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A NOTE ON INEQUALITIES OF DIAZ-METCALF TYPE FOR ISOTONIC LINEAR FUNCTIONALS

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Abstract. Some refinements of the Beaseck-Pečarić generalization of the Diaz-Metcalf inequality are proved.

1. Let T be a nonempty set and let L be a linear class of real-valued functions $g\colon T \to \mathbf{R}$ having the properties:

L1: $f,\, g \in L \implies (af + bg) \in L$ for all $a,\, b \in \mathbf{R}$;

L2: $1 \in L$, that is if $f(t) = 1$ $(t \in T)$, then $f \in L$.

We also consider isotonic linear functionals $A\colon L \to \mathbf{R}$, that is, we suppose:

A1: $A(af + bg) = aA(f) + bA(g)$ for all $f,\, g \in L$, $a,\, b \in \mathbf{R}$;

A2: $f \in L$, $f(t) \geqslant 0$, $t \in T \implies A(f) \geqslant 0$ (i.e. A is isotonic).

We merely note here that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g\, d\mu \quad \text{or} \quad A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case, and E is a subset of $\mathbf{N}$ with $p_k > 0$ in the second case.

In the paper [1] Beaseck and Pečarić have proved the following generalization of Diaz-Metcalf inequality [3, pp. 61–63]:

THEOREM 1. *Let L and A satisfy L1, L2 and A1, A2 on a base set T. Suppose $p > 1$, $q = p/(p-1)$ and $w,\, f,\, g \geqslant 0$ on T with wf^p, wg^q, $wfg \in L$. If, in addition, we have $0 < m \leqslant f(t)g^{-q/p}(t) \leqslant M < \infty$ for all $t \in T$ $(m,\, M \in \mathbf{R})$, then*

$$(M - m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \leqslant (M^p - m^p)A(wfg). \tag{1}$$

If $p < 0$, (1) holds provided either $A(wf^p) > 0$ or $A(wg^q) > 0$; while if $0 < p < 1$, the opposite inequality to (1) holds if either $A(wf^p) > 0$ or $A(wg^q) > 0$.

For $p = q = 2$, $w = 1$ and $A(f) = \sum_{k=1}^{n} f_k$ or $A(f) = \int_a^b f(x)\, dx$ one gets Diaz-Metcalf's inequality.

2. Further on we shall give some similar results.

THEOREM 2. *Let L and A be as above, and suppose $w \geqslant 0$ on T with wf, wg, $wfg \in L$. If in addition we have:*

$$m_1 \leqslant f(t) \leqslant M_1, \quad m_2 \leqslant g(t) \leqslant M_2 \quad \text{for all } t \in T, \tag{2}$$

then

$$(m_1 + M_1)A(wg) + (m_2 + M_2)A(wf) - (m_1m_2 + M_1M_2)A(w) \leqslant 2A(wfg)$$
$$\leqslant (m_1 + M_1)A(wg) + (m_2 + M_2)A(wf) - (m_1M_2 + M_1m_2)A(w). \tag{3}$$

Proof. From (2) we get:

$$(M_1 - f(t))(M_2 - g(t)) + (f(t) - m_1)(g(t) - m_2) \geqslant 0$$

for all $t \in T$, giving:

$$2f(t)g(t) \geqslant (m_1 + M_1)g(t) + (m_2 + M_2)f(t) - (m_1m_2 + M_1M_2),$$

for all $t \in T$. Applying to this inequality the functional A, we can derive the first part of (3).

The second part follows from the inequality

$$(M_1 - f(t))(g(t) - m_2) + (f(t) - m_1)(M_2 - g(t)) \geqslant 0$$

by a similar argument as above, and we shall omit the details. ∎

In the paper [2, Theorem 1.1, p. 16] S. S. Dragomir has proved the following result in connection to Pólya-Szegö inequality for real numbers and integrals.

THEOREM 3. *Let $(a_k)_{k=\overline{1,n}}$, $(b_k)_{k=\overline{1,n}}$ be such that $0 < m \leqslant a_k/b_k \leqslant M < \infty$ and f, g be two integrable functions on $[a,b]$ with $0 < \gamma \leqslant f(x)/g(x) \leqslant \Gamma < \infty$, $x \in [a,b]$. Then the following estimates hold:*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \leqslant \frac{M}{m}\Big(\sum_{k=1}^{n} a_k b_k\Big)^2, \tag{4}$$

$$\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx \leqslant \frac{\Gamma}{\gamma}\Big(\int_a^b f(x)g(x)\,dx\Big)^2. \tag{5}$$

It is also proved that the inequality of Pólya-Szegö for integrals

$$\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx \leqslant \left(\frac{\sqrt{M_1M_2/m_1m_2} + \sqrt{m_1m_2/M_1M_2}}{2}\right)^2, \tag{6}$$

where $0 < m_1 \leqslant f(x) \leqslant M_1 < \infty$, $0 < m_2 \leqslant g(x) \leqslant M_2 < \infty$, $x \in [a,b]$, and inequality (5) are uncomparable to each other, i.e. there exists a pair of functions (f_1, g_1) such that (6) is stronger than (5) and also a pair of functions (f_2, g_2) such that (5) is stronger than (6).

Now we will obtain a generalization of Theorem 3 to isotonic linear functionals:

THEOREM 4. *Let L and A satisfy L1, L2 and A1, A2 on a base set T. Assume $w \geqslant 0$ on T, wf^2, wg^2, $wfg \in L$ and that there exist two positive numbers γ, Γ such that $0 < \gamma \leqslant f(t)/g(t) \leqslant \Gamma < \infty$ for all $t \in T$. Then the following inequality holds:*

$$A(f^2w)A(g^2w) \leqslant \frac{\Gamma}{\gamma}A^2(fgw). \tag{7}$$

Proof. It is easy to see that for all t, τ in T we have:

$$\frac{f^2(t)g^2(\tau)}{f(t)g(t)f(\tau)g(\tau)} = \frac{f(t)/g(t)}{f(\tau)g(\tau)} \leqslant \frac{\Gamma}{\gamma}$$

which yields

$$f^2(t)w(t)g^2(\tau)w(\tau) \leqslant \frac{\Gamma}{\gamma}f(t)g(t)w(t)f(\tau)g(\tau)w(\tau).$$

Apllying the functional A to this inequality, first with respect to the variable t and then with respect to the variable τ, we obtain without difficulty the relation (7). ∎

Finally, we prove:

THEOREM 5. *Let L and A be as above, w, $v \geqslant 0$ on T such that fw, gv, w, g^2v, v, $f^2w \in L$. If the following condition is satisfied:*

$$0 \leqslant \gamma \leqslant f(t) \leqslant \Gamma < \infty, \quad 0 < \varphi \leqslant g(t) \leqslant \Phi < \infty \quad \textit{for all } t \in T,$$

then we have the inequality:

$$(\Gamma\Phi + \gamma\varphi)A(fw)A(gv) \geqslant \Gamma\gamma A(w)A(g^2v) + \Phi\varphi A(v)A(f^2w). \tag{8}$$

Proof. For all t, τ in T we can write:

$$\left(\frac{g(\tau)}{f(t)} - \frac{\varphi}{\Gamma}\right)\left(\frac{\Phi}{\gamma} - \frac{g(\tau)}{f(t)}\right)w(t)v(\tau) \geqslant 0.$$

The proof runs on the lines of argument used in the proof of the above theorem and we shall omit the details. ∎

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