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COMMON FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS SATISFYING AN IMPLICIT RELATION

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Abstract. We prove common fixed point theorems in metric spaces for expansive mappings satisfying an implicit relation without non-decreasing assumption and surjectivity using the concept of weak compatibility which generalize some theorems appearing in the recent literature.

1. Introduction

Let S and T be self-mappings of a metric space (X, d). S and T are commuting if $STx = TSx$ for all $x \in X$.

Sessa [15] defined S and T to be weakly commuting if for all $x \in X$

$$d(STx, TSx) \leq d(Tx, Sx)$$

Jungck [6] defined S and T to be compatible as a generalization of weakly commuting if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is easy to show that commutativity implies weak commutativity and this implies compatibility, and there are examples in the literature verifying that the inclusions are proper, see [6] and [15].

Jungck et al [7] defined S and T to be compatible mappings of type (A) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(TSx_n, S^2x_n) = 0.$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Examples are given to show that the two concepts of compatibility are independent, see [7].

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Recently, Pathak and Khan [11] defined S and T to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, T^2x_n)] \text{ and}$$
$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, S^2x_n)]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true, see [11]. However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent if S and T are continuous, see [11].

Pathak et al. [12] defined S and T to be compatible mappings of type (P) if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if S and T are continuous, see [12].

Pathak et al. [13] defined S and T to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, S^2x_n) + \lim_{n\to\infty} d(Tt, T^2x_n)]$$

and

$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, T^2x_n) + \lim_{n\to\infty} d(St, S^2x_n)]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if S and T are continuous, see [13].

2. Preliminaries

Definition 2.1. [8] Mappings $S, T : X \to X$ are said to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$ implies $STu = TSu$.

Lemma 2.2. [6, 7, 11–13]. *If S and T are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.*

The converses are not true in general, see [2].

Definition 2.3. [9] Mappings $S, T : X \to X$ are said to be R-weakly commuting if there exists an $R > 0$ such that

$$d(STx, TSx) \leq Rd(Tx, Sx) \text{ for all } x \in X. \tag{2.1}$$

Definition 2.4. [10] Mappings $S, T : X \to X$ are said to be pointwise R-weakly commuting if for each $x \in X$, there exists an $R > 0$ such that (2.1) holds.

It is proved in [10] that R-weak commutativity is equivalent to commutativity at coincidence points; i.e., S and T are pointwise R-weakly commuting if and only if they are weakly compatible.

Let $\mathbb{R}_+$ be the set of all non-negative real numbers and $\mathcal{G}_6$ the family of all continuous mappings $G : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(G_1): G is non-decreasing in the fifth and sixth variables.

(G_2): there exists $\theta > 1$ such that for all $u, v \geq 0$ with

(G_a): $G(u, v, u, v, u + v, 0) \geq 0$ or (G_b): $G(u, v, v, u, 0, u + v) \geq 0$

we have $u \geq \theta v$.

(G_3): $G(u, u, 0, 0, u, u) < 0$ for all $u > 0$.

The following theorem was proved in [5].

Theorem 2.5. *Let A, B, S and T be self-mappings of a complete metric space (X, d) satisfying the following conditions:*

i) *A and B are surjective.*

ii) *The pairs (A, S) and (B, T) are weakly compatible.*

iii) *$G(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By)) \geq 0$ for all x, y in X and some $G \in \mathcal{G}_6$.*

Then A, B, S and T have a unique common fixed point in X.

Remark 2.6. A similar theorem is proved in [1].

It is our goal in this paper to prove common fixed point theorems in metric spaces for expansive mappings satisfying an implicit relation without non-decreasing assumption and surjectivity using the concept of weak compatibility which generalizes theorems of [4], [5] and [14].

3. Implicit relations

Let $\mathcal{F}_6$ be the family of all continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(C_1): there exists $h > 1$ such that for all $u, v, w \geq 0$ with

(C_a): $F(u, v, v, u, 0, w) \geq 0$ or (C_b): $F(u, v, u, v, w, 0) \geq 0$

we have $u \geq hv$.

(C_2): $F(u,u,0,0,u,u) < 0$ for all $u > 0$.

Example 3.1.

$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\max\{t_2,t_3,t_4\} - b(t_5+t_6)$, $a > 1$ and $b > 0$.

(C_1): Let $u,v,w \geq 0$. We have $F(u,v,v,u,0,w) = u - a\max\{v,u\} - bw \geq 0$. If $v \leq u$, then $u > u$ which is a contradiction. Therefore, $u \geq av$. Similarly, if $F(u,v,u,v,w,0) \leq 0$, then $u \geq av$.

(C_2): $F(u,u,0,0,u,u) = (1-a-2b)u < 0$ for all $u > 0$.

Example 3.2.

$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\max\{t_2,t_3,t_4\} - bt_5t_6$, $a > 1$ and $b > 0$.

(C_1) and (C_2) as in Example 3.1.

Example 3.3.

$F(t_1,t_2,t_3,t_4,t_5,t_6) = (1+pt_2)t_1 - pt_3t_4 - a\max\{t_2,t_3,t_4\} - b(t_5+t_6)$, $a > 1$, $b > 0$ and $p \geq 0$.

(C_1) and (C_2) as in Example 3.1.

Example 3.4.

$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 + b\dfrac{t_3^2+t_4^2}{t_5+t_6+1}$, $0 < a,b$ and $a > 2b+1$.

(C_1): Let $u,v,w \geq 0$ and $0 \leq F(u,v,v,u,0,w) = u^2 - av^2 + b\dfrac{(u^2+v^2)}{w+1} \leq u^2 - av^2 + b(u^2+v^2)$. Then, $u^2 \geq \dfrac{a-b}{1+b}v^2$. Hence, $u \geq hv$, $h = \left(\dfrac{a-b}{1+b}\right)^{\frac{1}{2}} > 1$.

Similarly, if $F(u,v,u,v,w,0) \geq 0$, then $u \geq hv$.

(C_2): For all $u > 0$, $F(u,u,0,0,u,u) = (1-a)u^2 < 0$.

Example 3.5.

$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 + b\dfrac{t_3^2+t_4^2}{t_5t_6+1}$, $0 < a,b$ and $a > 2b+1$.

(C_1) and (C_2) as in Example 3.4.

Example 3.6.

$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - bt_3 + c\dfrac{t_4t_5}{t_5+t_6+1}$, $a > 1, 0 \leq b < 1, c > 0$ and $a+b-c > 1$.

(C_1): Let $u,v,w \geq 0$ and $F(u,v,v,u,0,w) = u - av - bv \geq 0$. Then $u \geq h_1v$, $h_1 = a+b > 1$.

$0 \leq F(u,v,u,v,w,0) = u - av - bu + c\dfrac{vw}{w+1} \leq u - av - bu + cv$ implies $u \geq h_2v$. Hence, $h_2 = \dfrac{a-c}{1-b} > 1$. We take $h = \min\{h_1,h_2\}$.

(C_2): $F(u,u,0,0,u,u) = (1-a)u < 0$ for all $u > 0$.

Example 3.7.

$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 + b\dfrac{t_3t_6}{t_5+t_6+1} - ct_4$, $a > 1, b > 0, 0 \leq c < 1$ and $a+c-b > 1$.

(C_1): Let $u, v, w \geq 0$ and $0 \leq F(u, v, v, u, 0, w) = u - av + b\dfrac{vw}{w+1} - cu \leq u - av + bv - cu$. Then $u \geq h_1 v$, $h_1 = \dfrac{a-b}{1-c} > 1$. $F(u, v, u, v, w, 0) = u - av - cv \geq 0$ implies $u \geq h_2 v$. Hence, $h_2 = a + c > 1$. We take $h = \min\{h_1, h_2\}$.

(C_2): For all $u > 0$, $F(u, u, 0, 0, u, u) = (1 - a - c)u < 0$.

4. Main results

Theorem 4.1. *Let A, B, S and T be self-mappings of a metric space (X, d) satisfying the following conditions*

$$S(X) \subset B(X) \text{ and } T(X) \subset A(X), \tag{4.1}$$

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By)) \geq 0 \tag{4.2}$$

for all $x, y \in X$ and some $F \in \mathcal{F}_6$. Suppose that $A(X)$ or $B(X)$ or $S(X)$ or $T(X)$ is complete and the pairs (A, S) and (B, T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (4.1), we can define inductively a sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1} \quad \text{and} \quad y_{2n+1} = Ax_{2n+2} = Tx_{2n+1} \tag{4.3}$$

for all $n = 0, 1, 2, \dots$. Using (4.2) and (4.3) we have

$$\begin{aligned} 0 \leq F(&d(Ax_{2n}, Bx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}),\\ &d(Ax_{2n}, Tx_{2n+1}), d(Sx_{2n}, Bx_{2n+1}))\\ = F(&d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}), 0). \end{aligned}$$

By (C_b) we get $d(y_{2n-1}, y_{2n}) \geq hd(y_{2n}, y_{2n+1})$. Similarly, we obtain by (C_a), $d(y_{2n+1}, y_{2n}) \geq hd(y_{2n+2}, y_{2n+1})$. Therefore

$$d(y_n, y_{n+1}) \leq \frac{1}{h} d(y_{n-1}, y_n).$$

Now, assume that $A(X)$ is complete. Then, $\{y_{2n+1}\} = \{Ax_{2n+2}\} \subset A(X)$ converges to a point $z = Au$ for some $u \in X$ and the subsequences $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ converge also to z.

If $z \neq Su$, using (4.2) we have

$$\begin{aligned} 0 \leq F(&d(Au, Bx_{2n+1}), d(Su, Tx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}),\\ &d(Au, Tx_{2n+1}), d(Su, Bx_{2n+1})). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$F(0, d(Su, z), d(Su, z), 0, 0, d(Su, z)) \geq 0.$$

By (C_a), we get $z = Su = Au$. Since $S(X) \subset B(X)$ there exists $v \in X$ such that $z = Su = Bv$.

If $z \neq Tv$, using (4.2) we get

$$\begin{aligned} 0 &\leq F(d(Au,Bv),d(Su,Tv),d(Au,Su),d(Bv,Tv),d(Au,Tv),d(Su,Bv)) \\ &= F(0,d(z,Tv),0,d(z,Tv),d(z,Tv),0). \end{aligned}$$

By (C_b), we obtain $z = Tv = Bv = Au = Su$. As the pairs (A,S) and (B,T) are weakly compatible, we get $Az = Sz$ and $Bz = Tz$.

If $z \neq Az$, using (4.2) we have

$$\begin{aligned} 0 &\leq F(d(Az,Bv),d(Sz,Tv),d(Az,Sz),d(Bv,Tv),d(Az,Tv),d(Sz,Bv)) \\ &= F(d(Az,z),d(Az,z),0,0,d(Az,z),d(Az,z)), \end{aligned}$$

which is a contradiction with (C_2). Therefore, $z = Az = Sz$. Similarly, we can prove that $z = Bz = Tz$. Hence, z is a common fixed point of A,B,S and T. The uniqueness of z follows from (4.2) and (C_2).

In a similar manner, Theorem 4.1 holds if $B(X)$ or $S(X)$ or $T(X)$ is complete instead of $A(X)$. ■

Remark 4.2. As the function F in Theorem 4.1 is non-decreasing in variables t_5 and t_6, Theorem 2.5 of [5] and theorems of [4] and [14] are not applicable.

Theorem 4.3. *Let $\{g_i\}_{i\geq 1}$, S and T be self-mappings of a metric space (X,d) satisfying the following conditions:*

$$S(X) \subset g_{i+1}(X) \text{ and } T(X) \subset g_i(X), i \geq 1 \tag{4.4}$$

$$F(d(g_ix,g_{i+1}y),d(Sx,Ty),d(g_ix,Sx),d(g_{i+1}y,Ty),d(g_ix,Ty),d(Sx,g_{i+1}y)) \geq 0 \tag{4.5}$$

for all $x,y \in X$ and some $F \in F_6$. Suppose that $g_i(X)$ or $g_{i+1}(X)$ or $S(X)$ or $T(X)$ is complete and the pairs (g_i,S) and (g_i,T) are weakly compatible. Then $\{g_i\}_{i\geq 1}$, S and T have a unique common fixed point in X.

Proof. It follows as in the proof of Theorem 4.3 of [4]. ■

Theorem 4.3 generalizes Theorem 4.3 of [4].

Theorem 4.4. *Let A,B,S and T be self-mappings of a complete metric space (X,d) satisfying (4.1) and (4.2). Suppose that $A(X)$ or $B(X)$ or $S(X)$ or $T(X)$ is closed and the pairs (A,S) and (B,T) are weakly compatible. Then, A,B,S and T have a unique common fixed point in X.*

Proof. As in the proof of Theorem 4.1, $\{y_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, it converges to a point $z \in X$ and the sub-sequences $\{Ax_{2n+2}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ converge also to z. Now, assume that $A(X)$ is closed. Then, $z = Au$ for some $u \in X$. The rest of the proof follows as in Theorem 4.1. ■

Remark 4.5. As the function F in Theorem 4.4 is non-decreasing in variables t_5 and t_6, Theorem 2.5 of [5] and theorems of [4] and [14] are not applicable.

Theorem 4.6. *Let $\{g_i\}_{i\geq 1}$, S and T be self-mappings of a complete metric space (X,d) satisfying (4.4) and (4.5). Suppose that $g_i(X)$ or $g_{i+1}(X)$ or $S(X)$ or $T(X)$ is closed and the pairs (g_i,S) and (g_i,T) are weakly compatible. Then $\{g_i\}_{i\geq 1}$, S and T have a unique common fixed point in X.*

Proof. It follows as in the proof of Theorem 4.3. ■

The following example supports our Theorem 4.4.

Example 4.7. Let $X=[1,\infty)$, $d(x,y)=|x-y|$, A,B,S and T be self-mappings of X defined by:

$$Ax=\begin{cases}2x^6 & \text{if } x\in[1,\infty) \text{ and } x\neq 2\\ 2 & \text{if } x=2\end{cases}$$
$$Sx=\begin{cases}x^3+1 & \text{if } x\in[1,\infty) \text{ and } x\neq 2\\ 2 & \text{if } x=2\end{cases}$$
$$Bx=\begin{cases}2x^4 & \text{if } x\in[1,\infty) \text{ and } x\neq 2\\ 2 & \text{if } x=2\end{cases}$$
$$Tx=\begin{cases}x^2+1 & \text{if } x\in[1,\infty) \text{ and } x\neq 2\\ 2 & \text{if } x=2\end{cases}$$

and

$$F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-at_2+b\frac{t_3t_6}{t_5+t_6+1}-ct_4,$$

$a>1$, $b>0$, $0\leq c<1$ and $a+c-b>1$. It is easy to see that the pairs (A,S) and (B,T) are weakly compatible.

If $x=y=2$ or $x=y=1$, we have

$$F(d(Ax,By),d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(Sx,By))=0.$$

If $x\in[1,\infty)$, $x\neq 2$ and $y\in[1,\infty)$, $y\neq 2$, we get

$$d(Ax,By)=2\left|x^6-y^4\right|=2(x^3+y^2)\left|x^3-y^2\right|\geq 4d(Sx,Ty).$$

If $x\in(1,\infty)$, $x\neq 2$ and $y=2$, we get

$$d(Ax,By)=2\left|x^6-1\right| \quad \text{and} \quad d(Sx,Ty)=\left|x^3-1\right|.$$

It follows that

$$\frac{d(Ax,By)}{d(Sx,Ty)}=\frac{2\left|x^6-1\right|}{|x^3-1|}>4.$$

Hence $d(Ax,By)>4d(Sx,Ty)$.

Similarly, if $x=2$ and $y\in[1,\infty)$, $y\neq 2$ we get $d(Ax,By)>4d(Sx,Ty)$. Then, for all $x,y\in X$

$$d(Ax,By)\geq 4d(fx,gy)-b\frac{d(Ax,Sx)d(Sx,By)}{d(Ax,Ty)+d(Sx,By)+1}+cd(By,Ty),$$

and so

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By)) \geq 0.$$

Thus, all conditions of Theorem 4.4 hold and 2 is the unique common fixed point of A, B, S and T. Note that Theorem 2.5 of [5] is not applicable since the mappings A and B are not surjective.

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REFERENCES

[**1**] M. Akkouchi, *Common fixed points for weakly compatible mappings satisfying implicit relations*, Demonstratio Math., **44** (1) (2011), 151–158.

[**2**] A. Aliouche, *A common fixed point theorem for weakly compatible mappings in compact metric spaces satisfying an implicit relation*, Sarajevo J. Math., **3** (1) (2007), 1–8.

[**3**] A. Aliouche and A. Djoudi, *Common fixed point theorems for mappings satisfying an implicit relation without decreasing assumption*, Hacettepe J. Math. Stat., **36** (1) (2007), 11–18.

[**4**] A. Djoudi, *A common fixed point theorem for compatible mappings of type (B) in complete metric spaces*, Demonstratio Math., **36** (2) (2003), 463–470.

[**5**] A. Djoudi, *General fixed point theorems for weakly compatible maps*, Demonstratio Math., **38** (1) (2005), 197–205.

[**6**] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. and Math. Sci., **9** (1986), 771–779.

[**7**] G. Jungck, P. P Murthy and Y. J. Cho, *Compatible mappings of type (A) and common fixed points*, Math. Japonica **38** (2) (1993), 381–390.

[**8**] G. Jungck, *Common fixed points for non-continuous non-self maps on non metric spaces*, Far East J. Math. Sci., **4** (2) (1996), 199–215.

[**9**] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl., **188** (1994), 436–440.

[**10**] R. P. Pant, *Common fixed points for four mappings*, Bull. Calcutta Math. Soc., **9** (1998), 281–286.

[**11**] H. K. Pathak and M. S. Khan, *Compatible mappings of type (B) and common fixed point theorems of Gregus type*, Czechoslovak Math. J. **45** (120) (1995), 685–698.

[**12**] H. K. Pathak, Y. J. Cho, S. M. Kang and B. S. Lee, *Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming*, Le Matematiche, **1** (1995), 15–33.

[**13**] H. K. Pathak, Y. J. Cho, S. M. Khan and B. Madharia, *Compatible mappings of type (C) and common fixed point theorems of Gregus type*, Demonstratio Math., **31** (3) (1998), 499–518.

[**14**] V. Popa, *On unique common fixed point for compatible mappings of type (A)*, Demonstratio Math., **30** (4) (1997), 931–936.

[**15**] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math., **32** (46) (1982), 149–153.

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