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COMPOSITION OPERATORS AND THEIR PRODUCTS ON $L^2(\Sigma)$

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Abstract. The paper gives measure-theoretic characterizations of classical properties of composition operators and their products on $L^2(\Sigma)$ such as complex symmetric and semi-Kato type operators.

1. Introduction and preliminaries

Let (X, Σ, μ) be a complete sigma finite measure space. For any sub-sigma finite algebra $\mathcal{A} \subseteq \Sigma$, the L^2-space $L^2(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is abbreviated by $L^2(\mathcal{A})$, and its norm is denoted by $\|.\|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. Let φ be a nonsingular measurable transformation from X into X; that is, $\mu\circ\varphi^{-1}$ is absolutely continuous with respect to μ and write $\mu\circ\varphi^{-1} \ll \mu$. Let h be the Radon-Nikodym derivative $d\mu\circ\varphi^{-1}/d\mu$. The pair (Σ, μ) is said to be normal invariant if $\varphi(\Sigma) \subseteq \Sigma$ and $\mu \ll \mu\circ\varphi^{-1}$. The composition operator $C_\varphi : L^2(\Sigma) \to L^0(\Sigma)$ induced by φ is given by $C_\varphi(f) = f\circ\varphi$, for each $f \in L^2(\Sigma)$. Here, the non-singularity of φ guarantees that C_φ is well defined. It is well known fact that for $u \in L^0(\Sigma)$, the multiplication operator $M_u : L^2(\Sigma) \to L^0(\Sigma)$ is bounded if and only if $u \in L^\infty(\Sigma)$, and in this case, $\|M_u\| = \|u\|_\infty$. Now, by the change of variables formula; $\int_X |f\circ\varphi|^2 d\mu = \int_X h|f|^2 d\mu$, $\|C_\varphi f\|_2 = \|M_{\sqrt{h}} f\|_2$ for each $f \in L^2(\Sigma)$. It follows that C_φ maps $L^2(\Sigma)$ boundedly into itself, if and only if $h \in L^\infty(\Sigma)$, and in this case, $\|C_\varphi\| = \|h\|_\infty^{1/2}$. Some other basic facts about composition operators can be found in [8, 19, 22].

For each f in $L^2(\Sigma)$ there is a unique function in $L^2(\mathcal{A})$, denoted $E^{\mathcal{A}}(f)$, such that, for every set $A \in \mathcal{A}$ of finite measure, $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$. $E^{\mathcal{A}}(f)$ is called

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the conditional expectation of f with respect to $\mathcal{A}$, and $E^{\mathcal{A}}$ is the conditional expectation operator. As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is the contractive orthogonal projection onto $L^2(\mathcal{A})$. Take $\mathcal{A} = \varphi^{-1}(\Sigma)$. So for each function f in $L^2(\Sigma)$ there is a Σ measurable function F such that $E^{\varphi^{-1}(\Sigma)}f = F \circ \varphi$. Moreover, F is uniquely determined in $\sigma(h)$ (see [ca]). Therefore, even though φ is not invertible the expression $F = (E^{\varphi^{-1}(\Sigma)}f) \circ \varphi^{-1}$ is well defined. Note that domain of $E^{\mathcal{A}}$ contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$. For further discussion of the conditional expectation operator see [13] and [17]. A result of Hoover, Lambert and Quinn [9] shows that the adjoint C_φ^* of C_φ on $L^2(\Sigma)$ is given by $C_\varphi^*(f) = hE^{\varphi^{-1}(\Sigma)}(f) \circ \varphi^{-1}$. From this it easily follows that $C_\varphi^* C_\varphi = M_h$ and $C_\varphi C_\varphi^* = M_{h\circ\varphi} E^{\varphi^{-1}(\Sigma)}$. The product $M_u \circ C_\varphi$ of M_u and C_φ is called a weighted composition operator.

Products of operators appear more often in the service of the study of other operators. More precisely, for any operator T, there exists a decomposition $T = (U+K)S$, where U is a partial isometry, K is a compact operator and S is a strongly irreducible operator [20]. Composition operators and their products have been used to provide examples and illustrations of many operator theoretic properties. In several cases major conjectures in operator theory have been reduced to the (weighted) composition operators. The purpose of this note is to find some new characterizations of composition operators on $L^2(\Sigma)$ and present a relationship between C_{φ_3} and their products.

2. On some classic properties of composition operators

Let $\mathcal{H}$ be the infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, the range and the null-space of A are denote by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively.

Throughout this paper we assume that for $i = 1, 2$, $\varphi_i : X \to X$ is a nonsingular measurable transformation and $\varphi_i^{-1}(\Sigma)$ is a relatively μ-complete sub-sigma finite algebra of Σ. Put $\varphi_3 = \varphi_1 \circ \varphi_2$, $h_i = d\mu \circ \varphi_i^{-1}/d\mu$ and $E_i = E^{\varphi_i^{-1}(\Sigma)}$. Then $\mu \circ \varphi_3^{-1}$ is absolutely continuous with respect to μ, because the assumption $\mu \circ \varphi_i^{-1} \ll \mu$ implies that for each $A \in \Sigma$ with $\mu(A) = 0$ we have $\mu(\varphi_1^{-1}(A)) = 0$, and so $\mu \circ \varphi_3^{-1}(A) = \mu(\varphi_2^{-1}(\varphi_1^{-1}(A))) = 0$. It follows that C_{φ_3} is a well-defined operator. Take $h_3 = d\mu \circ \varphi_3^{-1}/d\mu$ and $E_3 = E^{\varphi_3^{-1}(\Sigma)}$. Note that if h_1 and h_2 are essentially bounded, then for some $M_1 > 0$ and $M_2 > 0$, $\mu(\varphi_3^{-1}(A)) \leq M_2\mu(\varphi_2^{-1}(A)) \leq M_2 M_1 \mu(A)$ for each $A \in \Sigma$. Hence h_3 is essentially bounded and thus $\varphi_3^{-1}(\Sigma)$ is a sub-sigma finite algebra of Σ.

Proposition 2.1. *For nonsingular measurable transformations φ_1 and φ_2, let $\varphi_3^{-1}(\Sigma)$ be a sub-sigma finite algebra of Σ. Then the following assertions hold.*

(a) $C_{\varphi_3} = C_{\varphi_2} \circ C_{\varphi_1} \in \mathcal{B}(L^2(\Sigma))$ if and only if $h_3 = h_1 E_1(h_2) \circ \varphi_1^{-1} \in L^\infty(\Sigma)$ and in this case $\|C_{\varphi_3}\| = \|h_1 E_1(h_2) \circ \varphi_1^{-1}\|_\infty^{1/2}$.

(b) C_{φ_3} is injective if and only if C_{φ_1} is injective and $\sigma(E_1(h_2)) = X$.

Proof. (a) Recall that C_{φ_3} is bounded if and only if $h_3 \in L^\infty(\Sigma)$, and in this case, $\|C_{\varphi_3}\| = \|h_3\|_\infty^{1/2}$. Let $A \in \Sigma$. By using of conditional expectation operator and change of variables formula we have

$$\begin{aligned}\int_A h_3\, d\mu &= \int_{\varphi_1^{-1}(A)} d\mu \circ \varphi_2^{-1} = \int_{\varphi_1^{-1}(A)} E_1(h_2)\, d\mu \\ &= \int_A E_1(h_2) \circ \varphi_1^{-1}\, d\mu \circ \varphi_1^{-1} = \int_A h_1 E_1(h_2) \circ \varphi_1^{-1}\, d\mu.\end{aligned}$$

It follows that $h_3 = h_1 E_1(h_2) \circ \varphi_1^{-1}$. Note that C_{φ_2} is not necessarily bounded.

(b) Let $f \in L^2(\Sigma)$. It is easy to check that $\|C_{\varphi_i} f\|_2 = \|M_{\sqrt{h_i}} f\|_2$. So $\mathcal{N}(C_{\varphi_i}) = \mathcal{N}(M_{h_i}) = L^2(A, \Sigma_A, \mu_{|\Sigma_A})$, where $A = X \setminus \sigma(h_i) := (\sigma(h_i))^c$ and $\Sigma_A = \{B \cap A : B \in \Sigma\}$. Thus, C_{φ_i} is injective if and only if $\sigma(h_i) = X$. Now, let $A = \{x \in X : E_1(h_2) = 0\}$. Then $A = \varphi_1^{-1}(B)$, for some $B \in \Sigma$. If $\mu(A) > 0$, then $\mu(B) > 0$ because $\mu \circ \varphi_1^{-1} \ll \mu$. Hence

$$\int_B h_1 E_1(h_2) \circ \varphi_1^{-1}\, d\mu = \int_A E_1(h_2)\, d\mu = 0,$$

and so $h_1 = 0$ or $E_1(h_2) \circ \varphi_1^{-1} = 0$ on B. Therefore, $h_1 > 0$ and $E_1(h_2) \circ \varphi_1^{-1} > 0$ implies that $E_1(h_2) > 0$. Now, let $E_1(h_2) > 0$. Since $E_1(h_2)$ is a $\varphi_1^{-1}(\Sigma)$-measurable, then there exists a unique $g \in L^0(\Sigma)$, with $\sigma(g) \subseteq \sigma(h_1)$, such that $E_1(h_2) = g \circ \varphi_1$ (see [[]Lemma 2]ca). It follows that $0 < \int_X g \circ \varphi_1 d\mu = \int_X h_1 g d\mu$, and so $E_1(h_2) \circ \varphi_1^{-1} = g > 0$ on $\sigma(h_1)$. We conclude that $\sigma(h_3) = X$ if and only if $\sigma(h_1) = \sigma(E_1(h_2)) = X$. ∎

Lemma 2.2. *Let $C_{\varphi_i} \in \mathcal{B}(L^2(\Sigma))$ and $\varphi_3 = \varphi_1 \circ \varphi_2$. Then the following assertions hold.*

(a) $h_3 \circ \varphi_3 = (h_1 \circ \varphi_3) E_1(h_2) \circ \varphi_2$.

(b) $C^*_{\varphi_3}(f) = h_1 E_1(h_2 E_2(f) \circ \varphi_2^{-1}) \circ \varphi_1^{-1}$.

(c) $C^*_{\varphi_3} C_{\varphi_3}(f) = (h_1 E_1(h_2) \circ \varphi_1^{-1}) f$.

(d) $C_{\varphi_3} C^*_{\varphi_3}(f) = (h_1 \circ \varphi_3) E_1(h_2 (E_2 f) \circ \varphi_2^{-1}) \circ \varphi_2$.

(e) $C^*_{\varphi_3} C_{\varphi_3} C_{\varphi_3}(f) = (h_1 E_1(h_2) \circ \varphi_1^{-1}) f \circ \varphi_3$.

(f) $C_{\varphi_3} C^*_{\varphi_3} C_{\varphi_3}(f) = ((h_1 \circ \varphi_3) E_1(h_2) \circ \varphi_2) f \circ \varphi_3$.

Proof. Part (a) follows from Proposition 2.1(a). To prove (b), let $f \in L^2(\Sigma)$. Then

$$C^*_{\varphi_3}(f) = C^*_{\varphi_1}(C^*_{\varphi_2}(f)) = C^*_{\varphi_1}(h_2 E_2(f) \circ \varphi_2^{-1}) = h_1 E_1(h_2 E_2(f) \circ \varphi_2^{-1}) \circ \varphi_1^{-1}.$$

The remainder of the proof is left to the reader. ∎

Let $[T, S] = TS - ST$ for T and S in $\mathcal{B}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $[T, T^*] = 0$, quasinormal if $[T, T^*T] = 0$ and hyponormal if $[T, T^*] \geq 0$.

Lemma 2.3. *Let $C_\varphi \in \mathcal{B}(L^2(\Sigma))$. Then the following assertions hold.*

(a) C_φ *is normal if and only if* $\varphi^{-1}(\Sigma)=\Sigma$ *and* $h=h\circ\varphi$ [22].

(b) C_φ *is quasinormal if and only if* $h=h\circ\varphi$ [22].

(c) C_φ *is hyponormal if and only if* $h>0$ *and* $(h\circ\varphi)E^{\varphi^{-1}(\Sigma)}(\frac{1}{h})\leq 1$ [12].

LEMMA 2.4. *Let* $1\leq i,j\leq 3$, $C_{\varphi_i}\in\mathcal{B}(L^2(\Sigma))$ *and* $\varphi_3=\varphi_1\circ\varphi_2$. *Then* $\sigma(E_i(h_j)\circ\varphi_j)=X$ *and for each* $f\in L^2(\Sigma)$,

$$E_3(f)=\frac{E_1(h_2E_2(f)\circ\varphi_2^{-1})\circ\varphi_2}{E_1(h_2)\circ\varphi_2}.$$

Proof. It is easy to see that $\sigma(E_i(h_j)\circ\varphi_j)=\varphi_j^{-1}\sigma(E_i(h_j))\supseteq\varphi_j^{-1}\sigma(h_j)=\sigma(h_j\circ\varphi_j)=X$. Now, from $C_{\varphi_3}C^*_{\varphi_3}=M_{h_3\circ\varphi_3}E_3$ and Lemma 2.2(c) we obtain

$$E_3(f)=\frac{(h_1\circ\varphi_3)E_1(h_2E_2(f)\circ\varphi_2^{-1})\circ\varphi_2}{h_3\circ\varphi_3}.$$

But, by Lemma 2.2(a) we get that

$$\frac{h_1\circ\varphi_3}{h_3\circ\varphi_3}=\frac{\chi_{(E_1(h_2)\circ\varphi_2)}}{E_1(h_2)\circ\varphi_2}=\frac{1}{E_1(h_2)\circ\varphi_2}.$$

This completes the proof. ■

Assertion (a) of the following proposition is known (see [16]). However for completeness, we provide a new proof.

PROPOSITION 2.5. *Let* $C_{\varphi_i}\in\mathcal{B}(L^2(\Sigma))$ *with* $h_1\circ\varphi_2=h_1$ *and* $h_2\circ\varphi_1=h_2$.

(a) If C_{φ_1} *and* C_{φ_2} *are normal (quasinormal), then* C_{φ_3} *is a normal (quasinormal) operator.*

(b) If C_{φ_1} *and* C_{φ_2} *are hyponormal and* $E_2(h_2)\circ\varphi_2^{-1}$ *is a* $\varphi_1^{-1}(\Sigma)$*-measurable function, then* C_{φ_3} *is a hyponormal operator.*

Proof. (a) Let C_{φ_1} and C_{φ_2} are normal operators. Obviously $\varphi_3^{-1}(\Sigma)=\Sigma$. By hypotheses we get that

$$h_1\circ\varphi_3=(h_1\circ\varphi_1)\circ\varphi_2=h_1\circ\varphi_2=h_1,$$
$$E_1(h_2)\circ\varphi_2=h_2\circ\varphi_2=h_2=E_1(h_2)\circ\varphi_1^{-1}.$$

Now, by Lemma 2.2(a), we get that

$$h_3\circ\varphi_3=(h_1\circ\varphi_3)E_1(h_2)\circ\varphi_2=h_1E_1(h_2)\circ\varphi_1^{-1}=h_3.$$

(b) By hypotheses we have

$$(h_i\circ\varphi_i)E_i(\frac{1}{h_i})\leq 1,\quad h_i>0\quad (i=1,2),$$
$$E_2(h_1)=h_1=E_2(h_1)\circ\varphi_2^{-1},\quad E_1(h_2)=h_2=E_1(h_2)\circ\varphi_1^{-1},$$
$$E_2(\frac{1}{h_2})\circ\varphi_2^{-1}=\frac{1}{E_2(h_2)\circ\varphi_2^{-1}}\in L^0(\varphi_1^{-1}(\Sigma)),$$

and by Proposition 2.1(b), $\sigma(h_3) = X$ because $\sigma(h_1) = X$ and $\sigma(E_1(h_2)) \supseteq \sigma(h_2) = X$. Then we have

$$\begin{aligned}(h_3 \circ \varphi_3)E_3(\frac{1}{h_3}) &= (h_1 \circ \varphi_3)E_1(h_2 E_2(\frac{1}{h_1 E_1(h_2) \circ \varphi_1^{-1}}) \circ \varphi_2^{-1}) \circ \varphi_2 \\ &= (h_1 \circ \varphi_3)(h_2 \circ \varphi_2)E_1(\frac{1}{E_2(h_1) \circ \varphi_2^{-1}} E_2(\frac{1}{E_1(h_2) \circ \varphi_1^{-1}}) \circ \varphi_2^{-1}) \circ \varphi_2 \\ &= (h_1 \circ \varphi_3)(h_2 \circ \varphi_2)E_2(\frac{1}{h_2})E_1(\frac{1}{h_1}) \circ \varphi_2 \\ &= \{(h_2 \circ \varphi_2)E_2(\frac{1}{h_2})\}\{(h_1 \circ \varphi_1)E_1(\frac{1}{h_1})\} \circ \varphi_2 \leq 1. \blacksquare\end{aligned}$$

In [6], Douglas proved that when $A, B \in \mathcal{B}(\mathcal{H})$, then $AA^* \leq \lambda BB^*$ for some $\lambda \geq 0$; if and only if $A = BC$ for some $C \in \mathcal{B}(\mathcal{H})$.

PROPOSITION 2.6. *Let $C_{\varphi_i} \in \mathcal{B}(L^2(\Sigma))$. Then $h_3 \leq \lambda_1 h_1$ and $h_3 \circ \varphi_3 \leq \lambda_2(h_2 \circ \varphi_2)$ for some $\lambda_i \geq 0$.*

Proof. Since $C_{\varphi_3} = C_{\varphi_2} \circ C_{\varphi_1}$, by Douglas' theorem, there exists $\lambda_i \geq 0$ such that $C^*_{\varphi_3} C_{\varphi_3} \leq \lambda_1 C^*_{\varphi_1} C_{\varphi_1}$ and $C_{\varphi_3} C^*_{\varphi_3} \leq \lambda_2 C_{\varphi_2} C^*_{\varphi_2}$. Then for each $f, g \in L^2(\Sigma)$ we have $\langle h_3 f, f\rangle \leq \langle \lambda_1 h_1 f, f\rangle$ and $\langle h_3 \circ \varphi_3 E_3(g), g\rangle \leq \langle \lambda_2(h_2 \circ \varphi_2)E_2(g), g\rangle$. For $A \in \Sigma$, take $f = \chi_A$ and $g = \chi_{\varphi_3^{-1}(A)}$. Since $E_3(g) = g = E_2(g)$, we get that $\int_A h_3 d\mu \leq \int_A \lambda_1 h_1 d\mu$ and $\int_{\varphi_3^{-1}(A)} (h_3 \circ \varphi_3) d\mu \leq \int_{\varphi_3^{-1}(A)} \lambda_2(h_2 \circ \varphi_2) d\mu$. This completes the proof. $\blacksquare$

Write $X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_n, is non-atomic (see [23]). In [4], Chan proved that M_u is compact on $L^2(\Sigma)$ if and only if for any $\varepsilon > 0$, the set $\{x \in X : |u(x)| \geq \varepsilon\}$ consists of finitely many atoms. In the following, we give a sufficient condition for the product of a composition operator C_{φ_1} with the adjoint of a composition operator $C^*_{\varphi_2}$ on $L^2(\Sigma)$ to be compact. The order of the product gives rise to two different cases (see [5] and [21]).

PROPOSITION 2.7. *Let $C_{\varphi_i} \in \mathcal{B}(L^2(\Sigma))$ for $i = 1, 2$. Then the following assertions hold.*

(a) If for each $\varepsilon > 0$, the set $\{x \in X : h_2(\varphi_1(x)) \geq \varepsilon\}$ consists of finitely many atoms, then $C_{\varphi_1} C^*_{\varphi_2}$ is compact.

(b) If for each $\varepsilon > 0$, the set $\{x \in X : h_1(x)(E_1(h_2 \circ \varphi_2) \circ \varphi_1^{-1})(x) \geq \varepsilon\}$ consists of finitely many atoms, then $C^*_{\varphi_2} C_{\varphi_1}$ is compact.

Proof. For $f \in L^2(\Sigma)$ it is seen that

$$C_{\varphi_1} C^*_{\varphi_2}(f) = h_2 \circ \varphi_1(E_2(f) \circ \varphi_2^{-1}) \circ \varphi_1;$$
$$C^*_{\varphi_2} C_{\varphi_1}(f) = h_2 E_2(f \circ \varphi_1) \circ \varphi_2^{-1}.$$

Using change of variable formula and inequality $|E_2(f)|^2 \leq E_2(|f|^2)$, we obtain

$$\|C_{\varphi_1} C^*_{\varphi_2}(f)\|^2 = \int_X h_2^2 \circ \varphi_1 |E_2(f) \circ \varphi_2^{-1}|^2 \circ \varphi_1 \, d\mu = \int_X h_1 h_2^2 |E_2(f) \circ \varphi_2^{-1}|^2 \, d\mu$$

$$= \int_X h_1 h_2 |E_2(f)|^2 \circ \varphi_2^{-1}\, d\mu \circ \varphi_2^{-1} = \int_X (h_1 \circ \varphi_2)(h_2 \circ \varphi_2)|E_2(f)|^2\, d\mu$$
$$\leq \int_X (h_1 \circ \varphi_2)(h_2 \circ \varphi_2) E_2(|f|^2)\, d\mu = \|M_{\sqrt{(h_1 \circ \varphi_2)(h_2 \circ \varphi_2)}} f\|^2.$$

Similarly,

$$\begin{aligned}\|C^*_{\varphi_2} C_{\varphi_1}(f)\|^2 &= \int_X h_2^2 |E_2(f \circ \varphi_1) \circ \varphi_2^{-1}|^2\, d\mu = \int_X h_2 |E_2(f \circ \varphi_1)|^2 \circ \varphi_2^{-1}\, d\mu \circ \varphi_2^{-1} \\ &= \int_X (h_2 \circ \varphi_2)|E_2(f \circ \varphi_1)|^2\, d\mu \leq \int_X (h_2 \circ \varphi_2) E_2(|f|^2 \circ \varphi_1)\, d\mu \\ &= \int_X (h_2 \circ \varphi_2)|f|^2 \circ \varphi_1\, d\mu = \int_X h_1 E_1(h_2 \circ \varphi_2) \circ \varphi_1^{-1} |f|^2\, d\mu \\ &= \|M_{\sqrt{h_1 E_1(h_2 \circ \varphi_2) \circ \varphi_1^{-1}}} f\|^2.\end{aligned}$$

Now, the desired conclusions follows from the compactness criteria for multiplication operators. ■

Proposition 2.8. *Let $C_{\varphi_i} \in \mathcal{B}(L^2(\Sigma))$ for $i = 1, 2$. Then the following assertions hold.*

(a) If h_3 is bounded away from zero on $\sigma(h_3)$ and $\sigma(E_1(h_2) \circ \varphi_1^{-1}) = X$, then $\mathcal{R}(C_{\varphi_1})$ is closed.

(b) Let C_{φ_1} and C_{φ_2} have closed range. If $\sigma(h_2) = X$ or $\sigma(h_2)^c = A_{i_1} \cup \cdots \cup A_{i_k}$, then C_{φ_3} has closed range.

Proof. (a) Let $f \in L^2(\Sigma)$. Then $\|M_{\sqrt{h_3}} f\|_2 \leq \|C_{\varphi_2}\|\ \|M_{\sqrt{h_1}} f\|_2$. Recall that for $u \in L^\infty(\Sigma)$, $\mathcal{R}(M_u)$ is closed in $L^2(\Sigma)$ if and only if u is bounded away from zero on $\sigma(u)$ [18]. Thus there exists $\lambda \geq 0$ such that $\lambda \|f\|_2 \leq \|M_{\sqrt{h_1}} f\|_2$ for each $f \in L^2(\sigma(h_3))$. Since $\sigma(E_1(h_2) \circ \varphi_1^{-1}) = X$, then $\sigma(h_1) = \sigma(h_3)$, and so $\mathcal{R}(C_{\varphi_1})$ is closed

(b) It is a classical fact that C_{φ_3} has closed range if and only if $\mathcal{N}(C_{\varphi_2}) + L^2(\varphi_1^{-1}(\Sigma))$ is closed (see [15, Corollary 1]. Now, by assumptions, $\mathcal{N}(C_{\varphi_2}) = \{0\}$ or $\mathcal{N}(C_{\varphi_2})$ is a finite dimensional subspace of $L^2(\Sigma)$, and hence C_{φ_3} has closed range. ■

3. Complex symmetric, semi-Kato type and polar decomposition of composition operators

A conjugation on a Hilbert space $\mathcal{H}$ is an anti-linear operator $S : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Sx, Sy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $S^2 = I$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be complex symmetric if there exists a conjugation S on $\mathcal{H}$ such that $T = ST^*S$. The class of complex symmetric operators is unexpectedly large. We refer the reader to [7, 10] for more details, including historical comments and references.

Theorem 3.1. *Let $\varphi^2 = id$, the identity transformation, and let $0 < h \in L^\infty(\varphi^{-1}(\Sigma))$. Then the operator C_φ is complex symmetric.*

Proof. Put $E = E^{\varphi^{-1}(\Sigma)}$ and define $S(f) := \frac{\bar{f}\circ\varphi}{\sqrt{h\circ\varphi}}$. Then for each $f \in L^2(\Sigma)$ we have

$$SC_\varphi^* S(f) = S(hE(S(f))\varphi^{-1}) = \frac{(h\circ\varphi)E(S(\bar{f}))}{\sqrt{h\circ\varphi}} = \sqrt{h\circ\varphi}\, S(\bar{f}) = C_\varphi(f),$$

and

$$\langle S(f), S(g)\rangle = \int_X \frac{(\bar{f}\circ\varphi)(g\circ\varphi)}{h\circ\varphi}\, d\mu = \int_X \frac{h\bar{f}g}{h}\, d\mu = \langle g, f\rangle.$$

It follows that C_φ is a complex symmetric operator if and only if

$$S^2(f) = \frac{f\circ\varphi^2}{\sqrt{(h\circ\varphi)(h\circ\varphi^2)}} = f, \qquad f \in L^2(\Sigma).$$

Since by hypotheses $\varphi^2 = id$ and $E(h) = h$, then

$$1 = \frac{d\mu\circ\varphi^{-2}}{d\mu} = hE(h)\circ\varphi^{-1} = h(h\circ\varphi^{-1}).$$

Thus, $(h\circ\varphi^2)(h\circ\varphi) = 1$ and so $S^2 = I$. ∎

Corollary 3.2. *Let φ be a measure preserving transformation, i.e., $\mu(\varphi^{-1}(A)) = \mu(A)$ for all $A \in \Sigma$. Then C_φ is a complex symmetric operator on $L^2(\Sigma)$ if and only if $C_\varphi^2 = I$, the identity operator.*

In [11], the authors obtain some necessary conditions for C_φ and C_φ^* acting on H^2 for which $C_\varphi^* C_\varphi$ and $C_\varphi + C_\varphi^*$ commute. They prove that if φ is an automorphism of the open unit disk $\mathbb{D}$, then $[C_\varphi^* C_\varphi, C_\varphi + C_\varphi^*] = 0$ if and only if C_φ is normal. In the following we obtain a similar result for $C_\varphi \in \mathcal{B}(L^2(\Sigma))$.

Proposition 3.3. *Let $C_\varphi \in \mathcal{B}(L^2(\Sigma))$. If C_φ is quasinormal, then $[C_\varphi^* C_\varphi, C_\varphi + C_\varphi^*] = 0$.*

Proof. Let $f \in L^2(\Sigma)$. Since $h = h\circ\varphi$, then we have

$$\begin{aligned}(C_\varphi + C_\varphi^*)(C_\varphi^* C_\varphi(f)) &= (h\circ\varphi)(f\circ\varphi) + hE(hf)\circ\varphi^{-1} \\ &= h(f\circ\varphi) + hE((h\circ\varphi)f)\circ\varphi^{-1} = h(f\circ\varphi) + h^2E(f)\circ\varphi^{-1} \\ &= C_\varphi^* C_\varphi(C_\varphi + C_\varphi^*)(f),\end{aligned}$$

and so $[C_\varphi^* C_\varphi, C_\varphi + C_\varphi^*](f) = 0$. ∎

Definition 3.4. We say that $A \in \mathcal{B}(\mathcal{H})$ is an operator of semi-Kato type, if the null space of A is contained in $\cap_{n=1}^\infty \overline{\mathcal{R}(A^n)}$.

Recall that $A \in \mathcal{B}(\mathcal{H})$ is said to be Kato if $\mathcal{R}(A)$ is closed and $\mathcal{N}(A) \subseteq \bigcap_{n=1}^\infty \mathcal{R}(A^n)$. Any bounded operator that is either onto or bounded below is Kato (see [14]). The set of all semi-Kato and Kato type operators will be denoted by $\mathcal{SK}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ respectively. Obviously, $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{SK}(\mathcal{H})$. Also, if $A \in \mathcal{SK}(\mathcal{H})$ and for each $n \in \mathbb{N}$, A^n has closed range, then $A \in \mathcal{K}(\mathcal{H})$.

Theorem 3.5. *For $i = 1, 2, 3$, put $\Sigma_\infty^i = \bigcap_{n=1}^\infty \varphi_i^{-n}(\Sigma)$ and let $C_{\varphi_i} \in \mathcal{B}(L^2(\Sigma))$. Then the following assertions hold.*

(a) If $C_{\varphi_i} \in \mathcal{SK}(L^2(\Sigma))$ if and only if $\Sigma \cap (\sigma(h_i))^c \subseteq \Sigma_\infty^i$.

(b) $C_{\varphi_i} \in \mathcal{K}(L^2(\Sigma))$ if and only if, for each $n \in \mathbb{N}$, $h_{i,n}$ is bounded away from zero on $\sigma(h_{i,n})$ and $\Sigma \cap (\sigma(h_i))^c \subseteq \Sigma_\infty^i$.

(c) If $C_{\varphi_i} \in \mathcal{SK}(L^2(\Sigma))$, $\sigma(E_1(h_2) \circ \varphi_1^{-1}) = \sigma(E_1(h_2))$ and $\Sigma_\infty^1 \cup \Sigma_\infty^2 \subseteq \Sigma_\infty^3$, then $C_{\varphi_3} \in \mathcal{SK}(L^2(\Sigma))$.

(d) If $C_{\varphi_3} \in \mathcal{SK}(L^2(\Sigma))$ and $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, then $C_{\varphi_1} \in \mathcal{SK}(L^2(\Sigma))$.

Proof. (a) Note that $C_{\varphi_i} \in \mathcal{SK}(L^2(\Sigma))$, if $\mathcal{N}(C_{\varphi_i}) \subseteq \bigcap_{n=1}^\infty \overline{\mathcal{R}(C_{\varphi_i^n})}$. Since $\overline{\mathcal{R}(C_{\varphi_i^n})} = L^2(\varphi_i^{-n}(\Sigma))$ and $\mathcal{N}(C_{\varphi_i}) = L^2(\Sigma \cap (\sigma(h_i))^c)$, it follows that if $C_{\varphi_i} \in \mathcal{SK}(L^2(\Sigma))$ then $L^2(\Sigma \cap (\sigma(h_i))^c) \subseteq \bigcap_{n=1}^\infty L^2(\varphi_i^{-n}(\Sigma)) = L^2(\bigcap_{n=1}^\infty \varphi_i^{-n}(\Sigma)) = L^2(\Sigma_\infty^i)$, and so $\Sigma \cap (\sigma(h_i))^c \subseteq \Sigma_\infty^i$. Conversely, if $\Sigma \cap (\sigma(h_i))^c \subseteq \Sigma_\infty^i$ then $\mathcal{N}(C_{\varphi_i}) = L^2(\Sigma \cap (\sigma(h_i))^c) \subseteq L^2(\Sigma_\infty^i) = \bigcap_{n=1}^\infty L^2(\varphi_i^{-n}(\Sigma)) = \bigcap_{n=1}^\infty \overline{\mathcal{R}(C_{\varphi_i}^n)}$.

(b) Let $C_{\varphi_i} \in \mathcal{K}(L^2(\Sigma))$. Then for each $n \in \mathbb{N}$, $\mathcal{R}(C_{\varphi_i}^n)$ is closed and so $h_{i,n} := d\mu \circ \varphi_i^{-n}/d\mu$ is bounded away from zero on $\sigma(h_{i,n})$. Also, we have $L^2(\Sigma \cap (\sigma(h_i))^c) = \mathcal{N}(C_{\varphi_i}) \subseteq \bigcap_{n=1}^\infty \overline{\mathcal{R}(C_{\varphi_i}^n)} = L^2(\Sigma_\infty^i)$. On the other hand, if for each $n \in \mathbb{N}$, $h_{i,n}$ is bounded away from zero on its support and $\Sigma \cap (\sigma(h_i))^c \subseteq \Sigma_\infty^i$, then $\overline{\mathcal{R}(C_{\varphi_i}^n)} = \mathcal{R}(C_{\varphi_i}^n)$ and $\mathcal{N}(C_{\varphi_i}) \subseteq \bigcap_{n=1}^\infty \mathcal{R}(C_{\varphi_i}^n)$.

(c) For $i = 1, 2$, let $C_{\varphi_i} \in \mathcal{SK}(L^2(\Sigma))$ and $\sigma(E_1(h_2) \circ \varphi_1^{-1}) = \sigma(E_1(h_2))$. Then we have $(\sigma(h_3))^c \cap \Sigma = \{\sigma(h_1) \cap \sigma(E_1(h_2))\}^c \cap \Sigma \subseteq \{(\sigma(h_1))^c \cup (\sigma(h_2))^c\} \cap \Sigma \subseteq \Sigma_\infty^1 \cup \Sigma_\infty^2 \subseteq \Sigma_\infty^3$. hence by (a), $C_{\varphi_3} \in \mathcal{SK}(L^2(\Sigma))$. Note that $\Sigma_\infty^3 \subseteq \Sigma_\infty^1 \cup \Sigma_\infty^2$, because, in general, $\Sigma_\infty^3 \subseteq \Sigma_\infty^2$.

(d) If $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, then $\Sigma_\infty^3 \subseteq \Sigma_\infty^1$. Hence if $C_{\varphi_3} \in \mathcal{SK}(L^2(\Sigma))$ and $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, then $C_{\varphi_1} \in \mathcal{SK}(L^2(\Sigma))$ because $\sigma(h_3) \subseteq \sigma(h_1)$. ■

We recall that every operator $T \in \mathcal{B}(\mathcal{H})$ can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$, then this decomposition is called the polar decomposition. Notice that the parts of the polar decomposition V, $|W|$ for $W = M_u \circ C_\varphi$ are given by

$$V = M_{\frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J\circ\varphi}}} W, \qquad |W| = M_{\sqrt{J}},$$

where $J = hE^{\varphi^{-1}(\Sigma)}(|u|^2) \circ \varphi^{-1}$ (see [3]). Hence by Proposition 2.1(a) the parts of the polar decomposition $C_{\varphi_3} = U_{\varphi_3}|C_{\varphi_3}|$ are given by

$$|C_{\varphi_3}|(f) = \sqrt{h_1 E_1(h_2) \circ \varphi_1^{-1}}\ f,$$

$$U_{\varphi_3}(f) = \frac{f \circ \varphi_3}{\sqrt{(h_1 \circ \varphi_3) E_1(h_2) \circ \varphi_2}}.$$

Since $h_3 \circ \varphi_3 > 0$, then $\mathcal{N}(U_{\varphi_3}) = \mathcal{N}(C_{\varphi_3}) = L^2((\sigma(h_3))^c) = \mathcal{N}(|C_{\varphi_3}|)$. It is easy to see that $U_{\varphi_3} U_{\varphi_3}^* U_{\varphi_3} = U_{\varphi_3}$, and so U_{φ_3} is a partial isometry. Also, the second part of $C^*_{\varphi_3} = U_{\varphi_3}|C^*_{\varphi_3}|$ is given by

$$|C^*_{\varphi_3}|(f) = \sqrt{(h_1 \circ \varphi_3)E_1(h_2) \circ \varphi_2}\ E_3(f).$$

Recall that the Aluthge transform of $T = U|T| \in \mathcal{B}(\mathcal{H})$ is the operator $\tilde{T}$ given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. For $0 < r \leq 1$, put $T_r := |T|^r U|T|^{1-r}$ (see [1]). Then $T_{\frac{1}{2}} = \tilde{T}$. Now, take $T = C_{\varphi_3}$. Then we have

$$T_r f = |T|^r U(h_3^{\frac{1-r}{2}} f) = |T|^r ((\frac{1}{h_3 \circ \varphi_3})^{\frac{r}{2}} f \circ \varphi_3) = (\frac{h_3}{h_3 \circ \varphi_3})^{\frac{r}{2}} f \circ \varphi_3.$$

Put $\omega_r := (\frac{h_3}{h_3 \circ \varphi_3})^{r/2}$. Then $T_r f = \omega_r.f \circ \varphi_3$ is a weighted composition operator. The parts of the polar decomposition V_r, $|T_r|$ for T_r are given by

$$|T_r|(f) = \sqrt{h_3 E_3(\omega_r^2) \circ \varphi_3^{-1}}\ f;$$
$$V_r f = \frac{\chi_{\sigma(E_3(\omega_r^2))} T_r f}{\sqrt{(h_3 \circ \varphi_3)E_3(\omega_r^2)}}.$$

Example 3.6. Let $X = \mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$ and let μ be the counting measure. Define

$$\varphi_1(n) = \begin{cases} 1, & n = 1, 2 \\ n - 1, & n \geq 3, \end{cases}$$

and $\varphi_2(n) = n + 1$. Then

$$h_1(n) = \mu(\varphi_1^{-1}(n)) = \begin{cases} 2, & n = 1 \\ 1, & n \geq 2; \end{cases} \qquad h_2(n) = \mu(\varphi_2^{-1}(n)) = \begin{cases} 0, & n = 1 \\ 1, & n \geq 2; \end{cases}$$
$$E_1(h_2)(n) = \begin{cases} \frac{1}{2}, & n = 1, 2 \\ 1, & n \geq 3; \end{cases} \qquad h_3(n) = \sum_{k \in \varphi_1^{-1}(n)} h_2(k) = 1.$$

It follows that the injectivity condition for C_{φ_2} in Proposition 2.1 is not necessary. Moreover, since φ_3 is the identity function, by Corollary 3.2, C_{φ_3} is a complex symmetric operator but neither C_{φ_1} nor C_{φ_2} is a complex symmetric operator.

Example 3.7. (a) Let $X = [0, 1]$, $d\mu = dx$ and Σ be the Lebesgue sets. Define the non-singular transformations $\varphi_i : X \to X$ by

$$\varphi_1(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2x - 1, & x \in (\frac{1}{2}, 1], \end{cases} \qquad f_2(x) = \begin{cases} 1 - 2x, & x \in [0, \frac{1}{2}] \\ 2x - 1, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then

$$\varphi_3(x) = \begin{cases} 1 - 4x, & x \in [0, \frac{1}{4}) \\ 2 - 4x, & x \in [\frac{1}{4}, \frac{1}{2}) \\ 4x - 2, & x \in [\frac{1}{2}, \frac{3}{4}) \\ 4x - 3, & x \in [\frac{3}{4}, 1], \end{cases}$$

and so $h_1(x) = h_2(x) = h_3(x) = 1$. Moreover, we can obtain from direct computations that

$$E_1(f)(x) = \frac{1}{2}\{f(x) + f(\frac{1+2x}{2})\}\chi_{[0,\frac{1}{2}]} + \frac{1}{2}\{f(\frac{2x-1}{2}) + f(x)\}\chi_{(\frac{1}{2},1]},$$
$$E_2(f)(x) = \frac{1}{2}\{f(x) + f(1-x)\},$$

and

$$|C^*_{\varphi_3}|(f) = E_3(f)(x) = \frac{1}{2}\begin{cases} f(\frac{1-4x}{2}) + f(1-2x), & x \in [0, \frac{1}{4}) \\ f(1-2x) + f(\frac{3-4x}{2}), & x \in [\frac{1}{4}, \frac{1}{2}) \\ f(2x-1) + f(\frac{4x-1}{2}), & x \in [\frac{1}{2}, \frac{3}{4}) \\ f(\frac{4x-3}{2}) + f(2x-1), & x \in [\frac{3}{4}, 1], \end{cases}$$

for all $f \in L^2(\Sigma)$. Note that for each $i = 1, 2, 3$ and $0 < r \leq 1$, $(C_{\varphi_i})_r = C_{\varphi_i} = \widetilde{C_{\varphi_i}}$. Furthermore, C_{φ_i} is injective, quasinormal and has closed range but not normal operator.

(b) Let $X = \mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$ and let $\mu(\{n\}) = m_n$, where $w := \{m_n\}_n \subset (0, \infty)$. Let $f = \{f_n\} \in l^2(w)$. Then for $i = 1, 2$ we have

$$h_i(k) = \frac{1}{m_k}\int_{\{k\}} h_i \, d\mu = \frac{1}{m_k}\int_{\varphi_i^{-1}(\{k\})} d\mu = \frac{1}{m_k}\sum_{j\in\varphi_i^{-1}(\{k\})} m_j,$$
$$(h_i E_i(f) \circ \varphi_i^{-1})(k) = \frac{1}{m_k}\int_{\varphi_i^{-1}(\{k\})} E_i(f)\, d\mu = \frac{1}{m_k}\sum_{j\in\varphi_i^{-1}(\{k\})} f_j m_j.$$

Now, by these computations we obtain

$$h_3(k) = \frac{1}{m_k}\sum_{j\in\varphi_1^{-1}(\{k\})} h_2(j) m_j = \frac{1}{m_k}\sum_{j\in\varphi_1^{-1}(\{k\})}\sum_{l\in\varphi_2^{-1}(\{j\})} m_l.$$

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