a real Banach space and* J *be the normalized duality map on* E. *Then* $J(\lambda x) = \lambda J(x), \forall \lambda \in \mathbb{R}, \forall x \in E$.

LEMMA 2.5. [16] *Let* C *be a nonempty closed convex subset of a Banach space* E *with a uniformly Gâteaux differentiable, and* $T : C \to C$ *be a continuous pseudocontractive mapping with a fixed point. If there exists a bounded sequence* $\{x_n\}$ *such that* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *and* $p = \lim_{t\to 0} z_t$ *exists, where* $\{z_t\}$ *is defined by* $z_t = tu + (1-t)Tz_t$. *Then*

$$\limsup_{n\to\infty} \langle u - p, j(x_n - p)\rangle \leq 0.$$

LEMMA 2.6. [7] *Let* E *be a real Banach space with Fréchet differentiable norm. For* $x \in E$, *let* $\beta^*(t)$ *be defined for* $0 < t < \infty$ *by*

$$\beta^*(t) = \sup\Big\{\Big|\frac{\|x+ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x)\rangle\Big| : \|y\| = 1\Big\}. \tag{2.1}$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0$ *and*

$$\|x+h\|^2 \leq \|x\|^2 + 2\langle h, j(x)\rangle + \|h\|\beta^*(\|h\|)$$

for all $h \in E \setminus \{0\}$.

REMARK 2.7. In a real Hilbert space, we see that $\beta^*(t) = t$ for $t > 0$.

In the result of Cholamjiak and Suantai [7], the authors assumed that $\beta^*(t) \leq 2t$ for $t > 0$. This naturally leads to this important question.

QUESTION. What uniformly smooth Banach spaces (except Hilbert spaces) satisfy the assumption $\beta^*(t) \leq 2t$ for $t > 0$? In particular, do L_p spaces, $1 < p < \infty$ satisfy it?

In $E = L_p, 2 \leq p < \infty$, we know that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + (p-1)\|y\|^2,\ \forall x, y \in E.$$

Then β^* in (2.1) is estimated by $\beta^*(t) \leq (p-1)t$ for $t > 0$.

In our more general setting, throughout this paper, we will assume that

$$\beta^*(t) \leq ct,\ t > 0 \text{ and for some } c > 1,$$

where β^* is the function appearing in (2.1).

LEMMA 2.8. *Let* C *be a nonempty convex subset of a real Banach space* E *with Fréchet differentiable norm and* $T : C \to C$ *be a* λ*-strict pseudo-contraction. For*

$\alpha \in (0,1)$, we define $T_\alpha x := (1-\alpha)x + \alpha Tx$. Then, as $\alpha \in (0,\mu]$, $\mu := \min\left\{1, \frac{2\lambda}{c}\right\}$, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.

Proof. For any $x, y \in C$, we compute

$$\begin{aligned}
\|T_\alpha x - T_\alpha y\|^2 &= \|(1-\alpha)(x-y) + \alpha(Tx - Ty)\|^2 \\
&= \|(x-y) - \alpha(x - y - (Tx - Ty))\|^2 \\
&\le \|x-y\|^2 - 2\alpha\langle x - y - (Tx - Ty), j(x-y)\rangle \\
&\quad + \alpha\|x - y - (Tx - Ty)\|\beta^*(\|x - y - (Tx - Ty)\|) \\
&\le \|x-y\|^2 - 2\alpha\langle x - y - (Tx - Ty), j(x-y)\rangle \\
&\quad + c\alpha^2\|x - y - (Tx - Ty)\|^2 \\
&\le \|x-y\|^2 - \alpha(2\lambda - c\alpha)\|x - y - (Tx - Ty)\|^2 \\
&\le \|x-y\|^2,
\end{aligned}$$

which shows that T_α is a nonexpansive mapping.

It is obvious that $x = T_\alpha x \Leftrightarrow x = Tx$. This proves the assertion. ∎

Remark 2.9. Our Lemma 2.8 extends Lemma 2.2 of Zhang and Su [22] from q-uniformly smooth Banach space to real Banach space E with Fréchet differentiable norm and Proposition 4.1 of Sahu and Petrusel [15] from uniformly smooth Banach space to real Banach space E with Fréchet differentiable norm. Furthermore, boundedness assumption imposed on C in [15, Proposition 4.1] is dispensed with in this our more general setting.

3. Main results

Using our Lemma 2.8 in place of Lemma 2.2 of Zhang and Su [22] and following the same line of proof of Theorem 3.1 of [22], the following theorem can easily be proved.

Theorem 3.1. *Let K be a nonempty closed convex subset of a uniformly smooth real Banach space E and let $T_i : K \to K$, $i = 1, 2, \ldots, N$ be a finite family of λ_i-strict pseudocontractive mappings such that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\lambda := \min\{\lambda_i : 1 \le i \le N\}$. Assume that, for each n, $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \ge 1$ and $\inf_{n\ge1} \eta_i^{(n)} > 0$, for all $1 \le i \le N$. For arbitrary fixed $u \in K$, define a sequence $\{x_n\}_{n=1}^\infty$ by $x_1 \in K$,*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n \sum\limits_{i=1}^N \eta_i^{(n)} T_i x_n \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \end{cases}$$

for all $n \ge 1$, where $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ are sequences in $(0,1)$ satisfying: (i) $\lim_{n\to\infty} \beta_n = 0$, (ii) $\sum_{n=1}^\infty \beta_n = \infty$, (iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$, (iv) $\sum_{n=1}^\infty \sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}| < \infty$, (v) $0 < \liminf_{n\to\infty} \gamma_n \le$

$\limsup_{n\to\infty} \gamma_n < 1$, *(vi)* $\beta_n + \gamma_n + \delta_n = 1$, *(vii)* $0 < a \le \alpha_n \le \mu$, $\mu = \min\left\{1, \frac{2\lambda}{c}\right\}$. *Then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a common fixed point* z *of* $\{T_i\}_{i=1}^{N}$, *where* $z = Q_F u$ *and* $Q_F : K \to F$ *is the unique sunny nonexpansive retraction from* K *onto* F.

Remark 3.2. Our Theorem 3.1 extends the results of Zhang and Su [22, 23] from q-uniformly smooth Banach spaces to uniformly smooth Banach spaces.

Furthermore, using our Lemma 2.8 in place of Proposition 4.1 of [15] and following the same line of proof of Theorems 4.5 and 4.7 of [15], the following theorems can easily be proved.

Theorem 3.3. *Let* C *be a nonempty, closed and convex subset of a real uniformly smooth Banach space* E *and let* $T : C \to C$ *be a* λ*-strictly pseudocontractive mapping. Given* $u, x_1 \in C$, *a sequence* $\{x_n\}$ *in* C *is defined by*

$$x_{n+1} = T_w[(1-\alpha_n)x_n + \alpha_n u],$$

where $T_w = (1-w)I + wT$ *for some* $w \in (0,\mu], \mu := \min\left\{1, \frac{2\lambda}{c}\right\}$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1]$ *satisfying the following condition*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and either* $\lim_{n\to\infty} \left|1 - \frac{\alpha_n}{\alpha_{n+1}}\right| = 0$ *or* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $Q_{F(T)}(u)$, *where* $Q_{F(T)}$ *is the sunny nonexpansive retraction from* C *onto* $F(T)$.

Theorem 3.4. *Let* C *be a nonempty, closed and convex subset of a real uniformly smooth Banach space* E *and let* $T : C \to C$ *be a* λ*-strictly pseudocontractive mapping. Given* $u, x_1 \in C$, *a sequence* $\{x_n\}$ *in* C *is defined by*

$$x_{n+1} = T_w[(1-\alpha_n)x_n + \alpha_n u],$$

where $T_w = (1-w)I + wT$ *for some* $w \in (0,\mu]$, $\mu := \min\left\{1, \frac{2\lambda}{c}\right\}$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1]$ *satisfying the condition* $(C1)$. *Then* $\{x_n\}$ *converges strongly to* $Q_{F(T)}(u)$, *where* $Q_{F(T)}$ *is the sunny nonexpansive retraction from* C *onto* $F(T)$.

Remark 3.4. The boundedness assumption on Theorem 4.5 and Theorem 4.7 of [15] is dispensed within our Theorems 3.3 and 3.4.

Lemma 3.6. *Let* C *be a nonempty, closed and convex subset of a real Banach space* E *with Fréchet differentiable norm and* $T : C \to C$ *be a* λ*-strict pseudo-contraction such that* $F(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two real sequences in* $(0,1)$. *Assume that the following conditions are satisfied:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(C2)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C3)$ $\beta_n \in [\epsilon, \mu(1-\alpha_n))$, $\mu := \min\left\{1, \frac{2\lambda}{c}\right\}$ *for some* $\epsilon > 0$.

For a fixed $u \in C$, *let the sequence* $\{x_n\}_{n=1}^{\infty}$ *be generated iteratively by* $x_1 \in C$,

$$x_{n+1} = (1-\beta_n)x_n + \beta_n T x_n - \alpha_n(x_n - u),\ n \ge 1. \tag{3.1}$$

Then the sequence $\{x_n\}$ *is bounded.*

Proof. Take $p \in F(T)$, then we have from (3.1) that

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(Tx_n - p) + \alpha_n(u-p)\| \\ &\le \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\| + \alpha_n\|u-p\| \\ &= \|(1-\alpha_n)(x_n - p) - \beta_n(x_n - Tx_n)\| + \alpha_n\|u-p\|. \end{aligned} \tag{3.2}$$

Furthermore, we obtain from 3.2, (1.2) and Lemma 2.4 that

$$\begin{aligned}&\|(1-\alpha_n)(x_n-p) - \beta_n(x_n - Tx_n)\|^2 \\ &\quad \le (1-\alpha_n)^2\|x_n - p\|^2 + \beta_n^2 c\|x_n - Tx_n\|^2 - 2\beta_n(1-\alpha_n)\langle x_n - Tx_n, j(x_n - p)\rangle \\ &\quad \le (1-\alpha_n)^2\|x_n - p\|^2 + \beta_n^2 c\|x_n - Tx_n\|^2 - 2\lambda\beta_n(1-\alpha_n)\|x_n - Tx_n\|^2 \\ &\quad = (1-\alpha_n)^2\|x_n - p\|^2 - \beta_n(2\lambda(1-\alpha_n) - c\beta_n)\|x_n - Tx_n\|^2 \\ &\quad \le (1-\alpha_n)^2\|x_n - p\|^2. \end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|u-p\| \\ &\le \max\{\|x_n - p\|, \|u-p\|\} \\ &\le \vdots \\ &\le \max\{\|x_n - p\|, \|u-p\|\}. \end{aligned}$$

Hence $\{x_n\}$ is bounded and also is $\{Tx_n\}$. ■

Theorem 3.7. *Let C be a nonempty, closed and convex subset of a uniformly smooth real Banach space E and $T : C \to C$ be a λ-strict pseudo-contraction such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$. Assume that the following conditions are satisfied:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(C2)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C3)$ $\beta_n \in [\epsilon, \mu(1-\alpha_n))$, $\mu := \min\left\{1, \frac{2\lambda}{c}\right\}$ *for some* $\epsilon > 0$.

For a fixed $u \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by $x_1 \in C$,

$$x_{n+1} = (1-\beta_n)x_n + \beta_n Tx_n - \alpha_n(x_n - u), \; n \ge 1.$$

Then the sequence $\{x_n\}$ converges strongly to a point of $F(T)$.

Proof. Using Lemmas 2.2 and 2.6, and (3.1), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p) - \alpha_n(x_n - u)\|^2 \\ &= \|(x_n - p) - \beta_n(Tx_n - p) - \alpha_n(x_n - u)\|^2 \\ &\le \|x_n - p\|^2 - 2\beta_n\langle x_n - Tx_n, j(x_n - p)\rangle \\ &\qquad + c\beta_n^2\|x_n - Tx_n\|^2 - 2\alpha_n\langle x_n - u, j(x_{n+1} - p)\rangle \end{aligned}$$

$$
\begin{aligned}
&\leq \|x_n - p\|^2 - 2\beta_n\lambda\|x_n - Tx_n\|^2 + c\beta_n^2\|x_n - Tx_n\|^2 \\
&\quad - 2\alpha_n\langle x_n - u, j(x_{n+1} - p)\rangle \\
&= \|x_n - p\|^2 - \beta_n(2\lambda - c\beta_n)\|x_n - Tx_n\|^2 \\
&\quad - 2\alpha_n\langle x_n - u, j(x_{n+1} - p)\rangle.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, then there exists $M > 0$ such that

$$\|x_{n+1} - p\|^2 - \|x_n - p\|^2 \leq \alpha_n M - \beta_n(2\lambda - c\beta_n)\|x_n - Tx_n\|^2.$$

This implies that

$$
\begin{aligned}
0 &< \epsilon(2\lambda(1-\alpha_n) - c\beta_n)\|x_n - Tx_n\|^2 \\
&\leq \beta_n(2\lambda - c\beta_n)\|x_n - Tx_n\|^2 \\
&\leq \alpha_n M + \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \qquad (3.4)
\end{aligned}
$$

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - p\|\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\|x_n - p\|\}_{n=0}^{\infty}$ converges and $\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0,\ n \to \infty$. This implies from (3.4) and condition $(C3)$ that

$$\|x_n - Tx_n\| \to 0,\ n \to \infty.$$

By Lemma 2.5, we have that

$$\limsup_{n\to\infty}\langle u - p, j(x_n - p)\rangle \leq 0.$$

Using Lemma 2.2 and (3.1) in (3.1), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n)(x_n - p) + \beta_n(Tx_n - p) + \alpha_n(u - p)\|^2 \\
&\leq \|(1 - \alpha_n - \beta_n)(x_n - p) + \beta_n(Tx_n - p)\|^2 + 2\alpha_n\langle u - p, j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle u - p, j(x_{n+1} - p)\rangle.
\end{aligned}
$$

By Lemma 2.3, we have that $x_n \to p$ as $n \to \infty$.

Case 2. Assume that $\{\|x_n - p\|\}$ is not monotonically decreasing sequence. Set $\Gamma_n := \|x_n - p\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ for some n_0 large enough by

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (3.4), it is easy to see that

$$\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \leq \frac{\alpha_{\tau(n)}M}{\epsilon(2\lambda(1 - \alpha_{\tau(n)}) - c\beta_{\tau(n)})} \to 0,$$

thus $\|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0$. By similar argument as above in Case 1, we conclude immediately that

$$\limsup_{n\to\infty}\langle u - p, j(x_{\tau(n)} - p)\rangle \leq 0.$$

At the same time, we note that for all $n \geq n_0$,

$$\begin{aligned}0 &\leq \|x_{\tau(n)+1} - p\|^2 - \|x_{\tau(n)} - p\|^2 \\ &\leq \alpha_{\tau(n)}(\langle u - p, j(x_{\tau(n)+1} - p)\rangle - \|x_{\tau(n)} - p\|^2).\end{aligned}$$

Hence, we deduce that $\lim_{n\to\infty} \|x_{\tau(n)} - p\| = 0$. Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to p. This completes the proof. ■

Corollary 3.8. *Let C be a nonempty, closed and convex subset of a 2-uniformly smooth real Banach space E and $T : C \to C$ be a λ-strict pseudo-contraction such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$. Assume that the following conditions are satisfied:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(C2)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C3)$ $\beta_n \in [\epsilon, \mu(1 - \alpha_n)), \mu := \min\left\{1, \frac{2\lambda}{c}\right\}$ *for some* $\epsilon > 0$.

For a fixed $u \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by $x_1 \in C$,

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n T x_n - \alpha_n(x_n - u),\ n \geq 1.$$

Then the sequence $\{x_n\}$ converges strongly to a point of $F(T)$.

By following the same line of proof of Theorem 3.6, we can prove the following corollary.

Corollary 3.9. [10] *Let H be a real Hilbert space. Let $T : H \to H$ be a λ-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$. Assume that the following conditions are satisfied:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(C2)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C3)$ $\beta_n \in [\epsilon, 2\lambda(1 - \alpha_n))$ *for some* $\epsilon > 0$.

Let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by $x_1 \in H$,

$$x_{n+1} = (1 - \beta_n - \alpha_n)x_n + \beta_n T x_n,\ n \geq 1.$$

Then the sequence $\{x_n\}$ converges strongly to a point of $F(T)$.

We next apply the result of Theorem 3.6 to approximate the common fixed point of a finite family of λ-strict pseudocontractive mappings in real Banach spaces.

Theorem 3.10. *Let C be a nonempty, closed and convex subset of a uniformly smooth real Banach space E. For each $i = 1, 2, \dots, N$, let $T_i : C \to C$ be a λ_i-strict pseudocontractive mapping such that $\cap_{i=1}^N F(T_i) \neq \emptyset$. Assume that $\{k_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N k_i = 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$. Assume that the following conditions are satisfied:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(C2)$ $\sum_{n=1}^\infty \alpha_n = \infty$;

$(C3)$ $\beta_n \in [\epsilon, \mu(1-\alpha_n)), \mu := \min\left\{1, \frac{2\lambda}{c}\right\}$ *for some* $\epsilon > 0$.

For a fixed $u \in C$, let the sequence $\{x_n\}_{n=1}^\infty$ be generated iteratively by $x_1 \in C$,

$$x_{n+1} = (1-\beta_n)x_n + \beta_n \sum_{i=1}^N k_i T_i x_n - \alpha_n(x_n - u), \ n \geq 1. \tag{3.5}$$

Then the sequence $\{x_n\}$ converges strongly to a common point p in $\cap_{i=1}^N F(T_i)$.

Proof. Define $A := \sum_{i=1}^N k_i T_i$. Then, by Lemma 2.6, A is λ-strict pseudocontractive mapping and $F(A) = \bigcap_{i=1}^N F(T_i)$. We can rewrite the scheme (3.5) as

$$x_{n+1} = (1-\beta_n)x_n + \beta_n A x_n - \alpha_n(x_n - u), \ n \geq 1.$$

Now, Theorem 3.6 guarantees that $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$. ■

Remark 3.11. Our Corollary 3.9 extends the result of [10] from approximation of fixed points of a λ-strictly pseudocontractive mapping in a Hilbert space to approximation of fixed points of a λ-strictly pseudocontractive mapping in a uniformly smooth real Banach space.

Remark 3.12. The prototypes of our control sequences in Theorem 3.6 are

$$\alpha_n = \frac{1}{n+1}, \ n \geq 1 \quad \text{and} \quad \beta_n = \epsilon + \frac{n}{n+1}\Big(\frac{2\lambda}{c}\frac{n}{n+1} - \epsilon\Big), \ n \geq 1.$$

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