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COARSE TOPOLOGIES ON THE REAL LINE

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Abstract. Let $c = |\mathbb{R}|$ denote the cardinality of the continuum and let η denote the Euclidean topology on $\mathbb{R}$. Let $\mathcal{L}$ denote the family of all Hausdorff topologies τ on $\mathbb{R}$ with $\tau \subset \eta$. Let $\mathcal{L}_1$ resp. $\mathcal{L}_2$ resp. $\mathcal{L}_3$ denote the family of all $\tau \in \mathcal{L}$ where $(\mathbb{R}, \tau)$ is *completely normal* resp. *second countable* resp. *not regular*. Trivially, $\mathcal{L}_1 \cap \mathcal{L}_3 = \emptyset$ and $|\mathcal{L}_i| \leq |\mathcal{L}| \leq 2^c$ and $|\mathcal{L}_2| \leq c$. For $\tau \in \mathcal{L}$ the space $(\mathbb{R}, \tau)$ is metrizable if and only if $\tau \in \mathcal{L}_1 \cap \mathcal{L}_2$. We show that, up to homeomorphism, both $\mathcal{L}_1$ and $\mathcal{L}_3$ contain precisely 2^c topologies and $\mathcal{L}_2$ contains precisely c completely metrizable topologies. For 2^c non-homeomorphic topologies $\tau \in \mathcal{L}_1$ the space $(\mathbb{R}, \tau)$ is *Baire*, but there are also 2^c non-homeomorphic topologies $\tau \in \mathcal{L}_1$ and c non-homeomorphic topologies $\tau \in \mathcal{L}_1 \cap \mathcal{L}_2$ where $(\mathbb{R}, \tau)$ is of *first category*. Furthermore, we investigate the *complete lattice* $\mathcal{L}_0$ of all topologies $\tau \in \mathcal{L}$ such that τ and η coincide on $\mathbb{R} \setminus \{0\}$. In the lattice $\mathcal{L}_0$ we find 2^c (non-homeomorphic) immediate predecessors of the maximum η, whereas the minimum of $\mathcal{L}_0$ is a compact topology without immediate successors in $\mathcal{L}_0$. We construct chains of homeomorphic topologies in $\mathcal{L}_0 \cap \mathcal{L}_1 \cap \mathcal{L}_2$ and in $\mathcal{L}_0 \cap \mathcal{L}_2 \cap \mathcal{L}_3$ and in $\mathcal{L}_0 \cap (\mathcal{L}_1 \setminus \mathcal{L}_2)$ and in $\mathcal{L}_0 \cap (\mathcal{L}_3 \setminus \mathcal{L}_2)$ such that the length of each chain is c (and hence maximal). We also track down a chain in $\mathcal{L}_0$ of length 2^λ where λ is the smallest cardinal number κ with $2^\kappa > c$.

1. Introduction

Write $|S|$ for the cardinality (the size) of a the set S and let $c = |\mathbb{R}|$ denote the cardinality of the continuum. Let η denote the Euclidean topology on $\mathbb{R}$ and let $\mathcal{L}$ denote the family of all topologies τ on $\mathbb{R}$ where τ is coarser than η (i.e. τ is a subset of η) and $(\mathbb{R}, \tau)$ is a Hausdorff space. If $\tau \in \mathcal{L}$ and B is a nonempty bounded subset of $\mathbb{R}$ then the relative topologies of τ and η coincide on B. (Because they coincide on the interval $[\inf B, \sup B]$ due to the well-known fact that a topology cannot be T_2 if it is strictly coarser than a T_2-compact topology.) Nevertheless, on the whole space $\mathbb{R}$ the two topologies τ and η need not coincide. In fact, as we will see, $|\mathcal{L}| = 2^c$. (Note that $|\mathcal{L}| \leq 2^c$ is trivial because $|\eta| = c$.) Moreover, as we will prove in Section 4, $\mathcal{L}$ contains 2^c mutually non-homeomorphic topologies τ such that $(\mathbb{R}, \tau)$ is a completely normal Baire space. In Section 8 we will prove that $\mathcal{L}$ also contains 2^c mutually non-homeomorphic topologies τ such that $(\mathbb{R}, \tau)$ is a completely normal space of first category.

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For every $\tau \in \mathcal{L}$ the space $(\mathbb{R}, \tau)$ is separable and arcwise connected and σ-compact. Separability is trivial since $\mathbb{Q}$ is clearly a dense set in $(\mathbb{R}, \tau)$. Arcwise connectedness and σ-compactness follow immediately from the coincidence of η and τ on each Euclidean compact interval. Whereas the Euclidean space $\mathbb{R}$ is second countable, for arbitrary $\tau \in \mathcal{L}$ the space $(\mathbb{R}, \tau)$ need not be second countable. In fact, there cannot be more than c second countable topologies in the family $\mathcal{L}$ since $|\eta| = c$ and a set of size c has precisely c countable subsets. Due to separability, for $\tau \in \mathcal{L}$ the space $(\mathbb{R}, \tau)$ is metrizable if and only if it is regular and second countable. In particular, there are at most c metrizable topologies in the family $\mathcal{L}$. In Section 7 we will prove that there exist c mutually non-homeomorphic topologies $\tau \in \mathcal{L}$ such that $(\mathbb{R}, \tau)$ is completely metrizable. In Section 9 we will prove that there exist c mutually non-homeomorphic topologies $\tau \in \mathcal{L}$ such that $(\mathbb{R}, \tau)$ is a metrizable space of first category.

Let us call the image $g(\mathbb{R})$ of any continuous one-to-one mapping g from the Euclidean space $\mathbb{R}$ into a Hausdorff space X a *real arc*. There is a natural correspondence between topologies in the family $\mathcal{L}$ and real arcs. Because, with g and X as above, evidently the family τ_g of all sets $g^{-1}(U)$ where U is an open subset of X is a topology in the family $\mathcal{L}$ and g defines a homeomorphism between the space $(\mathbb{R}, \tau_g)$ and the subspace $g(\mathbb{R})$ of X. Conversely, for each $\tau \in \mathcal{L}$ the space $(\mathbb{R}, \tau)$ is a real arc since the identity is a continuous mapping from $(\mathbb{R}, \eta)$ onto $(\mathbb{R}, \tau)$. As a consequence of our enumeration results mentioned above and proved in Sections 4 and 7, up to homeomorphism there are precisely 2^c completely normal real arcs and precisely c completely metrizable real arcs.

2. Locally and globally coarse topologies

If τ is a topology on the set $\mathbb{R}$ and $a \in \mathbb{R}$ then let $\mathcal{N}_\tau(a)$ denote the filter of the neighborhoods of the point a in the space $(\mathbb{R}, \tau)$. Trivially, $\mathcal{N}_\tau(a) \subset \mathcal{N}_\eta(a)$ for every $\tau \in \mathcal{L}$. Let us call a topology τ in our family $\mathcal{L}$ *coarse at the point* $a \in \mathbb{R}$ if and only if $\mathcal{N}_\tau(a) \neq \mathcal{N}_\eta(a)$. A proof of the following lemma is straightforward.

Lemma 1. *If an injective mapping g with domain $\mathbb{R}$ defines a real arc $g(\mathbb{R})$ then the topology τ_g in $\mathcal{L}$ corresponding with g is coarse at $a \in \mathbb{R}$ if and only if the bijection g^{-1} from $g(\mathbb{R})$ onto $\mathbb{R}$ is not continuous at $g(a)$.*

The following proposition makes it easy to detect whether a topology $\tau \in \mathcal{L}$ is coarse at a point $a \in \mathbb{R}$.

Proposition 1. *A topology $\tau \in \mathcal{L}$ is coarse at a point $a \in \mathbb{R}$ if and only if every set in the filter $\mathcal{N}_\tau(a)$ is an unbounded subset of $\mathbb{R}$.*

Proof. Let $\tau \in \mathcal{L}$ and $a \in \mathbb{R}$ and assume that some $U \in \mathcal{N}_\tau(a)$ is bounded. Fix $\delta > 0$ so that $[a - \delta, a + \delta] \subset U$ and let $0 < \varepsilon \leq \delta$ be arbitrary. The Euclidean compact set $A = [\inf U, a - \varepsilon] \cup [a + \varepsilon, \sup U]$ is compact and hence closed in the space $(\mathbb{R}, \tau)$. Consequently, $]a - \varepsilon, a + \varepsilon[= U \setminus A$ is τ-open whenever $0 < \varepsilon \leq \delta$ and hence $\mathcal{N}_\tau(a) = \mathcal{N}_\eta(a)$. ■

The following proposition provides a nice and very useful characterization of the first-category topologies in the family $\mathcal{L}$.

Proposition 2. *For $\tau \in \mathcal{L}$ the space $(\mathbb{R}, \tau)$ is of first category if and only if every nonempty open set in the space $(\mathbb{R}, \tau)$ is an unbounded subset of $\mathbb{R}$.*

Proof. Assume firstly that $\tau \in \mathcal{L}$ and every nonempty τ-open set is unbounded. Then for each $n \in \mathbb{N}$ the set $[-n, n]$ is nowhere dense in the space $(\mathbb{R}, \tau)$. (Note that the Euclidean compact set $[-n, n]$ is τ-compact and hence τ-closed.) Thus the space $(\mathbb{R}, \tau)$ is of first category since $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n, n]$. Assume secondly that $\tau \in \mathcal{L}$ and that $(\mathbb{R}, \tau)$ is a space of first category and suppose indirectly that there exists a nonempty τ-open set U which is bounded. As an open subspace of a space of first category, the set U equipped with the relative topology of τ is a space of first category. But this space is identical with U equipped with the relative topology of η (since U is bounded) and, naturally, the Euclidean space U is of second category. This is a contradiction. ■

Remark. As a trivial consequence of Propositions 1 and 2, for $\tau \in \mathcal{L}$ the space $(\mathbb{R}, \tau)$ is of first category if and only if τ is *everywhere* coarse. In [5] we construct 2^{2^c} non-homeomorphic connected topologies τ on $\mathbb{R}$ with certain properties where τ is finer than η. In [5] it is not explicitly stated that all these topologies τ are actually *everywhere* finer than η, i.e. $\mathcal{N}_\eta(a)$ is a proper subset of $\mathcal{N}_\tau(a)$ for every $a \in \mathbb{R}$. However, some of these 2^{2^c} topologies are of first category, but some of them are of second category.

For $\tau \in \mathcal{L}$ let $C(\tau)$ denote the set of all points a such that τ is coarse at a. Clearly, if $C(\tau) \neq \mathbb{R}$ then the subspace topologies of τ and η coincide on the set $\mathbb{R} \setminus C(\tau)$. The following proposition shows that the set $C(\tau)$ is always of a very special form.

Proposition 3. *Let $\tau \in \mathcal{L}$. Then $C(\tau)$ is a closed subset of the Euclidean space $\mathbb{R}$. Moreover, the set $C(\tau)$ is closed and meager in the space $(\mathbb{R}, \tau)$.*

Proof. Let $\tau \in \mathcal{L}$. Firstly we verify that $C(\tau)$ is closed in the space $(\mathbb{R}, \tau)$. (Then, of course, $C(\tau)$ is closed in the Euclidean space automatically.) Assume that $x \in \mathbb{R}$ is a limit point of the set $C(\tau)$ in the space $(\mathbb{R}, \tau)$. Then $U \cap C(\tau) \neq \emptyset$ for every τ-open set U in the filter $\mathcal{N}_\tau(x)$ and hence every set in the filter $\mathcal{N}_\tau(x)$ lies in the filter $\mathcal{N}_\tau(a)$ for some $a \in C(\tau)$. Thus every set in $\mathcal{N}_\tau(x)$ is unbounded by Proposition 1. Hence $x \in C(\tau)$ by Proposition 1. Therefore the set $C(\tau)$ is τ-closed. Since $[-n, n]$ is compact and hence closed in the space $(\mathbb{R}, \tau)$ for every $n \in \mathbb{N}$, all sets $C(\tau) \cap [-n, n]$ are closed in the space $(\mathbb{R}, \tau)$. No point in $C(\tau) \cap [-n, n]$ is an τ-interior point of $C(\tau) \cap [-n, n]$ because if $a \in C(\tau)$ then $S \not\subset [-n, n]$ for every $S \in \mathcal{N}_\tau(a)$ by Proposition 1. Consequently, $C(\tau) \cap [-n, n]$ is nowhere dense in the space $(\mathbb{R}, \tau)$ for every $n \in \mathbb{N}$ and hence the set $C(\tau) = \bigcup_{n=1}^{\infty}(C(\tau) \cap [-n, n])$ is meager in the space $(\mathbb{R}, \tau)$. ■

The following proposition generalizes the special fact that $(\mathbb{R}, \eta)$ is a Baire space with $C(\eta) = \emptyset$ and will be useful for the proof of the enumeration results in Sections 4 and 5.

Proposition 4. *If $\tau \in \mathcal{L}$ such that $C(\tau)$ is a meager set in the space $(\mathbb{R}, \eta)$ then $(\mathbb{R}, \tau)$ is a Baire space.*

Proof. For $\tau \in \mathcal{L}$ assume that $C(\tau)$ is a meager subset of Euclidean space $\mathbb{R}$. Then $C(\tau) \neq \mathbb{R}$ and hence $U := \mathbb{R} \setminus C(\tau)$ is nonempty. By Proposition 3 the set U is Euclidean open (even τ-open). As an open subspace of the Baire space $(\mathbb{R}, \eta)$, the space (U, η) is Baire. The spaces (U, η) and (U, τ) are identical in view of $U \cap C(\tau) = \emptyset$ and the definition of the set $C(\tau)$. In particular, the space (U, τ) is Baire. As the complement of a meager set, U is dense in the Euclidean space $\mathbb{R}$ and hence dense in the space $(\mathbb{R}, \tau)$ a fortiori. This is enough in view of the well-known fact (cf. [2] 3.9.J.b) that a Hausdorff space must be Baire if some dense subspace is Baire. ■

The following proposition, which implies that $\mathcal{L}$ contains c completely metrizable topologies, demonstrates that the converse of Proposition 4 would be far from being true.

Proposition 5. *For every $z \in \mathbb{R}$ there exists a topology $\tau_z \in \mathcal{L}$ with $C(\tau_z) =]-\infty, z]$ such that all spaces $(\mathbb{R}, \tau_z)$ are completely metrizable and homeomorphic.*

Proof. We work with real arcs and define for every $z \in \mathbb{R}$ an injective and continuous mapping g_z from the Euclidean space $\mathbb{R}$ into the Euclidean plane $\mathbb{R}^2$ by putting $g_z(t) = (t, 0)$ for $t \leq z$ and $g_z(t) = (z + (t-z)(z+1-t), t-z)$ for $z \leq t \leq z+1$ and $g_z(t) = (z + (z+1-t)|\sin(z+1-t)|, e^{z+1-t})$ for $t \geq z+1$. Clearly, $g_z(\mathbb{R})$ is a closed subset of the complete metric space $\mathbb{R}^2$. We observe that g_z^{-1} is continuous at $g_z(a)$ if and only if $a \in]z, \infty[$. (Hence $C(\tau_z) =]-\infty, z]$ for $\tau_z \in \mathcal{L}$ corresponding with g_z.) Finally, for every $z \in \mathbb{R}$ the space $g_z(\mathbb{R})$ is homeomorphic to the space $g_0(\mathbb{R})$ since the translation $(x, y) \mapsto (x - z, y)$ of the vector space $\mathbb{R}^2$ maps $g_z(\mathbb{R})$ onto $g_0(\mathbb{R})$. ■

3. Selecting non-homeomorphic topologies

Lemma 2. *If $\mathcal{H} \subset \mathcal{L}$ and all topologies in $\mathcal{H}$ are homeomorphic then $|\mathcal{H}| \leq c$.*

Proof. Firstly, if $\tau_1, \tau_2 \in \mathcal{L}$ then each continuous function from the space $(\mathbb{R}, \tau_1)$ into the space $(\mathbb{R}, \tau_2)$ is completely determined by its values at the points in the τ_1-dense set $\mathbb{Q}$. Secondly, there are precisely c functions from $\mathbb{Q}$ into $\mathbb{R}$. ■

The following lemma makes it very easy to provide mutually non-homeomorphic topologies in certain situations.

Lemma 3. *If the size of a family $\mathcal{K} \subset \mathcal{L}$ is greater than c then $\mathcal{K}$ contains a family $\mathcal{K}'$ equipollent to $\mathcal{K}$ such that all topologies in $\mathcal{K}$ are mutually non-homeomorphic.*

Proof. Define an equivalence relation $\sim$ on $\mathcal{K}$ by putting $\tau_1 \sim \tau_2$ for $\tau_i \in \mathcal{K}$ when the spaces $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$ are homeomorphic. By Lemma 2 the size of an equivalence class cannot exceed c. Consequently, from $|\mathcal{K}| > c$ we derive that the total number of the equivalence classes must be $|\mathcal{K}|$. So we are done by choosing for $\mathcal{K}'$ a set of representatives with respect to the equivalence relation $\sim$. ■

4. Completely normal Baire topologies

The following lemma is very useful in order to avoid a lengthy verification of complete normality by verifying regularity only.

LEMMA 4. *Let $z \in \mathbb{R}$ and $\tau \in \mathcal{L}$ with $C(\tau) = \{z\}$. Then the space $(\mathbb{R}, \tau)$ is second countable if and only if some local basis at the point z is countable. And the space $(\mathbb{R}, \tau)$ is completely normal if and only if it is regular.*

Proof. Clearly, $z \notin V \in \eta$ implies $V \in \tau$. This settles the first statement and has also the consequence that $U \cup V \in \tau$ whenever $z \in U \in \tau$ and $V \in \eta$. Assume that $(\mathbb{R}, \tau)$ is regular and that in the space $(\mathbb{R}, \tau)$ we have $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ for $A, B \subset \mathbb{R}$. If $z \notin A \cup B$ then A and B can be separated by η-open subsets of $\mathbb{R} \setminus \{z\}$ which must be τ-open. So assume $z \in A \cup B$ and, say, $z \in A$. Then we can find disjoint sets $U_1, V_1 \in \eta$ with $z \notin U_1 \cup V_1$ such that $A \setminus \{z\} \subset U_1$ and $B \subset V_1$. Furthermore, since the space $(\mathbb{R}, \tau)$ is regular, we can find disjoint sets $U_2, V_2 \in \tau$ with $z \in U_2$ and $\overline{B} \subset V_2$. Then $U_1 \cup U_2$ and $V_1 \cap V_2$ are disjoint τ-open sets and $A \subset U_1 \cup U_2$ and $B \subset V_1 \cap V_2$. ∎

Our first main result is the following theorem.

THEOREM 1. *There exists a family $\mathcal{T} \subset \mathcal{L}$ with $|\mathcal{T}| = 2^c$ such that $(\mathbb{R}, \tau)$ is a completely normal Baire space for each $\tau \in \mathcal{T}$ and two spaces $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \tau')$ are never homeomorphic for distinct topologies $\tau, \tau' \in \mathcal{T}$.*

Proof. The cardinal number 2^c indicates that the natural way to define $\mathcal{T}$ is to use ultrafilters on a countably infinite set. It is well-known (see [1]) that an infinite set of size κ carries precisely 2^{2^κ} free ultrafilters. In particular, there are 2^c free ultrafilters on $\mathbb{Z}$. Note that no free ultrafilter contains a finite set.

For each free ultrafilter $\mathcal{F}$ on $\mathbb{Z}$ define a topology $\tau = \tau[\mathcal{F}]$ on $\mathbb{R}$ by declaring $U \subset \mathbb{R}$ open if and only if U is Euclidean open and satisfies $0 \notin U$ or $U \cap \mathbb{Z} \in \mathcal{F}$. It is plain that τ is a well-defined topology on $\mathbb{R}$ coarser than η. Further, $(\mathbb{R}, \tau)$ is a Hausdorff space, whence $\tau \in \mathcal{L}$, because if $u < v$ then the intersection $\mathbb{Z} \setminus [u, v]$ of $\mathbb{Z}$ and the Euclidean open set $\mathbb{R} \setminus [u, v]$ must lie in $\mathcal{F}$ (since $\mathbb{Z} \cap [u, v]$ is a finite set and the ultrafilter $\mathcal{F}$ is free). By Proposition 1 we have $0 \in C(\tau)$ since $M \cap \mathbb{Z} \in \mathcal{F}$ for every $M \in \mathcal{N}_\tau(0)$ and every $S \in \mathcal{F}$ is an infinite set. Moreover, $C(\tau) = \{0\}$ since τ and η coincide on the Euclidean open set $\mathbb{R} \setminus \{0\}$. Hence $(\mathbb{R}, \tau)$ is a Baire space by Proposition 4.

We claim that $(\mathbb{R}, \tau)$ is completely normal. By Lemma 4 it is enough to check the T_3-separation property. Let $A \subset \mathbb{R}$ be τ-closed (and hence η-closed) and let $b \in \mathbb{R} \setminus A$. If $b \neq 0$ then we can find $\epsilon > 0$ and $U \in \eta$ disjoint from $V :=]b - \epsilon, b + \epsilon[$ with $0 \notin V$ and $A \subset U$. Then V is τ-open and $U_\epsilon := U \cup (\mathbb{R} \setminus [b - \epsilon, b + \epsilon])$ is τ-open and $b \in V$ and $A \subset U_\epsilon$ and $U_\epsilon \cap V = \emptyset$. (The set $U_\epsilon \cap \mathbb{Z}$ lies in the free ultrafilter $\mathcal{F}$ since $\mathbb{Z} \setminus U_\epsilon$ is finite.) If $b = 0$ then $B := \{0\} \cup (\mathbb{Z} \setminus A)$ is η-closed and disjoint from A and hence we can choose disjoint η-open sets U, V with $A \subset U$ and $b \in B \subset V$. The set U is τ-open because $0 \notin U$ since $0 \in V$ and $U \cap V = \emptyset$. The set V is τ-open because $\mathbb{Z} \setminus A \in \mathcal{F}$ (since A is τ-closed) and hence from $V \cap \mathbb{Z} \supset B \cap \mathbb{Z} \supset \mathbb{Z} \setminus A$ we derive $V \cap \mathbb{Z} \in \mathcal{F}$.

Finally we observe that $\tau[\mathcal{F}_1] \not\subset \tau[\mathcal{F}_2]$ (and hence $\tau[\mathcal{F}_1] \neq \tau[\mathcal{F}_2]$) whenever $\mathcal{F}_1$ and $\mathcal{F}_2$ are distinct free ultrafilters on $\mathbb{Z}$. Indeed, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are free ultrafilters on $\mathbb{Z}$ and $\tau[\mathcal{F}_1] \subset \tau[\mathcal{F}_2]$ and $S \in \mathcal{F}_1$ then the $\tau[\mathcal{F}_1]$-open set $W := \left]-\frac{1}{3}, \frac{1}{3}\right[\cup \bigcup_{s\in S} \left]s - \frac{1}{3}, s + \frac{1}{3}\right[$ is a $\tau[\mathcal{F}_2]$-open neighborhood of 0 and hence $S \cup \{0\} = W \cap \mathbb{Z}$ lies in $\mathcal{F}_2$, whence $S \in \mathcal{F}_2$. (Note that $\mathbb{Z} \setminus \{0\} \in \mathcal{F}_2$ since the ultrafilter $\mathcal{F}_2$ is free.) Thus $\mathcal{F}_1 \subset \mathcal{F}_2$ and hence $\mathcal{F}_1 = \mathcal{F}_2$ since $\mathcal{F}_1$ and $\mathcal{F}_2$ are ultrafilters. ∎

Remark. Since $\mathcal{L}$ contains only c second countable topologies, there are 2^c free ultrafilters $\mathcal{F}$ on $\mathbb{Z}$ such that the space $(\mathbb{R}, \tau[\mathcal{F}])$ is not second countable or, equivalently, that any local basis at 0 is uncountable. In fact, this is true for every free ultrafilter $\mathcal{F}$ on $\mathbb{Z}$. Indeed, assume indirectly that the countable family $\{B_1, B_2, B_3, \ldots\}$ is a local basis at 0 in the space $(\mathbb{R}, \tau[\mathcal{F}])$. Then we may choose a sequence $a_1, a_2, a_3, \ldots$ of distinct integers and $0 < \epsilon_n < \frac{1}{3} (n \in \mathbb{N})$ such that $a_n \in B_n \setminus \{a_k | k < n\}$ and $[a_n - \epsilon_n, a_n + \epsilon_n] \subset B_n$ for every $n \in \mathbb{N}$. Then with $S = \mathbb{Z} \setminus \{a_1, a_2, a_3, \ldots\}$ the set

$$U := \bigcup_{n=1}^{\infty}]a_n - \epsilon_n, a_n + \epsilon_n[\cup \bigcup_{s\in S}]s - \tfrac{1}{3}, s + \tfrac{1}{3}[$$

is a $\tau[\mathcal{F}]$-open $\tau[\mathcal{F}]$-neighborhood of 0 (since $U \cap \mathbb{Z} = \mathbb{Z} \in \mathcal{F}$) with $a_n + \epsilon_n \in B_n \setminus U$ and hence $B_n \not\subset U$ for every $n \in \mathbb{N}$. Thus $\{B_1, B_2, B_3, \ldots\}$ is not a local basis at 0.

5. Non-regular Baire topologies

In view of Theorem 1 and Lemma 4 there arises the question whether $\mathcal{L}$ contains also 2^c topologies τ which are Baire because of $C(\tau) = \{0\}$ and where $(\mathbb{R}, \tau)$ is not regular. This is indeed true.

Theorem 2. *There exist 2^c mutually non-homeomorphic topologies $\tau \in \mathcal{L}$ such that $(\mathbb{R}, \tau)$ is a Baire space which is not regular.*

Proof. It is enough to modify the proof of Theorem 1 in the following way. For any free ultrafilter $\mathcal{F}$ on $\mathbb{Z}$ define a topology $\sigma[\mathcal{F}]$ on $\mathbb{R}$ by declaring $U \subset \mathbb{R}$ open if and only if U is Euclidean open and $0 \notin U$ or $U \supset \bigcup_{s\in S} \left]s - \frac{1}{3}, s + \frac{1}{3}\right[$ for some $S \in \mathcal{F}$. Certainly, $\sigma[\mathcal{F}]$ is well-defined and Hausdorff. The space $(\mathbb{R}, \sigma[\mathcal{F}])$ is not regular since, for example, the point 0 and the obviously $\sigma[\mathcal{F}]$-closed set $\bigcup_{k=-\infty}^{k=\infty} \left[k + \frac{1}{3}, k + \frac{2}{3}\right]$ cannot be separated by $\sigma[\mathcal{F}]$-open sets. Finally, similarly as in the proof of Theorem 1, $\sigma[\mathcal{F}] \neq \sigma[\mathcal{F}']$ whenever $\mathcal{F}$ and $\mathcal{F}'$ are distinct free ultrafilters on $\mathbb{Z}$. ∎

Remark. In the proof of Theorem 1 or Theorem 2 one cannot avoid an application of Lemma 3 (or a similar transfinite counting argument). Actually, for every free ultrafilter $\mathcal{F}_0$ on $\mathbb{Z}$ there is an infinite family $\mathcal{U}$ of free ultrafilters on $\mathbb{Z}$ with $\mathcal{F}_0 \in \mathcal{U}$ such that all topologies $\tau[\mathcal{F}] (\mathcal{F} \in \mathcal{U})$ are homeomorphic and all topologies $\sigma[\mathcal{F}] (\mathcal{F} \in \mathcal{U})$ are homeomorphic. Indeed, put $\mathcal{U} := \{\mathcal{F}_k | k = 0, 1, 2, \ldots\}$ where

$\mathcal{F}_k := \{k+S|S \in \mathcal{F}_0\}$ for every integer $k \geq 0$. Clearly, $\mathcal{F}_m = \{(m-n)+S|S \in \mathcal{F}_n\}$ whenever $n, m \geq 0$ and each family $\mathcal{F}_k$ is a free ultrafilter on $\mathbb{Z}$. We have $\mathcal{F}_n \neq \mathcal{F}_m$ whenever $0 \leq n < m$ because firstly precisely one of the congruence classes modulo $2m$ lies in $\mathcal{F}_n$. (Note that a union of finitely many sets lies in an ultrafilter only if one of these sets lies in the ultrafilter.) And secondly, if a congruence class A modulo $2m$ lies in $\mathcal{F}_n$ then the congruence class $(m-n)+A$ lies in $\mathcal{F}_m$ but not in $\mathcal{F}_n$. (For A and $(m-n)+A$ are disjoint.) Finally, for each $k \in \mathbb{N}$ define an increasing bijection φ_k from $\mathbb{R}$ onto $\mathbb{R}$ so that $\varphi_k(0) = 0$ and $\varphi_k(n) = n+k$ for every $n \in \mathbb{Z} \setminus [-k, 0]$. Since φ_k is a homeomorphism from the Euclidean space $\mathbb{R} \setminus \{0\}$ onto itself, by considering the open neighborhoods of 0 it is evident that φ_k is a homeomorphism from the space $(\mathbb{R}, \tau[\mathcal{F}_0])$ onto the space $(\mathbb{R}, \tau[\mathcal{F}_k])$ and also a homeomorphism from the space $(\mathbb{R}, \sigma[\mathcal{F}_0])$ onto the space $(\mathbb{R}, \sigma[\mathcal{F}_k])$.

6. Counting Polish spaces

For the proof of our second main result in Section 7 we need the following enumeration theorem.

Theorem 3. *There is a family $\mathcal{H}$ of countably infinite G_δ-sets in the Euclidean space $\mathbb{R}$ such that the size of $\mathcal{H}$ is c and distinct members of $\mathcal{H}$ are always non-homeomorphic subspaces of $\mathbb{R}$.*

Proof. We work with Cantor derivatives and is enough to consider finite derivatives. (Note in the following that we regard $\mathbb{N}$ to be defined in the classical way, i.e. $0 \notin \mathbb{N}$.) If X is a Hausdorff space and $A \subset X$ then the first derivative A' of A is the set of all limit points of A. Further, with $A^{(1)} := A'$, for every $k = 2, 3, 4, \ldots$ the k-th derivative $A^{(k)}$ of A is given by $A^{(k)} = (A^{k-1)})'$. Naturally, the first derivative of any set is closed. Consequently, $A^{(m)} \supset A^{(n)}$ whenever $m \leq n$.

Now, define for each $n \in \mathbb{N}$ a compact and countably infinite subset K_n of the interval $[2n, 2n+1]$ with $\min K_n = 2n$ and $\max K_n = 2n+1$ such that $K_n^{(n)} = \{2n+1\}$. (Simply take for K_n an appropriate order-isomorphic copy of the well-ordered set of all ordinal numbers $\alpha \leq \omega^n$.) Thus for $m, n \in \mathbb{N}$ the derived set $K_n^{(m)}$ contains the point $2n+1$ if and only if $m \leq n$. Furthermore, define a discrete subset E_n of $]2n+1, 2n+\frac{7}{4}]$ via $E_n := \{2n+1+2^{-m}+2^{-m-k} \big| m, k \in \mathbb{N}\}$. For every nonempty $S \subset \mathbb{N}$ put $G_S := \bigcup_{n \in S}(K_n \cup E_n)$. Since G_S is the union of the closed set $\bigcup_{n \in S} K_n$ and the discrete set $\bigcup_{n \in S} E_n$, the set G_S is a countably infinite G_δ-set in $\mathbb{R}$. Obviously, $G_S^{(m)} = \bigcup_{n \in S} K_n^{(m)}$ for every $m \in \mathbb{N}$.

If $\emptyset \neq S \subset \mathbb{N}$ then let N_S denote the set of all $x \in G_S$ such that no neighborhood of the point x in the space G_S is compact. By construction, $x \in N_S$ if and only if $x = 2n+1$ for some $n \in S$. Hence a moment's reflection suffices to see that

$$\{m \in \mathbb{N} \mid (G_S^{(m)} \setminus G_S^{(m+1)}) \cap N_S \neq \emptyset\} = S,$$

for each nonempty set $S \subset \mathbb{N}$. Thus the set S can always be recovered from the space G_S purely topologically and hence two spaces G_{S_1} and G_{S_2} are never homeomorphic for distinct nonempty sets $S_1, S_2 \subset \mathbb{N}$. Thus the family $\mathcal{H} = \{G_S | \emptyset \neq S \subset \mathbb{N}\}$ is as desired and this concludes the proof of Theorem 3. ∎

Remark. Every Polish space is homeomorphic to a closed subspace of the product of countably infinitely many copies of the real line (cf. [3] 4.3.25). As a consequence, every uncountable Polish space is of size c and the size of a family of mutually non-homeomorphic Polish spaces cannot exceed c. Therefore, by virtue of Theorem 3, *there exist precisely c countably infinite Polish spaces up to homeomorphism.* In comparison, by [4] Theorem 1.3 there exist precisely c uncountable Polish spaces up to homeomorphism.

7. Completely metrizable topologies

Theorem 4. *There exist c mutually non-homeomorphic topologies τ on $\mathbb{R}$ coarser than the Euclidean topology such that $(\mathbb{R}, \tau)$ is completely metrizable (and hence Polish).*

Proof. Let $\mathcal{H}$ be a family as in Theorem 3. Our goal is to construct for each $H \in \mathcal{H}$ a real arc A_H which is a G_δ-subset of the Euclidean space $\mathbb{R}^3$ (and hence completely metrizable) so that $H \times \{0\} \times \{0\} \subset A_H$ and A_H and $A_{H'}$ are never homeomorphic for distinct $H, H' \in \mathcal{H}$.

For two points P, Q in the vector space $\mathbb{R}^3$ let $[P, Q]$ denote the closed straight segment which connects the points P and Q, $[P, Q] = \{\lambda P + (1-\lambda)Q | 0 \leq \lambda \leq 1\}$. Furthermore, for abbreviation, put $y(n) := 2^{-n} \cos 2^{-n}$ and $z(n) := 2^{-n} \sin 2^{-n}$ for $n \in \mathbb{N}$.

For every set $H = \{a_1, a_2, a_3, \dots\}$ in the family $\mathcal{H}$ with $a_i \neq a_j$ for $i \neq j$ we define an injective and continuous mapping $g = g_H$ from $\mathbb{R}$ into $\mathbb{R}^3$ by

$$g(t) = (t \sin t, -e^t, 0) \text{ for every real } t \leq 0$$

and so that $g([k, k+1]) = [g(k), g(k+1)]$ for every integer $k \geq 0$ where

$$g(0) = (0, -1, 0) \quad \text{and} \quad g((1) = (0, -1, 1) \quad \text{and}$$
$$g(2m) = (a_m, 0, 0) \text{ and } g(2m+1) = (a_m, y(m), z(m)) \text{ for every } m \in \mathbb{N}.$$

The injectivity of g is feasible because if E_m is the plane through the three points $g(2m), g(2m+1), g(2m+2)$ then $E_m \neq \mathbb{R} \times \mathbb{R} \times \{0\}$ and $E_m \cap E_n = \mathbb{R} \times \{0\} \times \{0\}$ whenever $m, n \in \mathbb{N}$ and $m \neq n$.

Let $H \in \mathcal{H}$ and put $A_H := g_H(\mathbb{R})$ and let $\overline{A_H}$ denote the closure of A_H in the Euclidean space $\mathbb{R}^3$. Trivially, $H \times \{0\} \times \{0\}$ is a G_δ-set in the space $\mathbb{R}^3$ and a subspace of $\mathbb{R}^3$ homeomorphic with H. Obviously, $\overline{A_H} = B \times \{0\} \times \{0\} \cup A_H$ for some $B \subset \mathbb{R}$. Hence $A_H = H \times \{0\} \times \{0\} \cup (\overline{A_H} \cap (\mathbb{R}^3 \setminus \mathbb{R} \times \{0\} \times \{0\}))$ is the union of a G_δ-set and a set which is the intersection of a closed set with an open set. Thus A_H is a G_δ-set in the space $\mathbb{R}^3$ and hence the Euclidean space A_H is completely metrizable.

A moment's reflection is sufficient to see that $H \times \{0\} \times \{0\}$ equals the set of all points a in the space A_H where no local basis at a contains only arcwise connected sets. Therefore, the space H can essentially be recovered from the space A_H and this finishes the proof. ∎

Remark. In the previous proof one cannot replace $\mathcal{H}$ with a family $\mathcal{H}'$ of mutually non-homeomorphic countably infinite and *closed* subspaces of the Euclidean space $\mathbb{R}$. Because in view of [4] Theorem 8.1 we have $|\mathcal{H}'| \leq \aleph_1$ for any such family $\mathcal{H}'$ and it is widely known (cf. [3]) that $\aleph_1 < c$ (i.e. the negation of the Continuum Hypothesis) is irrefutable. However, by applying a theorem not proved in this paper and with a bit greater effort concerning the notations it is not difficult to modify the previous proof starting with a family $\mathcal{H}^*$ of mutually non-homeomorphic closed subspaces of $\mathbb{R}$ such that $|\mathcal{H}^*| = c$ and every member of $\mathcal{H}^*$ is the union of infinitely many mutually exclusive intervals $[a, b]$ with $a < b$. (Such a family $\mathcal{H}^*$ exists by [6] Theorem 1.)

8. Completely normal spaces of first category

Theorem 5. *There exist 2^c mutually non-homeomorphic topologies $\tau \in \mathcal{L}$ such that $(\mathbb{R}, \tau)$ is a completely normal space of first category.*

Proof. Let B be an injective mapping from $\mathbb{Z}$ into the power set of $\mathbb{R}^3$ such that $B(k)$ is always a nonempty open ball in the Euclidean metric space $\mathbb{R}^3$ and that $\{B(k)|k \in \mathbb{Z}\}$ is a basis of the Euclidean topology of $\mathbb{R}^3$. We define a double sequence of distinct points

$$\ldots, P_{-3}, P_{-2}, P_{-1}, P_0, P_1, P_2, P_3, \ldots$$

in $\mathbb{R}^3$ by induction. Start with three distinct points P_{-1}, P_0, P_1 where P_{-1} does not lie in the straight line through P_0 and P_1. Suppose that for $n \in \mathbb{N}$ we have already chosen $2n+1$ distinct points P_k with $k \in \mathbb{Z}$ and $|k| \leq n$. Then choose $P_{n+1} \in B(n+1)$ and $P_{-n-1} \in B(-n-1)$ so that:

(i) three distinct points in $\{P_k||k| \leq n+1\}$ never lie in one straight line,

(ii) four distinct points in $\{P_k||k| \leq n+1\}$ never lie in one plane.

Such a choice is always possible since neither finitely many straight lines nor finitely many planes can cover any ball $B(k)$.

In this way we obtain a countable, dense subset $\{P_k|k \in \mathbb{Z}\}$ of the Euclidean space $\mathbb{R}^3$ (with $P_k \neq P_{k'}$ whenever $k \neq k'$) such that $[P_m, P_{m+1}]$ and $[P_n, P_{n+1}] \setminus \{P_n, P_{n+1}\}$ are disjoint whenever $m, n \in \mathbb{Z}$ and $m \neq n$.

Now define a mapping g from $\mathbb{R}$ into $\mathbb{R}^3$ so that $g(k) = P_k$ and g is a continuous bijection from $[k, k+1]$ into $\mathbb{R}^3$ with $g([k, k+1]) = [P_k, P_{k+1}]$ for every $k \in \mathbb{Z}$. Then $g : \mathbb{R} \to \mathbb{R}^3$ is injective and continuous and hence $g(\mathbb{R})$ is a real arc within $\mathbb{R}^3$ such that $g(\mathbb{Z})$ is dense in $\mathbb{R}^3$. Therefore the Euclidean compact spaces $[P_k, P_{k+1}]$ are closed subsets of the space $g(\mathbb{R})$ whose interior in the space $g(\mathbb{R})$ is empty and hence the space $g(\mathbb{R})$ is of first category. By construction, for any nonempty open set U in the Euclidean space $\mathbb{R}^3$ the set $g^{-1}(U)$ is an unbounded subset of $\mathbb{R}$. Thus the topology in $\mathcal{L}$ corresponding with $g(\mathbb{R})$ is one that satisfies the desired properties of Theorem 5. (Moreover, the topology is metrizable.)

The first step is done and now we are going to track down 2^c topologies as desired. Since $g(\mathbb{Z})$ is dense in $\mathbb{R}^3$ we may fix an infinite set $Z \subset g(\mathbb{Z})$ such that $g(0) \in Z$ and the Euclidean distance between any two points in Z is always greater than 1. (In particular, Z is an unbounded, countable subset of $\mathbb{R}^3$.) Similarly as in the proof of Theorem 1, for each of the 2^c free ultrafilters $\mathcal{F}$ on Z define a topology $\tilde{\tau}[\mathcal{F}]$ on $\mathbb{R}^3$ such that $U \subset \mathbb{R}^3$ lies in the family $\tilde{\tau}[\mathcal{F}]$ if and only if U is Euclidean open and satisfies $g(0) \notin U$ or $U \cap Z \in \mathcal{F}$.

Of course, by exactly the same arguments as in the proof of Theorem 1, for every free ultrafilter $\mathcal{F}$ on Z the topology $\tilde{\tau}[\mathcal{F}]$ is completely normal and coarser than the Euclidean topology on $\mathbb{R}^3$ (and strictly coarser precisely at the point $g(0)$).

Now let $\tau = \tilde{\tau}[\mathcal{F}]$ be any such topology on $\mathbb{R}^3$. Then the set $g(\mathbb{R})$ equipped with the subspace topology of $(\mathbb{R}^3, \tau)$ is completely normal. (Here it is essential that the property *completely normal* is, other than the property *normal*, hereditary.) Since g is a continuous one-to-one mapping from $(\mathbb{R}, \eta)$ into $(\mathbb{R}^3, \tau)$ a fortiori, the family $g^{-1}(\tau) := \{g^{-1}(V) | V \in \tau\}$ is a topology in the family $\mathcal{L}$ and g is a homeomorphism from the space $(\mathbb{R}, g^{-1}(\tau))$ onto the space $(g(\mathbb{R}), \tau)$. In particular, the space $(\mathbb{R}, g^{-1}(\tau))$ is completely normal. Furthermore, every nonempty open set in the space $(\mathbb{R}, g^{-1}(\tau))$ is unbounded in $\mathbb{R}$, whence $(\mathbb{R}, g^{-1}(\tau))$ is a space of first category by Proposition 2.

Trivially, $U \cap Z = (U \cap g(\mathbb{R})) \cap Z$ for every Euclidean open set $U \subset \mathbb{R}^3$. Therefore, by a similar argument as in the proof of Theorem 1, for distinct free ultrafilters $\mathcal{F}_1, \mathcal{F}_2$ on Z the relative topologies of $\tilde{\tau}[\mathcal{F}_1]$ and $\tilde{\tau}[\mathcal{F}_2]$ on the set $g(\mathbb{R})$ must be distinct. (We even have $\tau_1 \not\subset \tau_2$ for such distinct relative topologies τ_1, τ_2 on $g(\mathbb{R})$.) Thus by Lemma 3 we can track down a family $\mathcal{U}$ of free ultrafilters on Z such that $|\mathcal{U}| = 2^c$ and two spaces $(g(\mathbb{R}), \tilde{\tau}[\mathcal{F}_1])$ and $(g(\mathbb{R}), \tilde{\tau}[\mathcal{F}_2])$ are never homeomorphic for distinct $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{U}$. Hence the topologies $g^{-1}(\tilde{\tau}[\mathcal{F}_1])$ and $g^{-1}(\tilde{\tau}[\mathcal{F}_2])$ in the family $\mathcal{L}$ are never homeomorphic for distinct $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{U}$ since g is a homeomorphism from the space $(\mathbb{R}, g^{-1}(\tilde{\tau}[\mathcal{F}]))$ onto the space $(g(\mathbb{R}), \tilde{\tau}[\mathcal{F}])$ for every $\mathcal{F} \in \mathcal{U}$. This concludes the proof. ∎

9. Metrizable spaces of first category

THEOREM 6. *There exist c mutually non-homeomorphic topologies $\tau \in \mathcal{L}$ such that $(\mathbb{R}, \tau)$ is a metrizable space of first category.*

Proof. Let η_3 denote the Euclidean topology on $\mathbb{R}^3$ and for any continuous one-to-one mapping $g : \mathbb{R} \to \mathbb{R}^3$ let $g^{-1}(\eta_3) := \{g^{-1}(V) | V \in \eta_3\}$ denote the topology in $\mathcal{L}$ corresponding with the real arc $g(\mathbb{R})$. Let $\mathcal{H}$ be a family as in Theorem 3. Our goal is to construct a real arc $h_H(\mathbb{R})$ within the metrizable space $(\mathbb{R}^3, \eta_3)$ for every $H \in \mathcal{H}$ such that firstly $h_H(\mathbb{Z})$ is dense in $\mathbb{R}^3$, whence every nonempty open set in the space $(\mathbb{R}, h_H^{-1}(\eta_3))$ is unbounded, and secondly two real arcs $h_{H_1}(\mathbb{R})$ and $h_{H_2}(\mathbb{R})$ are never homeomorphic for distinct sets $H_1, H_2 \in \mathcal{H}$.

Let $H = \{a_1, a_3, a_5, \dots\}$ be a set in the family $\mathcal{H}$ where $a_i \neq a_j$ for distinct (and always odd) indices i, j. Again let $y(n) := 2^{-n} \cos 2^{-n}$ and $z(n) := 2^{-n} \sin 2^{-n}$ for $n \in \mathbb{N}$. We firstly define $h = h_H$ on the domain $[0, \infty[$. Choose an injective and continuous mapping h from $[0,\infty[$ into $\mathbb{R}^3$ so that $h([k,k+1]) = [h(k),h(k+1)]$ for every integer $k \geq 0$ where $h(k) = ((-2)^{k/2}, y(k), z(k))$ when k is even and $h(k) = (a_k, 0, 0)$ when k is odd. (Such a choice is clearly possible because if E_m is the plane through the three points $h(m-1), h(m), h(m+1)$ for any even $m \geq 2$ then $E_m \cap E_n = \mathbb{R}\times\{0\}\times\{0\}$ whenever $2 \leq m < n$.) Clearly, $H \times \{0\} \times \{0\}$ is the intersection of $h([0,\infty[)$ with the x-axis $\mathbb{R}\times\{0\}\times\{0\}$, and $h([0,\infty[) \cup \mathbb{R}\times\{0\}\times\{0\}$ is the closure of $h([0,\infty[)$ in $\mathbb{R}^3$.

For any Hausdorff space X let $W(X)$ denote the set of all points x in X such that no local basis at x contains only arcwise connected sets. By construction we have

$$W(h([0,\infty[)) = H \times \{0\} \times \{0\}.$$

In view of the definition of g in the proof of Theorem 5 it is plain to expand h to a continuous and injective mapping from $\mathbb{R}$ into $\mathbb{R}^3$ such that $h(\mathbb{Z}\setminus\mathbb{N})$ is a dense subset of the Euclidean space $\mathbb{R}^3$. As a consequence we have $W(h(\mathbb{R})) = h(\mathbb{R})$ and $(\mathbb{R}, h^{-1}(\eta_3))$ is a space of first category. Moreover, $W(h([t,\infty[)) = H \times \{0\} \times \{0\}$ for every real $t \leq 0$ and $W(h([t,\infty[)) \subset H \times \{0\} \times \{0\}$ and $W(h(]-\infty,t])) = h(]-\infty,t])$ for every $t \in \mathbb{R}$. In particular, for every $t \in \mathbb{R}$ the set $W(h([t,\infty[))$ is countable and the set $W(h(]-\infty,t]))$ is uncountable and we have $H\times\{0\}\times\{0\} = \bigcup\{W(h([t,\infty[))|t \in \mathbb{R}\}$.

We finish the proof by verifying that $H\times\{0\}\times\{0\}$ can be recovered from the space $h(\mathbb{R})$. (Note, again, that $H\times\{0\}\times\{0\}$ and H are homeomorphic.)

For any arcwise connected metrizable space X let $\mathcal{Y}(X)$ be the family of all sets $Y \subset X$ such that Y and $X \setminus Y$ are arcwise connected and $Y \setminus \{y\}$ is arcwise connected for some $y \in Y$. For the Euclidean space $\mathbb{R}$ we clearly have $Y \in \mathcal{Y}(\mathbb{R})$ if and only if $Y =]-\infty,t]$ or $Y = [t,\infty[$ for some $t \in \mathbb{R}$. While for an arbitrary real arc $g(\mathbb{R})$ it is not necessary that $\mathcal{Y}(g(\mathbb{R})) = \{g(Y)|Y \in \mathcal{Y}(\mathbb{R})\}$ (see the remark below), we observe that $Y \in \mathcal{Y}(h(\mathbb{R}))$ if and only if $Y = h(]-\infty,t])$ or $Y = h([t,\infty[)$ for some $t \in \mathbb{R}$. Therefore, $H\times\{0\}\times\{0\}$ equals the union of all sets $W(Y)$ where $Y \in \mathcal{Y}(h(\mathbb{R}))$ and $W(Y)$ is countable. ■

Remark. If $g(\mathbb{R}) \subset \mathbb{R}^3$ is a real arc and $a \in \mathbb{R}$ such that $g(x_n)$ converges to $g(a)$ whenever (x_n) is an unbounded and increasing sequence of reals then $g(\mathbb{R}) \setminus \{g(x)\}$ is arcwise connected for every $x > a$ and $g([u,v]) \in \mathcal{Y}(g(\mathbb{R}))$ whenever $a < u < v$.

10. A complete lattice of topologies

As any family of topologies on a fixed set, the family $\mathcal{L}$ is partially ordered by the relation $\subset$. A family $\mathcal{K} \subset \mathcal{L}$ is a *chain* if and only if $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$ whenever $\tau_1, \tau_2 \in \mathcal{K}$. The extreme opposite of chains of topologies are families of mutually incomparable topologies. (Two topologies τ_1, τ_2 are incomparable if and only if neither $\tau_1 \subset \tau_2$ nor $\tau_2 \subset \tau_1$.)

In order to prove Theorem 1 we considered topologies in $\mathcal{L}$ which are coarse at precisely one point $a \in \mathbb{R}$ (with $a = 0$). Let $\mathcal{L}_0 := \{\tau \in \mathcal{L}|C(\tau) \subset \{0\}\}$ be the family of all topologies in $\mathcal{L}$ which are either coarse precisely at the point 0 or equal to the Euclidean topology η. We have $|\mathcal{L}_0| = |\mathcal{L}| = 2^c$ by the proof of Theorem 1. Whereas, naturally, the family of all topologies on the set $\mathbb{R}$ coarser than η is a lattice with respect to the partial ordering $\subset$, the partially ordered family $(\mathcal{L}, \subset)$ is not a lattice. (See the remark below.) However, the partially ordered family $(\mathcal{L}_0, \subset)$ is a lattice. Moreover, $(\mathcal{L}_0, \subset)$ is a complete lattice (with η as its maximum) in view of the following proposition which also shows that for the minimum θ of the complete lattice $\mathcal{L}_0$ the space $(\mathbb{R}, \theta)$ has interesting properties. (Recall that a partially ordered set L is a complete lattice if and only if every nonempty subset of L has an infimum and a supremum.)

PROPOSITION 6. *If $\emptyset \neq \mathcal{S} \subset \mathcal{L}_0$ then $\bigcap \mathcal{S} \in \mathcal{L}_0$. If $\mathcal{K} \neq \emptyset$ is a chain in $\mathcal{L}_0$ then $\bigcup \mathcal{K}$ is a topology in $\mathcal{L}_0$, and $\bigcup \mathcal{K} \neq \eta$ when $\eta \notin \mathcal{K}$. If $\theta = \bigcap \mathcal{L}_0$ then the Hausdorff space $(\mathbb{R}, \theta)$ is compact and any locally connected, compact real arc with precisely one cut point is homeomorphic to the space $(\mathbb{R}, \theta)$.*

Proof. Let $\emptyset \neq \mathcal{S} \subset \mathcal{L}_0$. The family $\sigma := \bigcap \mathcal{S}$ is a topology on $\mathbb{R}$ coarser than η since, generally, the lattice of all topologies on any set is closed under arbitrary intersections. The topology σ is Hausdorff because σ and η coincide on $\mathbb{R} \setminus \{0\}$ and if, say, $x > 0$ then 0 and x can be separated by the σ-open sets $\mathbb{R} \setminus [\frac{x}{3}, 3x]$ and $]\frac{x}{2}, 2x[$. (Since $[\frac{x}{3}, 3x]$ is τ-compact for every $\tau \in \mathcal{L}$, the set $\mathbb{R} \setminus [\frac{x}{3}, 3x]$ is τ-open for every $\tau \in \mathcal{S}$.) If $\mathcal{S} \neq \{\eta\}$ then $C(\sigma) = \{0\}$ by Proposition 1. Hence, $\sigma \in \mathcal{L}_0$. Recall that if $\tau \in \mathcal{L}_0$ and $0 \in U \in \tau$ and $V \in \eta$ then $U \cup V \in \tau$. And, by Proposition 1, $]-1, 1[\in \tau$ for $\tau \in \mathcal{L}_0$ only if $\tau = \eta$. Consequently, the family $\bigcup \mathcal{S}$ is closed under arbitrary unions and we have $\bigcup \mathcal{S} \neq \eta$ when $\eta \notin \mathcal{S}$. And if $\mathcal{S}$ is a chain then $\bigcup \mathcal{S}$ is closed under finite intersections and hence $\bigcup \mathcal{S}$ is a topology on $\mathbb{R}$ coarser than η and finer than the Hausdorff topology $\bigcap \mathcal{S}$, whence $\bigcup \mathcal{S} \in \mathcal{L}_0$.

Define a topology $\tau_0 \in \mathcal{L}$ by declaring a set $U \subset \mathbb{R}$ τ_0-open if and only if the set U is η-open and either $0 \notin U$ or $U \supset \{0\} \cup (\mathbb{R} \setminus [-t, t])$ for some $t > 0$. Then $C(\tau_0) = \{0\}$ and hence $\tau_0 \in \mathcal{L}_0$. Let K be the union of two congruent circles in the plane $\mathbb{R}^2$ which meet in precisely one point. Then K (which looks like the digit 8 or the symbol ∞) is an arcwise connected and locally arcwise connected compact subspace of the Euclidean plane $\mathbb{R}^2$ with precisely one cut point. (Recall that x is a cut point of a connected space X if and only if $X \setminus \{x\}$ is not connected.) It is immediately obvious that K is a real arc which is homeomorphic to the space $(\mathbb{R}, \tau_0)$. (Of course, 0 is the unique cut point in the arcwise connected space $(\mathbb{R}, \tau_0)$.) It is well-known that any locally connected, compact real arc with precisely one cut point is homeomorphic to K (cf. [7]). Finally, the topologies τ_0 and $\bigcap \mathcal{L}_0$ must be identical because $\tau_0 \in \mathcal{L}_0$ and $\tau_0 \subset \tau$ for every $\tau \in \mathcal{L}_0$ since if $0 \in U \in \tau_0$ then $\mathbb{R} \setminus U$ is Euclidean compact and hence τ-closed for every $\tau \in \mathcal{L}_0$. ■

REMARK. If $a \in \mathbb{R}$ and $\varphi_a(x) = x + a$ for every $x \in \mathbb{R}$ and $\tau_0 \in \mathcal{L}_0$ is compact then $\tau_a := \{\varphi_a(U) | U \in \tau_0\}$ is a topology in $\mathcal{L}$ with $C(\tau_a) = \{a\}$ and hence $\tau_a \neq \tau_{a'}$ whenever $a \neq a'$. Each topology τ_a is compact since φ_a is a homeomorphism from $(\mathbb{R}, \tau_0)$ onto $(\mathbb{R}, \tau_a)$. Thus by Proposition 6, $\mathcal{L}$ contains c (homeomorphic) compact topologies. Therefore, the partially ordered family $(\mathcal{L}, \subset)$ is not a lattice because if τ, τ' are distinct compact topologies in $\mathcal{L}$ then $\{\tau, \tau'\}$ has no infimum in $(\mathcal{L}, \subset)$

since a topology cannot be T_2 if it is strictly coarser than a T_2-compact topology. (In particular, every nonempty chain of compact topologies in $\mathcal{L}$ is a singleton.) It is also worth mentioning that if for $\tau \in \mathcal{L}$ the space $(\mathbb{R}, \tau)$ is compact then it must be second countable. Because, naturally, the sets $]r_1, r_2[$ with $r_1, r_2 \in \mathbb{Q}$ form a network of τ and (cf. [2] 3.3.5.) any compact Hausdorff space has a countable basis if it has a countable network.

11. Long chains of homeomorphic topologies

The topologies in the family $\mathcal{T} \subset \mathcal{L}$ constructed in the proof of Theorem 1 are mutually non-homeomorphic and mutually incomparable. If $\tau_z \in \mathcal{L}$ are the completely metrizable topologies defined by the real arcs $g_z(\mathbb{R})$ in the proof of Proposition 5 then $\{\tau_z | z \in \mathbb{R}\}$ is a family of homeomorphic and mutually incomparable topologies. (They are mutually incomparable because if $r, s \in \mathbb{R}$ and $r \neq s$ then the sequence $(1 + r + \pi n)$ converges to r in the space $(\mathbb{R}, \tau_r)$, whereas in the space $(\mathbb{R}, \tau_s)$ the same sequence converges to s when $\frac{r-s}{\pi} \in \mathbb{Z}$ and diverges when $\frac{r-s}{\pi} \notin \mathbb{Z}$.) However, a simple modification of the real arc $g_z(\mathbb{R})$ makes it possible to track down a chain of homeomorphic topologies in $\mathcal{L}$.

Proposition 7. *There exists a chain $\mathcal{J} \subset \mathcal{L}$ such that $|\mathcal{J}| = c$ and all spaces $(\mathbb{R}, \tau)$ with $\tau \in \mathcal{J}$ are completely metrizable and homeomorphic.*

Proof. For $z \in \mathbb{R}$ consider the mapping $g_z : \mathbb{R} \to \mathbb{R}^2$ from the proof of Proposition 5 and for $-1 < a < 0$ put $\tilde{g}_a(t) = g_0(t)$ when $t \geq 0$ and $\tilde{g}_a(t) = (0, -t)$ when $a \leq t \leq 0$ and $\tilde{g}_a(t) = (t - a, -a)$ when $t \leq a$. For $-1 < a < 0$ let $\tilde{\tau}_a$ be the topology in $\mathcal{L}$ corresponding with the Euclidean continuous injective mapping $\tilde{g}_a : \mathbb{R} \to \mathbb{R}^2$. Then $C(\tilde{\tau}_a) = [a, 0]$ and $(\mathbb{R}, \tilde{\tau}_a)$ is completely metrizable since $\tilde{g}_a(\mathbb{R})$ is a G_δ-subset of $\mathbb{R}^2$. Obviously, $\tilde{\tau}_r$ is a proper subset of $\tilde{\tau}_s$ whenever $-1 < r < s < 0$. All spaces $(\mathbb{R}, \tilde{\tau}_a)$ with $-1 < a < 0$ are homeomorphic because a moment's reflection suffices to see that if $-1 < r < s < 0$ then there is a homeomorphism from the Euclidean plane $\mathbb{R}^2$ onto itself which maps $\tilde{g}_r(\mathbb{R})$ onto $\tilde{g}_s(\mathbb{R})$. ■

The chain $\mathcal{J}$ of homeomorphic topologies constructed in the previous proof is disjoint from the lattice $\mathcal{L}_0$. If $\mathcal{T}$ is a family as in Theorem 1 then $\mathcal{T} \subset \mathcal{L}_0$ but there is no chain $\mathcal{K} \subset \mathcal{T}$ with $|\mathcal{K}| > 1$. Nevertheless, the following theorem shows that the lattice $\mathcal{L}_0$ contains very long chains of homeomorphic topologies. (In the following, as usual, if $\mathcal{K}_2$ is a $\subset$-chain and $\mathcal{K}_1 \subset \mathcal{K}_2$ then $\mathcal{K}_1$ is dense in $\mathcal{K}_2$ if and only if for every pair $X, Y \in \mathcal{K}_2$ with $X \subset Y$ and $X \neq Y$ there exists a set Z in $\mathcal{K}_1 \setminus \{X, Y\}$ such that $X \subset Z \subset Y$.)

Theorem 7. *The lattice $\mathcal{L}_0$ contains four chains $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ of (the maximal possible) size c such that for $i \in \{0, 1, 2, 3\}$ all spaces $(\mathbb{R}, \tau)$ with $\tau \in \mathcal{K}_i$ are homeomorphic, and*

(i) *if $\tau \in \mathcal{K}_0$ then the space (R, τ) is second countable but not regular,*

(ii) *if $\tau \in \mathcal{K}_1$ then the space (R, τ) is neither regular nor first countable,*

(iii) *if $\tau \in \mathcal{K}_2$ then the space (R, τ) is completely normal but not first countable,*

(iv) *if* $\tau \in \mathcal{K}_3$ *then the space* (R, τ) *is completely metrizable,*

(v) $\mathcal{K}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ *is a chain and* $\mathcal{K}_i$ *is dense in* $\mathcal{K}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ *for every* $i \in \{0, 1, 2\}$,

(vi) *every topology in* $\mathcal{K}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ *is coarser than every topology in* $\mathcal{K}_3$.

Proof. The size of $\mathcal{K}_i$ cannot exceed c by Lemma 2. In order to obtain a chain $\mathcal{K}_3$ as desired, for real $\alpha \geq 0$ define an injective and Euclidean continuous mapping h_α from $\mathbb{R}$ into $\mathbb{R}^2$ by $h_\alpha(t) = (t, -t)$ for $t \leq 1$ and $h_\alpha(t) = (1, t-2)$ for $1 \leq t \leq 2$ and $h_\alpha(t) = (2t^{-1}, t^\alpha |\sin(\pi t)|)$ for $t \geq 2$.

Obviously $h_\alpha(\mathbb{R})$ is a G_δ-subset of $\mathbb{R}^2$ for every $\alpha \geq 0$. All sets $h_\alpha(\mathbb{R})$ with $\alpha \geq 0$ are homeomorphic subspaces of $\mathbb{R}^2$ because for every $\alpha \geq 0$ the mapping $(t, h_0(t)) \mapsto (t, h_\alpha(t))$ with t running through $\mathbb{R}$ is clearly a homeomorphism from the real arc $h_0(\mathbb{R})$ onto the real arc $h_\alpha(\mathbb{R})$. Let $\mu[\alpha]$ be the topology in $\mathcal{L}$ corresponding with h_α. Thus $\mu[\alpha] \in \mathcal{L}_0$ and in the space $(\mathbb{R}, \mu[\alpha])$ the family $\{B(\alpha, \varepsilon) | \varepsilon > 0\}$ is a local basis at the point 0 where

$$B(\alpha, \varepsilon) :=]-\varepsilon, \varepsilon[\cup \{t \in \mathbb{R} \mid t > \tfrac{2}{\varepsilon} \wedge t^\alpha |\sin(\pi t)| < \varepsilon\}.$$

(Obviously, $h_\alpha^{-1}(]-\varepsilon, \varepsilon[^2 \cap h_\alpha(\mathbb{R})) = B(\alpha, \varepsilon)$ for every positive $\varepsilon < 1$.) If $0 \leq \alpha_1 \leq \alpha_2$ then $B(\alpha_1, \varepsilon) \supset B(\alpha_2, \varepsilon)$ for every $\varepsilon > 0$ and hence $\mu[\alpha_1] \subset \mu[\alpha_2]$. If $0 \leq \alpha_1 < \alpha_2$ then $\mu[\alpha_1] \neq \mu[\alpha_2]$ because the $\mu[\alpha_2]$-open set $B(\alpha_2, 1)$ cannot be $\mu[\alpha_1]$-open since it is plain that $B(\alpha_1, \varepsilon) \not\subset B(\alpha_2, 1)$ for every $\varepsilon > 0$. So we define $\mathcal{K}_3 := \{\mu[\alpha] | \alpha \geq 0\}$.

In order to find appropriate chains $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2$ we define a family $\mathcal{D} \subset \mathcal{L}_0$ so that the partially ordered set $(\mathcal{D}, \subset)$ is a Boolean algebra isomorphic with the power set of $\mathbb{R}$. Write $x + Y := \{x + y | y \in Y\}$ for $x \in \mathbb{R}$ and $Y \subset \mathbb{R}$. For any set $D \subset [-\frac{1}{2}, \frac{1}{2}[$ define a topology $\tau(D) \in \mathcal{L}$ by declaring $U \subset \mathbb{R}$ open if and only if U is Euclidean open and either $0 \notin U$ or $U \supset \{0\} \cup \bigcup_{k=n}^\infty k + D$ for some $n \in \mathbb{N}$. It is plain that $\tau(D)$ is a well-defined topology on $\mathbb{R}$ and that $\tau(D) \in \mathcal{L}_0$.

Obviously, $\tau(\emptyset) = \eta$ and $\tau(B) \subset \tau(A)$ whenever $A \subset B \subset [-\frac{1}{2}, \frac{1}{2}[$. Furthermore $\tau(A) \neq \tau(B)$ when A, B are distinct subsets of $[-\frac{1}{2}, \frac{1}{2}[$. Moreover, if $B \not\subset A$ then $\tau(A) \not\subset \tau(B)$. Because if $z \in B \setminus A$ then it is clear that the Euclidean open set $\mathbb{R} \setminus (z + \mathbb{N})$ lies in $\tau(A)$ but not in $\tau(B)$. Therefore, if

$$\mathcal{D} := \{\tau(D) | D \subset [-\tfrac{1}{2}, \tfrac{1}{2}[\}$$

and g is a bijection from $\mathbb{R}$ onto $[-\frac{1}{2}, \frac{1}{2}[$ then $X \mapsto \tau([-\frac{1}{2}, \frac{1}{2}[\setminus g(X))$ is an isomorphism from the Boolean algebra of all subsets of $\mathbb{R}$ onto the partially ordered set $(\mathcal{D}, \subset)$.

A moment's reflection suffices to see that $\tau(D) \subset \mu[\alpha]$ for every $\alpha \geq 0$ if $D \subset [-\frac{1}{2}, \frac{1}{2}[$ and 0 is an interior point of D in the Euclidean space $\mathbb{R}$. Therefore, in order to achieve (vi) we choose mutually disjoint sets $\Lambda_0, \Lambda_1, \Lambda_2 \subset]0, \frac{1}{3}[$ of size c which are dense in $]0, \frac{1}{3}[$ and define $\mathcal{K}_0 := \{\tau([-\lambda, \lambda]) | \lambda \in \Lambda_0\}$ and $\mathcal{K}_1 := \{\tau([-\lambda, \lambda[) | \lambda \in \Lambda_1\}$ and $\mathcal{K}_2 := \{\tau(]-\lambda, \lambda[) | \lambda \in \Lambda_2\}$. The specific choice of $\Lambda_0, \Lambda_1, \Lambda_2$ is made for saving the density condition (v) because if $A \subset B \subset [-\frac{1}{2}, \frac{1}{2}[$ and $|B \setminus A| = 1$ then no topology from $\mathcal{D}$ lies strictly between $\tau(B)$ and $\tau(A)$. Clearly, if $0 < \lambda, \lambda' < \frac{1}{3}$ and f is any strictly increasing function from $\mathbb{R}$ onto $\mathbb{R}$ with $f(0) = 0$ and $f(n \pm \lambda) = n \pm \lambda'$ for every $n \in \mathbb{N}$ then f is a homeomorphism from

$(\mathbb{R}, \tau([-\lambda, \lambda]))$ onto $(\mathbb{R}, \tau([-\lambda', \lambda']))$ and from $(\mathbb{R}, \tau([-\lambda, \lambda[))$ onto $(\mathbb{R}, \tau([-\lambda', \lambda'[))$ and from $(\mathbb{R}, \tau(]-\lambda, \lambda[))$ onto $(\mathbb{R}, \tau(]-\lambda', \lambda'[))$. So the definitions of the four chains $\mathcal{K}_i$ do the job provided that (i) and (ii) and (iii) hold.

For $T \subset \mathbb{R}$ put $\Gamma(T) := \{e^{2\pi i t} | t \in T\}$. So $\Gamma(\mathbb{R}) = \Gamma([-\frac{1}{2}, \frac{1}{2}[)$ is the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ and $\Gamma(D) \subset \Gamma(\mathbb{R})$ for $D \subset [-\frac{1}{2}, \frac{1}{2}[$. We finish the proof by verifying the nice observation that for every $D \subset [-\frac{1}{2}, \frac{1}{2}[$,

(1) $(\mathbb{R}, \tau(D))$ *is second countable if and only if* $\Gamma(D)$ *is open in* $\Gamma(\mathbb{R})$,

(2) $(\mathbb{R}, \tau(D))$ *is regular if and only if* $\Gamma(D)$ *is closed in* $\Gamma(\mathbb{R})$.

Note that by Lemma 4 the space $(\mathbb{R}, \tau(D))$ is regular if and only if $(\mathbb{R}, \tau(D))$ is completely normal.

If $\Gamma(D)$ is open in $\Gamma(\mathbb{R})$ then $\{]-n^{-1}, n^{-1}[\cup \bigcup_{k=n}^{\infty} k+D | n \in \mathbb{N}\}$ is clearly a local basis at 0 in the space $(\mathbb{R}, \tau(D))$, whence $(\mathbb{R}, \tau(D))$ is second countable by Lemma 4. If $\Gamma(D)$ is not closed in $\Gamma(\mathbb{R})$ then for some $b \in [-\frac{1}{2}, \frac{1}{2}[\setminus D$ the point $e^{2\pi i b}$ is a limit point of $\Gamma(D)$ in $\Gamma(\mathbb{R})$. So the Euclidean closed set $b + \mathbb{N}$ is $\tau(D)$-closed and, obviously, the point 0 and the set $b + \mathbb{N}$ can not be separated by $\tau(D)$-open sets, whence $\tau(D)$ is not regular. If $\Gamma(D)$ is closed in $\Gamma(\mathbb{R})$ then, by the same arguments as in the proof of Theorem 1, the space $(\mathbb{R}, \tau(D))$ is regular. (One can adopt the proof line by line with the only modification that the set $B = \{0\} \cup (\mathbb{Z} \setminus A)$ is replaced by $B = \{0\} \cup \bigcup_{n=k}^{\infty} n + D$ where $k \in \mathbb{N}$ is chosen so that $A \cap (n + D) = \emptyset$ whenever $n \geq k$.)

Finally, assume that $\Gamma(D)$ is not open in $\Gamma(\mathbb{R})$ and choose $d \in D$ so that $e^{2\pi i d}$ is not an interior point of $\Gamma(D)$ in $\Gamma(\mathbb{R})$. Suppose that a countable family $\{B_1, B_2, B_3, \dots\}$ of Euclidean open sets is a local basis at 0 in the space $(\mathbb{R}, \tau(D))$. Let k_1 be the least positive integer n such that $B_1 \supset n + D$. If k_m is already defined then let k_{m+1} be the least integer $n > k_m$ such that $B_{m+1} \supset n + D$. For every $m \in \mathbb{N}$ choose a small $\epsilon_m > 0$ such that $\Gamma(]d - \epsilon_m, d + \epsilon_m[) \not\subset \Gamma(D)$ and $]k_m + d - \epsilon_m, k_m + d + \epsilon_m[\subset B_m$. Then for every $m \in \mathbb{N}$ we can choose a point x_m in $]k_m + d - \epsilon_m, k_m + d + \epsilon_m[\setminus (k_m + D)$. Then the set $V := \mathbb{R} \setminus \{x_m | m \in \mathbb{N}\}$ is $\tau(D)$-open and hence $V \supset B_n$ for some $n \in \mathbb{N}$. So we obtain the contradiction that $x_n \in B_n \subset V$ and $x_n \notin V$ for some $n \in \mathbb{N}$. Thus the assumption on $\{B_1, B_2, B_3, \dots\}$ is false and hence $\tau(D)$ is not first countable. This concludes the proof of Theorem 7. ∎

Remark. The maximum of the Boolean algebra $(\mathcal{D}, \subset)$ is $\tau(\emptyset) = \eta$. The topology $\tau([-\frac{1}{2}, \frac{1}{2}[)$ is the minimum of $\mathcal{D}$ and it is plain that $(\mathbb{R}, \tau([-\frac{1}{2}, \frac{1}{2}[))$ is homeomorphic to the subspace $\Gamma^* := \Gamma(\mathbb{R}) \cup \{0\} \times [1, \infty[$ of the Euclidean plane $\mathbb{R}^2$. It is well-known that any locally connected, locally compact but not compact real arc is homeomorphic either to Γ^* or to the real line (cf. [7]). In view of (1) and (2), the maximum and the minimum of the Boolean algebra $\mathcal{D}$ are the only metrizable topologies in $\mathcal{D}$. In view of (2) and Lemma 4 and $|\eta| = c$, precisely c topologies in $\mathcal{D}$ are completely normal, whence the proof of Theorem 1 is not dispensable. On the contrary, in view of Lemma 3 and Proposition 4 and the well-known fact that $\mathbb{R}^2$ has only c Euclidean closed subsets (and the trivial fact that $\Gamma(\mathbb{R})$ has 2^c subsets), an alternative proof of Theorem 2 (which does not use ultrafilters) is provided by (2).

12. Countably generated topologies

Only c topologies in the Boolean algebra $\mathcal{D}$ are first countable. But all topologies in $\mathcal{D}$ satisfy an interesting countability condition weaker than first countability. Let $\mathcal{L}_0^*$ denote the family of all topologies in $\mathcal{L}_0$ such that $\mathcal{N}_\tau(0) = \mathcal{N}_\eta(0) \cap \mathcal{F}$ for some filter $\mathcal{F}$ on $\mathbb{R}$ which is generated by a countable filter base. In other words, there is a countable filter base $\mathcal{B}$ of subsets of $\mathbb{R}$ such that $\eta \cap \mathcal{N}_\tau(0) = \{U \in \eta | \exists B \in \mathcal{B} : U \supset \{0\} \cup B\}$. So if $\tau \in \mathcal{L}_0$ is first countable then $\tau \in \mathcal{L}_0^*$. The converse is not true since $\mathcal{D} \subset \mathcal{L}_0^*$. In particular, $|\mathcal{L}_0^*| = |\mathcal{D}| = 2^c$. Whereas for $A \subset B \subset [-\frac{1}{2}, \frac{1}{2}[$ with $|B \setminus A| = 1$ there is no topology $\tau \in \mathcal{D}$ strictly between $\tau(B)$ and $\tau(A)$, the following theorem implies that between $\tau(B)$ and $\tau(A)$ there lie c comparable and c incomparable topologies from $\mathcal{L}_0^*$ and also 2^c incomparable topologies from $\mathcal{L}_0 \setminus \mathcal{L}_0^*$.

THEOREM 8. *If $\tau_1 \in \mathcal{L}_0^*$ is strictly coarser than $\tau_2 \in \mathcal{L}_0$ then there are a chain $\mathcal{R} \subset \mathcal{L}_0$ with $|\mathcal{R}| = c$ and two families $\mathcal{S} \subset \mathcal{L}_0$ and $\mathcal{T} \subset \mathcal{L}_0 \setminus \mathcal{L}_0^*$ of mutually incomparable topologies with $|\mathcal{S}| = c$ and $|\mathcal{T}| = 2^c$ such that $\tau_1 \subset \tau \subset \tau_2$ for every $\tau \in \mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$. Additionally $\mathcal{R}, \mathcal{S} \subset \mathcal{L}_0^*$ can be achieved if $\tau_2 \in \mathcal{L}_0^*$. For $\tau_2 = \eta$ it can be achieved that $\mathcal{R}, \mathcal{S} \subset \mathcal{L}_0^*$ and all topologies in $\mathcal{R} \cup \mathcal{S}$ are homeomorphic.*

Proof. First of all, if $\eta \cap \mathcal{N}_\tau(0) = \{U \in \eta | \exists B \in \mathcal{B} : U \supset \{0\} \cup B\}$ for $\tau \in \mathcal{L}_0$ and a filter base $\mathcal{B}$ then $\eta \cap \mathcal{N}_\tau(0) = \{U \in \eta | \exists B \in \mathcal{B} : U \supset \{0\} \cup (B \setminus [-1,1])\}$. Indeed, if $U \in \eta$ contains $\{0\} \cup (B_1 \setminus [-1,1])$ for some $B_1 \in \mathcal{B}$ then U contains $]-k^{-1}, k^{-1}[\cup (B_1 \setminus [-1,1])$ for some $k > 1$. Since $V_k := \mathbb{R} \setminus ([-k, -k^{-1}] \cup [k^{-1}, k])$ lies in $\eta \cap \mathcal{N}_\tau(0)$, we have $B_2 \subset V_k$ for some $B_2 \in \mathcal{B}$ and hence $U \supset B_1 \cap B_2$. Thus, since $\mathcal{B}$ is a filter base, we have $B \subset B_1 \cap B_2 \subset U$ for some $B \in \mathcal{B}$. There is an important consequence of the two representations of $\eta \cap \mathcal{N}_\tau(0)$. If $\eta \neq \tau \in \mathcal{L}_0$ and a filter base $\mathcal{B}$ generates a filter $\mathcal{F}$ with $\mathcal{N}_\tau(0) = \mathcal{N}_\eta(0) \cap \mathcal{F}$ then the family $\mathcal{B}' := \{B \setminus [-1,1] | B \in \mathcal{B}\}$ does not contain $\emptyset$ and hence $\mathcal{B}'$ is a filter base which generates a filter $\mathcal{F}'$ with $\mathcal{N}_\tau(0) = \mathcal{N}_\eta(0) \cap \mathcal{F}'$.

Let $\tau_1 \in \mathcal{L}_0^*$ be a proper subset of $\tau_2 \in \mathcal{L}_0$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be families of subsets of $\mathbb{R} \setminus [-1,1]$ such that $\mathcal{B}_1$ is a countable filter base and $\mathcal{B}_2$ is a filter base when $\tau_2 \neq \eta$ and $\mathcal{B}_2 = \{\emptyset\}$ when $\tau_2 = \eta$ and $\eta \cap \mathcal{N}_{\tau_i}(0) = \{U \in \eta | \exists B \in \mathcal{B}_i : U \supset \{0\} \cup B\}$ for $i \in \{1,2\}$. We may assume that $\mathcal{B}_1 = \{A_1, A_2, A_3, \dots\}$ where A_n is a proper subset of A_m whenever $m < n$. Since τ_1 is strictly coarser than τ_2, we can fix $D \in \mathcal{B}_2$ such that $A_n \not\subset D$ for every $n \in \mathbb{N}$. Since for every $k \in \mathbb{N}$ we have $A_n \subset V_k$ and hence $A_n \subset \mathbb{R} \setminus [-k,k]$ for some $n \in \mathbb{N}$, we can choose a sequence $a_1, a_2, a_3, \dots$ of distinct reals such that always $a_n \in A_n \setminus D$ and either $a_n > n$ for every $n \in \mathbb{N}$ or $a_n < -n$ for every $n \in \mathbb{N}$. Then $\{a_1, a_2, a_3, \dots\}$ is disjoint from $D \cup [-1,1]$ and Euclidean closed and discrete. Consequently, every subset of $\{a_1, a_2, a_3, \dots\}$ is τ_2-closed.

For every infinite set $S \subset \mathbb{N}$ define a topology $\rho[S] \in \mathcal{L}_0$ with $\rho[S] \subset \tau_2$ so that an τ_2-open neighborhood U of 0 is $\rho[S]$-open if and only if $U \supset \{a_n | k \leq n \in S\}$ for some $k \in \mathbb{N}$. We have $\tau_1 \subset \rho[S]$ since $\{a_n | n \geq k\} \subset A_k$ for every $k \in \mathbb{N}$. Obviously, $\rho[S_1] \subset \rho[S_2]$ when $S_1 \supset S_2$. Furthermore, if $S_2 \setminus S_1$ is an infinite set then $\rho[S_1] \not\subset \rho[S_2]$ because the τ_2-open set $\mathbb{R} \setminus \{a_n | n \notin S_1\}$ is $\rho[S_1]$-open but not

$\rho[S_2]$-open. Therefore, we define $\mathcal{R} := \{\rho[R_z] | z \in \mathbb{R}\}$ and $\mathcal{S} := \{\rho[S_z] | z \in \mathbb{R}\}$ where for every $z \in \mathbb{R}$ infinite sets $R_z, S_z \subset \mathbb{N}$ are defined so that if $x < y$ then on the one hand $R_x \supset R_y$ and $R_x \setminus R_y$ is an infinite set, and on the other hand $S_x \cap S_y$ is a finite set. (For example, choose a bijection φ from $\mathbb{N}$ onto $\mathbb{Q}$ and put $R_x := \{n \in \mathbb{N} | x \leq \varphi(n)\}$ for every $x \in \mathbb{R}$. Furthermore, for every $x \in \mathbb{R}$ choose a set $T_x \subset \mathbb{Q} \cap [x-1, x]$ with $T'_x = \{x\}$ and put $S_x := \varphi^{-1}(T_x)$.) Clearly, for every infinite set $S \subset \mathbb{N}$ the family $\{B \cup \{a_n | k \leq n \in S\} | B \in \mathcal{B}_2 \wedge k \in \mathbb{N}\}$ is a filter base which generates a filter $\mathcal{F}$ such that $\mathcal{N}_\eta(0) \cap \mathcal{F} = \mathcal{N}_{\rho[S]}(0)$. Thus $\mathcal{R}, \mathcal{S} \subset \mathcal{L}_0^*$ if $\mathcal{B}_2$ is countable. (So $\mathcal{R}, \mathcal{S} \subset \mathcal{L}_0^*$ can be achieved if $\tau_2 \in \mathcal{L}_0^*$.) If $\tau_2 = \eta$ (and hence $\mathcal{B}_2 = \{\emptyset\}$) then the topologies in $\mathcal{R} \cup \mathcal{S}$ are homeomorphic. Because if $S \subset \mathbb{N}$ is infinite then any increasing bijection from $\mathbb{R}$ onto $\mathbb{R}$ which maps 0 to 0 and $\{a_1, a_2, a_3, \dots\}$ onto $\{a_n | n \in S\}$ is clearly a homeomorphism from the space $(\mathbb{R}, \rho[\mathbb{N}])$ onto $(\mathbb{R}, \rho[S])$. So in order to conclude the proof it remains to define a family $\mathcal{T}$ as desired.

For every free ultrafilter $\mathcal{F}$ on $\mathbb{N}$ put $\rho[\mathcal{F}] := \bigcup_{S \in \mathcal{F}} \rho[S]$. Clearly, $\tau_1 \subset \rho[\mathcal{F}] \subset \tau_2$. We claim that $\rho[\mathcal{F}]$ is a topology in the lattice $\mathcal{L}_0$. Firstly, let $U_1, U_2 \in \rho[\mathcal{F}]$. Then $U_i \in \rho[S_i]$ for $S_i \in \mathcal{F}$. Since $S_1 \cap S_2$ is an infinite set in the ultrafilter $\mathcal{F}$ and $\rho[S_1 \cap S_2]$ is a topology containing $\rho[S_1]$ and $\rho[S_2]$, the intersection $U_1 \cap U_2$ lies in $\rho[S_1 \cap S_2]$ and hence in $\rho[\mathcal{F}]$. Since $U \in \rho[S]$ whenever $0 \notin U \in \eta$ and $S \in \mathcal{F}$, it is plain that the family $\rho[\mathcal{F}]$ is closed under arbitrary unions and furthermore that $\rho[\mathcal{F}] \in \mathcal{L}_0$. We also observe that for $U \in \eta \cap \mathcal{N}_\eta(0)$ we have $U \in \rho[\mathcal{F}]$ if and only if $U \supset B$ for some $B \in \mathcal{B}_2$ and $\{n \in \mathbb{N} | a_n \in U\} \in \mathcal{F}$. Let $\mathcal{F}_1, \mathcal{F}_2$ be free ultrafilters on $\mathbb{N}$ and $S \in \mathcal{F}_1$ and assume $\rho[\mathcal{F}_1] \subset \rho[\mathcal{F}_2]$. The set $V := \mathbb{R} \setminus \{a_n | n \notin S\}$ is τ_2-open and $\{n \in \mathbb{N} | a_n \in V\} = S$. Thus V is $\rho[\mathcal{F}_1]$-open and hence $\rho[\mathcal{F}_2]$-open and this implies $S \in \mathcal{F}_2$. So we derive $\mathcal{F}_1 \subset \mathcal{F}_2$ and hence $\mathcal{F}_1 = \mathcal{F}_2$. Thus the topologies $\rho[\mathcal{F}]$ are mutually incomparable and hence a family $\mathcal{T}$ as desired exists provided that we always have $\rho[\mathcal{F}] \notin \mathcal{L}_0^*$.

Assume indirectly that $\rho[\mathcal{F}] \in \mathcal{L}_0^*$ for a free ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Then we can choose a countable filter base $\{B_1, B_2, B_3, \dots\}$ of subsets of $\mathbb{R} \setminus [-1, 1]$ such that $B_n \supset B_{n+1}$ for every $n \in \mathbb{N}$ and $\eta \cap \mathcal{N}_{\rho[\mathcal{F}]}(0) = \{U \in \eta | \exists n \in \mathbb{N} : U \supset \{0\} \cup B_n\}$. Put $S_m := \{n \in \mathbb{N} | a_n \in B_m\}$ for every $m \in \mathbb{N}$. Trivially, $S_m \supset S_{m+1}$ for every $m \in \mathbb{N}$. Let S be any set in the ultrafilter $\mathcal{F}$. Then the set $\mathbb{R} \setminus \{a_n | n \notin S\}$ is $\rho[\mathcal{F}]$-open and hence it contains B_m for some $m \in \mathbb{N}$. So for some $m \in \mathbb{N}$ we have $B_m \cap \{a_n | n \notin S\} = \emptyset$ and hence $S_m \subset S$. Therefore, $\{S_m | m \in \mathbb{N}\}$ is a filter base for the filter $\mathcal{F}$. But this is impossible because a filter base for a free ultrafilter on $\mathbb{N}$ must be uncountable (cf. [1] 7.8.a). This concludes the proof of Theorem 8. ■

Remark. For achieving $\mathcal{R}, \mathcal{S} \subset \mathcal{L}_0^*$, the additional assumption $\tau_2 \in \mathcal{L}_0^*$ is essential in view of the following counterexample (τ_1, τ_2). Consider the topologies $\tau_1 := \tau(\{0\})$ and $\tau'_1 := \tau(]0, \frac{1}{2}[)$ in the Boolean algebra $\mathcal{D} \subset \mathcal{L}_0^*$. Let τ_2 be the supremum of $\{\tau_1, \tau'_1\}$ in the lattice $\mathcal{L}_0$. We observe that if $\tau_1 \neq \tau \in \mathcal{L}_0$ and $\tau_1 \subset \tau \subset \tau_2$ then $\tau \notin \mathcal{L}_0^*$. (Because for every $k \in \mathbb{N}$ and every sequence (u_n) with $0 < u_n \leq \frac{1}{2}$ the set $]-1, 1[\cup \bigcup_{n=k}^\infty]n, n + u_n[$ lies in $\tau \setminus \tau_1$.) In particular, $\tau_2 \notin \mathcal{L}_0^*$. Furthermore, this counterexample demonstrates that neither $\mathcal{D}$ nor $\mathcal{L}_0^*$ is a sublattice of $\mathcal{L}_0$.

The minimum $\theta = \bigcap \mathcal{L}_0$ of the complete lattice $\mathcal{L}_0$ lies in $\mathcal{L}_0^*$. Thus by Theorem 8 and since it is clear that $\mathcal{L}_0 = \{\tau \in \mathcal{L} | \tau \supset \theta\}$, the topology θ has no immediate successor in the lattice $\mathcal{L}_0$ or in the partially ordered set $(\mathcal{L}, \subset)$. On the other hand, the following proposition shows that the maximum $\eta = \bigcup \mathcal{L}_0$ of the lattice $\mathcal{L}_0$ has 2^c immediate predecessors in the lattice $\mathcal{L}_0$ which are also immediate predecessors of η in the partially ordered family $(\mathcal{L}, \subset)$.

PROPOSITION 8. *There exist 2^c (mutually non-homeomorphic) topologies $\vartheta \in \mathcal{L}_0$ such that no topology from $\mathcal{L}$ lies strictly between ϑ and η.*

Proof. For a free ultrafilter $\mathcal{F}$ on $\mathbb{Z}$ let $\tau[\mathcal{F}]$ denote the topology as defined in the proof of Theorem 1. If $\mathcal{K} \subset \mathcal{L}_0 \setminus \{\eta\}$ is a chain with $\tau[\mathcal{F}] \in \mathcal{K}$ then $\eta \neq \bigcup \mathcal{K} \in \mathcal{L}_0$ by Proposition 6. Therefore, by applying Zorn's lemma, for every free ultrafilter $\mathcal{F}$ on $\mathbb{Z}$ we can choose a maximal element $\vartheta[\mathcal{F}]$ in the partially ordered set $(\mathcal{L}_0 \setminus \{\eta\}, \subset)$ such that $\tau[\mathcal{F}] \subset \vartheta[\mathcal{F}]$. For distinct free ultrafilters $\mathcal{F}_1, \mathcal{F}_2$ we have $\vartheta[\mathcal{F}_1] \neq \vartheta[\mathcal{F}_2]$ because $\tau[\mathcal{F}_1] \neq \tau[\mathcal{F}_2]$ and η is the supremum of $\{\tau[\mathcal{F}_1], \tau[\mathcal{F}_2]\}$ in the lattice $\mathcal{L}_0$ in view of Proposition 1 since there are sets $U_i \in \tau[\mathcal{F}_i]$ with $U_1 \cap U_2 =]-1,1[$. (For example, choose $S_1 \in \mathcal{F}_1 \setminus \mathcal{F}_2$ and with $S_2 := \mathbb{Z} \setminus S_1$ put $U_i =]-1,1[\cup \bigcup_{n \in S_i}]n - \frac{1}{2}, n + \frac{1}{2}[$ for $i \in \{1,2\}$.) Finally, if $\eta \neq \tau \in \mathcal{L}$ and $\tau \supset \vartheta[\mathcal{F}]$ then $\tau = \vartheta[\mathcal{F}]$ since $\mathcal{L}_0 = \{\tau \in \mathcal{L} | \tau \supset \theta\}$ and $\vartheta[\mathcal{F}]$ is maximal in $\mathcal{L}_0 \setminus \{\eta\}$. ■

REMARK. By virtue of Theorem 8 every immediate predecessor of η in $\mathcal{L}_0$ must lie in $\mathcal{L}_0 \setminus \mathcal{L}_0^*$. This observation has two consequences in view of Proposition 8. Firstly we can be sure that $|\mathcal{L}_0 \setminus \mathcal{L}_0^*| = |\mathcal{L}_0^*| = 2^c$. Secondly, the central assumption $\tau_1 \in \mathcal{L}_0^*$ in Theorem 8 cannot be replaced with the weaker assumption $\tau_1 \in \mathcal{L}_0$.

13. Extremely long chains of topologies

Since both the existence of free ultrafilters and the existence of the topologies $\vartheta[\mathcal{F}]$ in the proof of Proposition 8 are based on a maximality principle equivalent with the Axiom of Choice, one might ask whether in the proof of Proposition 8 the topology $\tau[\mathcal{F}]$ is maximal in $\mathcal{L}_0 \setminus \{\eta\}$ already, whence $\vartheta[\mathcal{F}] = \tau[\mathcal{F}]$. This would be far from being true in view of the following theorem which affirmatively answers the interesting question whether the lattice $\mathcal{L}_0$ contains chains of size greater than c. Define $\lambda := \log(c^+)$, i.e. λ is the smallest cardinal number κ satisfying $2^\kappa > c$, whence $\aleph_1 \leq \lambda \leq c$ and $2^\lambda > c$.

THEOREM 9. *For every free ultrafilter $\mathcal{F}$ on $\mathbb{Z}$ there is a chain $\mathcal{K} \subset \mathcal{L}_0$ such that $|\mathcal{K}| = 2^\lambda$ and $\tau \supset \tau[\mathcal{F}]$ for every $\tau \in \mathcal{K}$.*

Proof. For $n \in \mathbb{N}$ define a strictly increasing real function φ_n by $\varphi_n(x) = 3^{-n}(x+1)$, whence φ_n maps $[0,1]$ onto $[3^{-n}, 2 \cdot 3^{-n}]$. For every set $A \subset [0,1]$ define

$$\Phi(A) := \{0\} \cup \bigcup_{k \in \mathbb{Z}} \big(k + \bigcup_{n=1}^{\infty} \varphi_n(A)\big).$$

Let $\mathcal{F}$ be a free ultrafilter $\mathcal{F}$ on $\mathbb{Z}$. For $A \subset [0,1]$ let $\tau[\mathcal{F}, A]$ denote the coarsest topology in the lattice $\mathcal{L}_0$ which is finer than $\tau[\mathcal{F}]$ and contains all Euclidean open sets $U \supset \Phi(A)$. (In particular, $\tau[\mathcal{F},\emptyset] = \eta$.) Since $\tau[\mathcal{F},A] \in \mathcal{L}_0$, it is plain that $W \in \eta$ is an open neighborhood of 0 in the space $(\mathbb{R},\tau[\mathcal{F},A])$ if and only if $W = U \cap V$ for some $U,V \in \eta$ with $U \supset \Phi(A)$ and $0 \in V$ and $V \cap \mathbb{Z} \in \mathcal{F}$.

Obviously, $\tau[\mathcal{F},B] \subset \tau[\mathcal{F},A]$ if $\emptyset \neq A \subset B \subset [0,1]$. Moreover, $\tau[\mathcal{F},B]$ is strictly coarser than $\tau[\mathcal{F},A]$ if $\emptyset \neq A \subset B \subset [0,1]$ and $A \neq B$. Because if $b \in B \setminus A$ then the Euclidean open set $Y :=]-1,1[\cup(\mathbb{R} \setminus (\mathbb{Z} \cup \Phi(\{b\})))$ is $\tau[\mathcal{F},A]$-open since $Y \supset \Phi(A)$. But Y is not $\tau[\mathcal{F},B]$-open because if $k \in \mathbb{Z}$ and $|k| \geq 2$ and $\varepsilon > 0$ then Y does not contain $]k,k+\varepsilon[\cap\Phi(B)$.

Therefore, $\mathcal{K} = \{\tau[\mathcal{F},A] | A \in \mathcal{A}\}$ is a chain as desired if $\mathcal{A}$ is a chain of subsets of $[0,1]$ with $|\mathcal{A}| = 2^\lambda$.

Such a chain $\mathcal{A}$ can easily be defined as follows. Choose a linearly ordered set $(L,\preceq)$ such that $|L| = 2^\lambda$ and L has a dense subset D with $|D| = c$. (This choice is possible in view of [1] Theorems 5.7.c and 5.8.b.) Define a bijection g from D onto $[0,1]$ and put $A_x := \{g(y) | x \prec y \in D\}$ for every $x \in L$. Finally define $\mathcal{A} := \{A_x | x \in L\}$. ■

Remark. One does not need Theorem 9 to track down chains in $\mathcal{L}_0$ of size 2^λ, it is enough to define $\mathcal{A}$ as above and to take into consideration that our Boolean algebra $\mathcal{D} \subset \mathcal{L}_0^*$ is isomorphic with the power set of $[0,1]$. The lattice $\mathcal{L}_0$ contains chains of the maximal possible size 2^c provided that $2^\lambda = 2^c$. Of course, $2^\lambda = 2^c$ trivially follows from the irrefutable hypothesis $\lambda = c$. (Conversely, $2^\lambda = 2^c$ does not imply $\lambda = c$.) The hypothesis $\lambda = c$ is irrefutable because $\lambda = c$ is obviously a consequence of the Continuum Hypothesis $\aleph_1 = c$. However, the hypothesis $\lambda = c$ is much weaker than the very restrictive hypothesis $\aleph_1 = c$ because it is consistent with ZFC set theory that $\lambda = c$ and $\aleph_1 < \mu < c$ for infinitely many cardinal numbers μ. Even more, roughly speaking, $\lambda = c$ cannot prevent an arbitrarily large deviation of c from $\aleph_1$. (Precisely, in view of [3] 16.13 and 16.20, if $\kappa > \aleph_1$ is an arbitrary regular cardinal in Gödel's Universe L then there is a generic extension E_κ of L preserving all cardinals such that $\lambda = c = \kappa$ holds in the ZFC-model E_κ.)

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