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THE COMPUTER MODELLING OF GLUING FLAT IMAGES ALGORITHMS

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Abstract. In this paper one of the important tasks of modern computer geometry is considered: creating effective algorithms for gluing different flat images of the same object. Images are obtained by central projection from different points of view. We use numerical simulation for comparison of three known algorithms for gluing—simple linear algorithm, normalized linear algorithm and direct algorithm. In each case stability to perturbations of the initial data and speed of calculations were estimated.

The results confirm hypothesis of G.V. Nosovskií and E.S. Skripka that the direct algorithm proposed in their work [Error estimation for the direct algorithm of projective mapping calculation in multiple view geometry, Proceedings of the Conference "Contemporary Geometry and Related Topics", Belgrade, Serbia-Montenegro, June 26–July 2, 2005, Faculty of Mathematics, University of Belgrade, 2006, pp. 399–408] is the most accurate and fast one.

1. Introduction

In recent years, the field of application of digital image processing has expanded considerably. Image analysis is used in the research, industry, medicine, space research and information systems.

In this paper, problem of constructing algorithms of gluing flat images of the same object is considered. Several images of the same object are obtained as central projections from the different viewing points (Figure 1).

We assume that the object is approximately flat with the respect to distances between the object and the viewing points. From the geometric point of view the problem can be formulated as a problem of finding the projective mapping F which bounds two domains $D1$ and $D2$ placed in the same affine coordinate map of a projective plane RP^2 (Figure 2).

In order to solve this problem it's necessary:

- To recognize enough quantity of pairs of points which reflect the same point on the image so called *conjugate points*;

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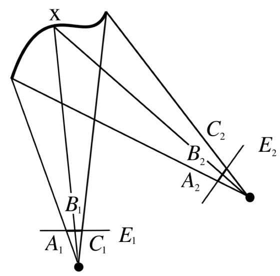


Fig. 1. Multiple images of the same object

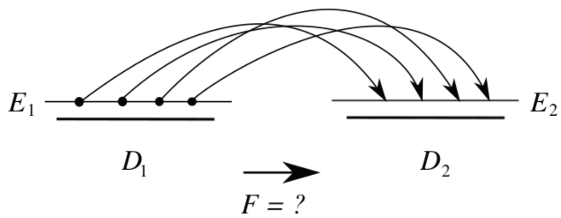


Fig. 2. Projective mapping F of two domains $D1$ and $D2$

- To create a robust algorithm for projective mapping calculation and to estimate its accuracy on the perturbed initial data.

In this paper, the second step is considered, i.e., it is assumed that some number of conjugate points have already been found with some error. The problem is to find the most robust algorithm for calculation the projective mapping. The following algorithms are analyzed:

- The simple linear algorithm.
- The normalized linear algorithm.
- The direct algorithm.

The simple and normalized linear algorithms are well-known algorithms which are extensively used in medicine, space research and robotics. The direct algorithm is a new algorithm which was proposed by Nosovskií and Skripka in [3].

In the paper of Nosovskií and Tolchennikov [2], it was shown that commonly used linear algorithms have a significant disadvantage: unstable work in the case of inaccurately found coordinates of conjugate points. Nosovskií and Skripka [3] formed the hypothesis that the direct algorithm is faster and more robust comparing to the known linear algorithms. In this paper, the results of the computer modelling confirm the hypothesis of Nosovskií and Skripka that their proposed direct algorithm is the most accurate and fast to perturbation coordinates of conjugate points.

2. Theoretical analysis of algorithms

We describe the considered algorithms of projective mapping calculation and define the fixed matrix representation for the projective mapping.

First of all, recall that the projective mapping F is fully determined by any 4 points $X_i \in RP^2$, $1 \le i \le 4$, situated in general position and their images $X_i' \in RP^2$, $1 \le i \le 4$, also situated in general position:

$$F(X_i) = X_i', \text{ where } 1 \le i \le 4,$$

and F is represented by a set of unknown variables (f_{jk}) $(1 \le j, k \le 3)$ which are organized in a square 3×3 matrix F representing a linear operator $R^3 \to R^3$ which corresponds to the mapping F.

We will assume that all 8 points X_i and X_i' belong to the affine chart $S_3 = \{(x_1, x_2, x_3) : x_3 \ne 0\} \subset RP^2$ and their coordinates are represented by the following three-dimensional vectors:

$$X_i = (x_i, y_i, z_i), \ 1 \le i \le 4,$$
$$X_i' = (x_i', y_i', z_i'), \ 1 \le i \le 4.$$

2.1. The simple linear algorithm

The equation $F(X_i) = X_i'$ is equivalent to $F(X_i) \times X_i' = 0$ in homogeneous coordinates. Let us denote the rows with index j as $(f^j)^T$. In this denotations:

$$F(X_i) \times X_i' = \begin{pmatrix} 0^T & -z_i' X_i^T & y_i' X_i^T \\ z_i' X_i^T & 0^T & -x_i' X_i^T \\ -y_i' X_i^T & x_i' X_i^T & 0^T \end{pmatrix} = C_i f \ \ (1 \le i \le 4).$$

The third row of matrix C_i is a linear combination of its first and second row. For projective mapping calculation we solve system of linear equations $Af = 0$ where A is a 8×9 matrix combined from 2×9 matrices A_i without the third row of matrix C_i:

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}; \quad A_i = \begin{pmatrix} 0^T & -z_i' X_i^T & y_i' X_i^T \\ z_i' X_i^T & 0^T & -x_i' X_i^T \end{pmatrix}.$$

Now, instead of the true points $X_i \in RP^2$, $1 \le i \le 4$ and $X_i' \in RP^2$, $1 \le i \le 4$, perturbed sets of points $\tilde{X}_i \in RP^2$, $1 \le i \le 4$ and $\tilde{X}'_i \in RP^2$, $1 \le i \le 4$ are given:

$$\tilde{X}_i = (\tilde{x}_i, \tilde{y}_i, \tilde{z}_i) = (x_i + \triangle x_i, y_i + \triangle y_i, z_i) \ \ 1 \le i \le 4,$$
$$\tilde{X}'_i = (\tilde{x}'_i, \tilde{y}'_i, \tilde{z}'_i) = (x_i' + \triangle x_i', y_i' + \triangle y_i', z_i') \ \ 1 \le i \le 4.$$

It is known that the points X_i and X_i' are linked by a projective mapping F. In order to calculate unknown projective mapping $\tilde{F}$ by the simple linear algorithm we do the following steps:

- According to the correspondences $\tilde{X}_i \leftrightarrow \tilde{X}'_i$, we build the matrices $\tilde{A}_i$ and $\tilde{A}$ using the above formulas.
- Calculate a vector f, at which the minimum of function $||\tilde{A}f||$ is reached with the condition $||f|| = 1$.
- Denote $\tilde{F} = \begin{pmatrix} \tilde{f}_1 & \tilde{f}_2 & \tilde{f}_3 \\ \tilde{f}_4 & \tilde{f}_5 & \tilde{f}_6 \\ \tilde{f}_7 & \tilde{f}_8 & \tilde{f}_9 \end{pmatrix}$.

Let us formulate an important lemma:

LEMMA 1. *Let A be a matrix of size $m \times n$ and let g be a column vector of size $n \times 1$. Assume that the minimum norm of $\|Ag\|$, provided $\|g\| = 1$, is reached on the vector g (where $\|\cdot\|$ is the Euclidean vector norm). Then g is the normalized eigenvector of the matrix $A^T A$, corresponding to its minimum eigenvalue.*

Proof. Contained in [2]. ■

The idea of simple linear algorithm is to find the normalized eigenvector of a square matrix of special type of size 9×9 corresponding to its minimum eigenvalue.

2.2. The normalized linear algorithm

In order to calculate unknown projective mapping $\tilde{F}$ by the normalized linear algorithm we do following steps:

- Let T_0 be a mapping such that the mass center of points $T_0\tilde{X}_i$ is at the origin, and the average distance from these points to the origin is equal to $\sqrt{2}$.
- Let T'_0 be a mapping such that for $T'_0\tilde{X}'_i$ points the same conditions are fulfilled.
- Run SLA for $T_0\tilde{X}_i \leftrightarrow T'_0\tilde{X}'_i$ for calculation of the mapping $\tilde{F}_0$.
- Denote $\tilde{F} = (T'_0)^{-1}\, \tilde{F}_0\, T_0$.

The simple linear algorithm is not invariant with respect to the plane motion. The normalized linear algorithm is invariant with the respect to the plane motion, which is the main difference between these algorithms.

2.3. The direct algorithm

We assume that unknown projective mapping F maps 4 points $P, Q, R, T \in RP^2$ situated in general position onto 4 points $P', Q', R', T' \in RP^2$ also situated in general position, respectively:

$$F(P) = P'; \;\; F(Q) = Q'; \;\; F(R) = R'; \;\; F(T) = T'.$$

The mapping F can be represented by a set of unknown variables (f_{ij}) $(1 \le i,\, j \le 3)$ which are organized in a square 3×3 matrix F of the linear operator $R^3 \to R^3$ which corresponds to the mapping F. We assume that all 8 points P, Q, R, T and P', Q', R', T' belong to the affine chart $S_3 = \{(x_1, x_2, x_3) : x_3 = 1\} \subset RP^2$ and their coordinates are represented by the following three-dimensional vectors:

$$P = (p_1, p_2, 1), \;\; Q = (q_1, q_2, 1), \;\; R = (r_1, r_2, 1), \;\; T = (t_1, t_2, 1),$$
$$P' = (p'_1, p'_2, 1), \;\; Q' = (q'_1, q'_2, q'_3), \;\; R' = (r'_1, r'_2, r'_3), \;\; T' = (t'_1, t'_2, t'_3).$$

The third coordinate of the first four conjugate points P, Q, R, T is equal to 1 due to the fact that S_3 is an affine chart. In order to obtain the unique presentation of the projective mapping by a matrix, we assume that the third coordinate of one of the second four conjugate points P', Q', R', T' is also equal to 1 (without loss of generality, $p'_3 = 1$). The other third coordinates of points Q', R', T' are arbitrary (in this case, they are denoted by q'_3, r'_3, t'_3).

We use the following notations:

$$a_p = p'_1,\ \ b_p = p'_2,\ \ a_q = \frac{q'_1}{q'_3},\ \ b_q = \frac{q'_2}{q'_3},$$
$$a_r = \frac{r'_1}{r'_3},\ \ b_r = \frac{r'_2}{r'_3},\ \ a_t = \frac{t'_1}{t'_3},\ \ b_t = \frac{t'_2}{t'_3}.$$

We consider the system of linear equations $Ax = y$, where x and y are vectors formed by the unknown variables:

$$x = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}, q'_3, r'_3, t'_3)^T,$$
$$y = (a_p, 0, 0, 0, b_p, 0, 0, 0, 1, 0, 0, 0)^T,$$

where A is given by:

$$A = \begin{pmatrix}
p_1 & p_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_1 & q_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -a_q & 0 & 0 \\
r_1 & r_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_r & 0 \\
t_1 & t_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_t \\
0 & 0 & 0 & p_1 & p_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q_1 & q_2 & 1 & 0 & 0 & 0 & -b_q & 0 & 0 \\
0 & 0 & 0 & r_1 & r_2 & 1 & 0 & 0 & 0 & 0 & -b_r & 0 \\
0 & 0 & 0 & t_1 & t_2 & 1 & 0 & 0 & 0 & 0 & 0 & -b_t \\
0 & 0 & 0 & 0 & 0 & 0 & p_1 & p_2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_1 & q_2 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & r_1 & r_2 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_1 & t_2 & 1 & 0 & 0 & -1
\end{pmatrix}$$

The system $Ax = y$ consists of 12 linear equations with 12 unknown variables: 9 elements f_{ij} of the projective mapping and 3 unknown point coordinates q'_3, r'_3, t'_3 (note that the matrix A contains known values only).

If the points $\{P, Q, R, T\}$ and $\{P', Q', R', T'\}$ in both sets are situated in the general position then the matrix A is non-degenerate. After perturbation of conjugate points we get a new system:

$$(A + \triangle A)(x + \triangle x) = (y + \triangle y)$$

We assume that the configuration of conjugate points was chosen in such a way that the determinant of perturbed matrix $A + \triangle A$ still differs from zero.

3. Computational analysis of algorithms

In the simple linear algorithm the eigenvalues of a square symmetric matrix are calculated. To do so, using rotation, the matrix is reduced to the almost triangular form. The QR algorithm for finding eigenvalues for almost triangular matrix is applied [1]. The convergence of the QR algorithm is accelerated by means of translations [1].

In the normalized linear algorithm the normalization of conjugate points is required. In the last step of the algorithm a matrix inversion is applied.

In the direct algorithm the matrix equation solution of type $Ax = y$ is required. The improved Gauss method for matrices with a large number of zero elements is applied.

4. The model of experiment

We fix an orthonormal basis $\{e_1, e_2, e_3\}$ of Euclidean space R^3. We consider the projective plane RP^2, the points of which are direct from R^3. Let us fix an affine chart $S_3 = \{(x_1, x_2, x_3) : x_3 = 1\} \subset RP^2$ with a basis $\{e_1, e_2\}$. Without loss of generality, we assume that the fields relating the projective mapping lie in S_3 map.

We introduce scale units. Let us assume that the precise coordinates of the first 4 conjugate points are located on the first screen with length and width equal to 1. The center of the screen coincides with the origin $(0, 0)$. We define a matrix of a projective mapping up to the scalar multiplication factor.

The experiment starts with the generation of the exact coordinates on the first screen. Under the influence of a given exact projective mapping matrix, we get new coordinates on the second screen.

After the preparatory phase the coordinates perturbations are generated. In our experiment these perturbations are independent normally distributed random variables with zero mean and variance σ^2, where $3\sigma = \epsilon$, where ϵ characterizes the error budget. We perturb only the first two coordinates of conjugate points, as the third coordinate of each point is equal to 1 (for the first four due to the affine chart, for a second four due to normalization of the third coordinate). After that the projective mapping matrix is calculated for each algorithm.

The sample of $N = 10^6$ eights perturbed coordinates of conjugate points is generated. The accuracy of projective mapping, characterized by the deviation from the mean matrix

$$\bar{F} = \frac{1}{n} \sum_{i=1}^{n} F_i$$

and

$$S_n^2 = \frac{1}{n} \sum_{i=1}^{n} \|F_i - \bar{F}\|^2,$$

is being estimated. The speed of the algorithms is being calculated.

We use different types of precision given matrices of size 3×3. The following cases have been considered:

- Plane motion case.

This case includes the following matrices: rotation matrices, translation matrices and composition of rotation and translation matrices.

- Perspective distortion case.
- Arbitrary case.

5. Results

The following conclusions were obtained:

- The direct algorithm shows better speed of work than the linear algorithms.

Several hundreds of experiments were performed. All experiments showed that direct algorithm is faster than linear algorithms. As a typical example, to present the results for the randomly generated matrix of arbitrary type (Figure 3)[1].

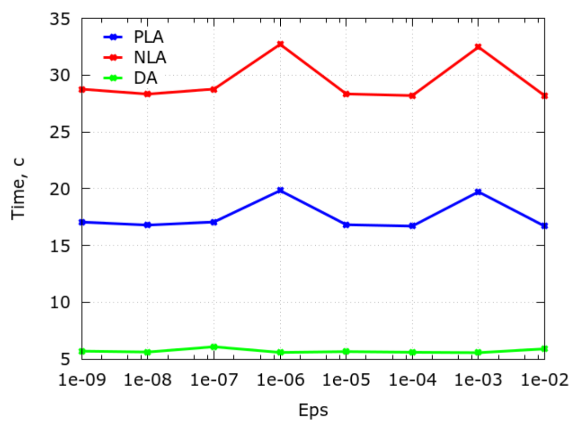


Fig. 3. Polygons of the elapsed algorithms time to compute $\tilde{F}$ (vertical axis) of error budget (horizontal axis). Red polygon represents the elapsed time of normalized linear algorithm (the slowest algorithm: in five times slower than direct algorithm), blue polygon paints the elapsed time of simple linear algorithm (in three times slower than direct algorithm), and green polygon depicts the elapsed time of direct algorithm (the fastest algorithm).

- The hypothesis of Nosovskií and Skripka that their proposed direct algorithm is the most accurate to perturbation coordinates of conjugate points has been confirmed.

Plane motion case

There were many experiments for the three different types of projective mapping matrices:

- Rotation matrices.

[1]The colours can be seen in the electronic versions of this paper on the sites of Matematički Vesnik.

- Translation matrices.
- Composition of rotation and translation matrices.

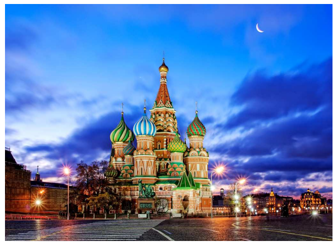
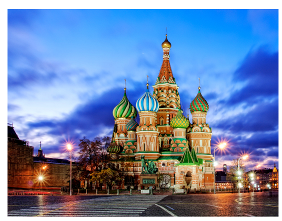


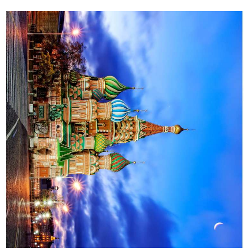
Fig. 4. Arbitrary image

Below the results are presented for all three types (see Figures 4 and 5). The results of algorithms are represented by Figure 6. For each algorithm accuracy is represented as a strip, the lower limit of which is characterized by the deviation from $\tilde{F}$ (color: simple linear algorithm is orange, normalized linear algorithm is red and direct algorithm is green), and the upper limit is characterized by three times of size of S_n^2.

(a)



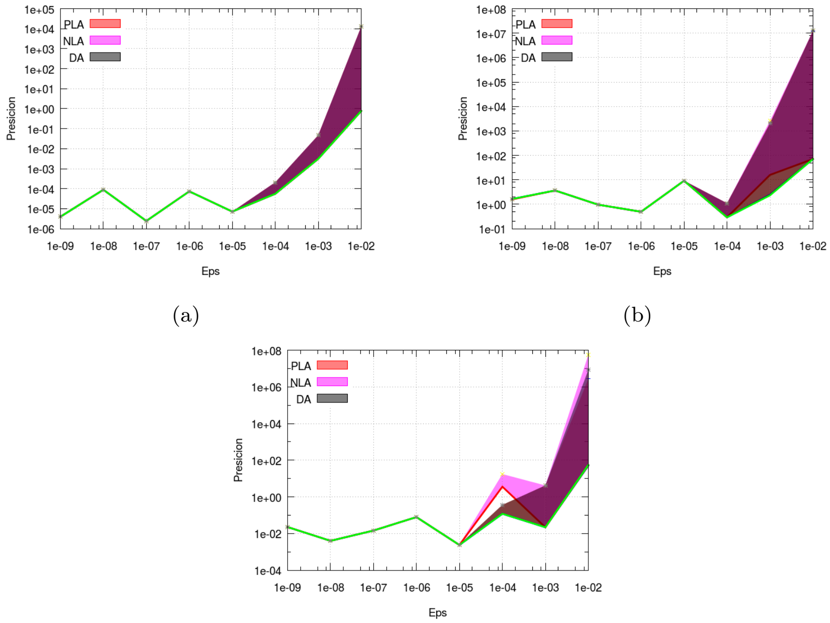
(b)



(c)

Fig. 5. Three types of matrices: (a) rotation matrix (by thirty degrees); (b) translation matrix (2D displacement vector); (c) composition of rotation and translation matrices (rotation by ninety degrees and 2D displacement vector)

The researched algorithms show the same results for $\epsilon \leq 1e-5$, and the value of S_n^2 vanishes. After increasing the value of ϵ to $1e-2$ (approximately 1% of the screen size) direct algorithm demonstrates the best results (see Figure 6).



(a) (b)

(c)

Fig. 6. Results for three types of matrices: (a) rotation matrix; (b) translation matrix; (c) composition of rotation and translation matrices

Perspective distortion case

There were many experiments for one type of projective transformation matrices (Figure 7).

(a) (b)

Fig. 7. (a) Distorted perspective; (b) Corrected perspective

The results of algorithms are represented by Figure 8. As in the previous case, accuracy is represented as a strip with color-marked at the upper left corner of each graph. The graph shows that direct algorithm demonstrates the best results (see Figure 8).

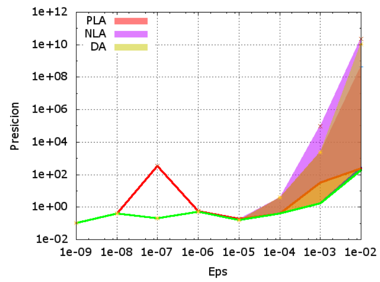
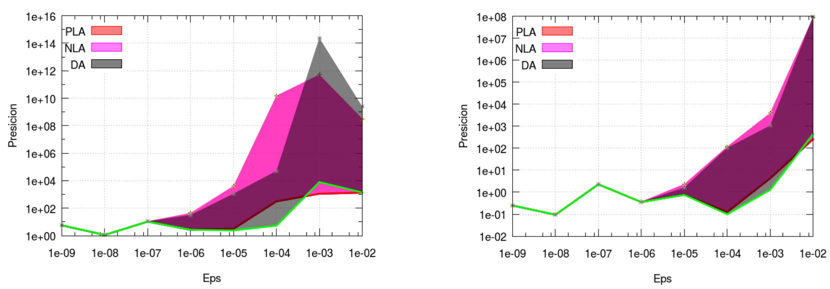


Fig. 8. Result for matrix of perspective distortion

Arbitrary case

The results of algorithms for two randomly generated matrices of arbitrary type are represented by Figure 9. On the left graph algorithms show similar results for $\epsilon = 1e - 7$, and on the right graph for $\epsilon = 1e - 6$ the strip degenerates to the line. For other values of ϵ direct algorithm works with the same accuracy as linear algorithms (Figure 9).



(a) (b)

Fig. 9. Results for two randomly generated matrices: (a) First random matrix; (b) Second random matrix

6. Conclusion

The hypothesis of Nosovskií and Skripka that direct algorithm is the fastest and most robust algorithm among known ones for calculating projective mapping was confirmed by computer simulation.

Programs written in the C++ computer language produce new results in the field of gluing flat images.

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