New York Journal of Mathematics

New York J. Math. **6** (2000) 95–106.

# An Eigenvalue Problem for Elliptic Systems

**Marco Squassina**

Abstract. By means of non-smooth critical point theory we prove existence of weak solutions for a general nonlinear elliptic eigenvalue problem under natural growth conditions.

Contents



## 1. Introduction

Let $\Omega$ be a bounded and open subset of $\mathbb{R}^n$ and $N \geq 1$. Existence and multiplicity results for quasilinear eigenvalue problems of the type :

$$\begin{cases} -\sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x,u)\frac{\partial u_k}{\partial x_i}) + \frac{1}{2}\sum\limits_{i,j=1}^{n}\sum\limits_{h=1}^{N} \frac{\partial a_{ij}}{\partial u_k}(x,u)\frac{\partial u_h}{\partial x_i}\frac{\partial u_h}{\partial x_j} = \lambda \frac{\partial G}{\partial u_k}(x,u) & \text{in } \Omega \\ k = 1, \dots, N \quad (u,\lambda) \in M \times \mathbb{R} & \end{cases}$$

on the submanifold of $H_0^1(\Omega, \mathbb{R}^N)$

$$M = \left\{ u \in H_0^1(\Omega, \mathbb{R}^N) : \int_\Omega G(x,u)\, dx = 1 \right\},$$

have been firstly studied in 1983 by M. Struwe [15] and recently by G. Arioli [1] via techniques of non-smooth critical point theory.

The goal of this paper is to study the following more general eigenvalue problem

$$(1) \qquad -\mathrm{div}(\nabla_\xi L(x,u,\nabla u)) + \nabla_s L(x,u,\nabla u) = \lambda \nabla_s G(x,u) \quad (u,\lambda) \in M \times \mathbb{R},$$

Received February 23, 2000.

*Mathematics Subject Classification.* 35J50, 47J10, 58E05.

*Key words and phrases.* Nonlinear eigenvalue problems, Non-smooth critical point theory.

ISSN 1076-9803/00

on the submanifold of $W_0^{1,p}(\Omega,\mathbb{R}^N)$

$$M = \left\{ u \in W_0^{1,p}(\Omega,\mathbb{R}^N) : \int_\Omega G(x,u)\,dx = 1 \right\}. \tag{2}$$

We shall consider functionals $f : W_0^{1,p}(\Omega,\mathbb{R}^N) \to \mathbb{R}$ defined by

$$f(u) = \int_\Omega L(x,u,\nabla u)\,dx\,, \tag{3}$$

with $1 < p < n$. In general, $f$ is not even locally Lipschitzian unless $L$ does not depend on $u$ or $n = 1$, so that classical critical point theory fails.

To overcome this difficulty, we shall use the non-smooth critical point theory developed in [8, 9, 10, 11, 12] and also the subdifferential for continuous functions recently introduced in [6].

We shall prove that problem (1) admits a nontrivial weak solution in $M \times \mathbb{R}$ by restricting $f$ to $M$ and looking for constrained critical points.

We assume that $M \neq \emptyset$, that $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is measurable in $x$ for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, of class $C^1$ in $(s,\xi)$ for a.e. $x \in \Omega$ and $L(x,s,\cdot)$ is strictly convex. Moreover, we shall assume that:

[$\mathrm{L}_1$] There exist $\nu > 0$ such that for each $\varepsilon > 0$ there is $a_\varepsilon \in L^1(\Omega)$ and $b_\varepsilon \in \mathbb{R}$ with

$$\nu|\xi|^p \le L(x,s,\xi) \le a_\varepsilon(x) + \varepsilon|s|^{p^*} + b_\varepsilon|\xi|^p \tag{4}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, where $p^*$ denotes the critical Sobolev's exponent.

[$\mathrm{L}_2$] There exists $b \in \mathbb{R}$ such that for each $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ with

$$|\nabla_s L(x,s,\xi)| \le a_\varepsilon(x) + \varepsilon|s|^{p^*} + b|\xi|^p \tag{5}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Moreover there is $a_1 \in L^{p'}(\Omega)$ with

$$|\nabla_\xi L(x,s,\xi)| \le a_1(x) + b|s|^{\frac{p^*}{p'}} + b|\xi|^{p-1} \tag{6}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$.

[$\mathrm{L}_3$] For a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$

$$\nabla_s L(x,s,\xi)\cdot s \ge 0\,. \tag{7}$$

[$\mathrm{L}_4$] If $N > 1$, there exists a bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\nabla_s L(x,s,\xi)\cdot \exp_\sigma(r,s) + \nabla_\xi L(x,s,\xi)\cdot \nabla\exp_\sigma(r,s,\xi) \le 0 \tag{8}$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$, $\sigma \in \{-1,1\}^N$ and $r,s \in \mathbb{R}^N$ where

$$(\exp_\sigma(r,s))_h := \sigma_h \exp[\sigma_h(\psi(r_h) - \psi(s_h))]$$

and

$$[\nabla\exp_\sigma(r,s,\xi)]_{hi} := -\exp[\sigma_h(\psi(r_h) - \psi(s_h))]\psi'(s_h)\xi_i^h$$

for each $h = 1,\dots,N$ and $i = 1,\dots,n$.

[G$_1$] $G(x,s)$ is measurable in $x$ and of class $C^1$ in $s$ with $G(x,0)=0$ a.e. in $\Omega$. If $g(x,s)$ denotes $\nabla_s G(x,s)$, for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{\frac{np}{n(p-1)+p}}(\Omega)$ such that

$$|g(x,s)| \leq a_\varepsilon(x) + \varepsilon |s|^{p^*-1} \tag{9}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^N$.

[G$_2$] For a.e. $x \in \Omega$ and for each $s \neq 0$ we have $g(x,s) \cdot s > 0$.

Under the preceding assumptions, the following is our main result.

**Theorem 1.** *The eigenvalue problem*

$$-\mathrm{div}(\nabla_\xi L(x,u,\nabla u)) + \nabla_s L(x,u,\nabla u) = \lambda g(x,u) \quad (u,\lambda) \in M \times \mathbb{R}\,, \tag{10}$$

*has at least one nontrivial weak solution* $(u,\lambda) \in M \times \mathbb{R}$.

In the vectorial case ($N > 1$), to my knowledge, problem (1) has only been considered in [15] and in [1] in the particular case

$$L(x,s,\xi) = \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x,s) \xi_i^h \xi_j^k\,, \tag{11}$$

for coefficients $a_{ij}^{hk} : \Omega \times \mathbb{R}^N \to \mathbb{R}^{n^2}$ of the type $a_{ij}^{hk}(x,s) = \delta^{hk} \alpha_{ij}(x,s)$.

In [15, Theorem 3.2] the statement is essentially of perturbative nature, since it says that if for each $k \in \mathbb{N}$ there exists a $\varrho_k > 0$ with

$$|\nabla_s \alpha_{ij}(x,s)| < \varrho_k \quad \text{for a.e. } x \in \Omega, \ \text{ for all } s \in \mathbb{R}^N, \tag{12}$$

then the problem has at least $k$ distinct weak solutions:

$$(u_\ell, \lambda_\ell) \in H_0^1(\Omega, \mathbb{R}^N) \times \mathbb{R}, \quad \ell = 1, \dots, k\,.$$

In other words, the less the coefficients $\alpha_{ij}(x,s)$ vary in $s$, the more solutions we get.

In [1] a new technical condition is introduced to be compared with (8). It is assumed that there exist $K > 0$ and an increasing bounded Lipschitz function $\psi$ from $[0,+\infty[$ to $[0,+\infty[$ with $\psi(0) = 0$, $\psi'$ non-increasing, $\psi(s) \to K$ as $s \to +\infty$ and such that

$$\sum_{i,j=1}^{n} \sum_{k=1}^{N} \left| \frac{\partial \alpha_{ij}}{\partial s_k}(x,s) \xi_i \xi_j \right| \leq 2e^{-4K} \psi'(|s|) \sum_{i,j=1}^{n} \alpha_{ij}(x,s) \xi_i \xi_j \tag{13}$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and for all $r, s \in \mathbb{R}^N$.

The proof itself of [2, Lemma 6.1] shows that this condition implies assumption (8) in the case of integrands $L$ like (11). On the other hand, if $N \geq 2$, the two conditions look quite similar. However, our condition (8) seems to be preferable, because when $N = 1$ and $L$ is given by (11), it reduces to the inequality

$$\left| \sum_{i,j=1}^{n} \frac{\partial \alpha_{ij}}{\partial s}(x,s) \xi_i \xi_j \right| \leq 2\psi'(s) \sum_{i,j=1}^{n} \alpha_{ij}(x,s) \xi_i \xi_j,$$

which is not so restrictive in view of the ellipticity of $\alpha_{ij}$, while (13) is in this case much stronger. For a general Lagrangian $L$, in the case $N=1$, condition (8) reduces to

$$|D_s L(x,s,\xi)| \leq \psi'(s)\nabla_\xi L(x,s,\xi)\cdot\xi$$

for a.e. $x\in\Omega$ and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$. This assumption has already been considered in literature in jumping problems (see e.g. [7]).

In Remarks 2 and 3 we will show examples of $L$ not of the form (11) and satisfying (8). Finally, we point out that (13) and (8) are not easily comparable to (12).

## 2. Recollections from non-smooth critical point theory

In this section, we want to recall the relationship between weak solutions to (1) and constrained critical points of $f$ to $M$. Let $a_0\in L^1(\Omega)$, $b_0\in\mathbb{R}$, $a_1\in L^1_{loc}(\Omega)$ and $b_1\in L^\infty_{loc}(\Omega)$ be such that for a.e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}^N\times\mathbb{R}^{nN}$

$$|L(x,s,\xi)|\leq a_0(x)+b_0|s|^{\frac{np}{n-p}}+b_0|\xi|^p\,, \tag{14}$$

$$|\nabla_s L(x,s,\xi)|\leq a_1(x)+b_1(x)|s|^{\frac{np}{n-p}}+b_1(x)|\xi|^p\,, \tag{15}$$

$$|\nabla_\xi L(x,s,\xi)|\leq a_1(x)+b_1(x)|s|^{\frac{np}{n-p}}+b_1(x)|\xi|^p\,. \tag{16}$$

Conditions (15) and (16) imply that for every $u\in W^{1,p}_0(\Omega,\mathbb{R}^N)$ we have

$$\nabla_\xi L(x,u,\nabla u)\in L^1_{loc}(\Omega,\mathbb{R}^{nN}),\quad \nabla_s L(x,u,\nabla u)\in L^1_{loc}(\Omega,\mathbb{R}^N)\,.$$

Therefore for every $u\in W^{1,p}_0(\Omega,\mathbb{R}^N)$ we have

$$-\mathrm{div}\,(\nabla_\xi L(x,u,\nabla u))+\nabla_s L(x,u,\nabla u)\in\mathcal{D}'(\Omega,\mathbb{R}^N)\,.$$

We shall now recall two definitions from [6], where a new notion of subdifferential for continuous functionals on normed spaces has been recently introduced by M. Degiovanni and I. Campa.

**Definition 1.** Let $X$ be a real normed space and $C\subseteq X$. For each $u\in C$, we denote with $T_C(u)$ the set of all $v\in X$ such that for each $\varepsilon>0$, there exist $\delta>0$ and

$$\nu:(B(u,\delta)\cap C)\times]0,\delta]\to B(v,\varepsilon)$$

continuous with

$$\xi+t\nu(\xi,t)\in C,$$

when $\xi\in B(u,\delta)\cap C$ and $t\in]0,\delta[$. $T_C(u)$ is said the cone tangent to $C$ at $u$.

**Definition 2.** For each $u\in X$, set

$$\partial f(u):=\{\alpha\in X^*:\,(\alpha,-1)\in N_{\mathrm{epi}f}(u,f(u))\}\,,$$

where

$$N_{\mathrm{epi}f}(u):=\{\nu\in X^*:\,\langle\nu,v\rangle\leq 0\ \text{ for all }\ v\in T_{\mathrm{epi}f}(u)\}\,.$$

$\partial f(u)$ is said to be the subdifferential of $f$ at $u$.

Via $\partial f(u)$ we shall connect critical points for functionals of calculus of variations (3) constrained to $M$, with weak solutions to the related eigenvalue problem. Define the submanifold of $W_0^{1,p}(\Omega,\mathbb{R}^N)$

$$M = \left\{ u \in W_0^{1,p}(\Omega,\mathbb{R}^N) : \ \ \Phi(u) = 1 \right\},$$

where $\Phi : W_0^{1,p}(\Omega,\mathbb{R}^N) \to W^{-1,p'}(\Omega,\mathbb{R}^N)$ is of class $C^1$, $M \neq \emptyset$, $0 \notin M$ and moreover $\nabla\Phi(u) \neq 0$ for each $u \in M$.

We now recall the fundamental definition of weak slope (see, [8, 9, 10, 11, 12]).

**Definition 3.** Let $(X,d)$ be a metric space, $f : X \to \mathbb{R}$ a continuous function and $u \in X$. We denote by $|df|(u)$ the supremum of $\sigma \in [0,+\infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B_\delta(u) \times [0,\delta] \longrightarrow X$$

such that for all $(v,t) \in B_\delta(u) \times [0,\delta]$

$$d(\mathcal{H}(v,t),v) \le t, \quad f(\mathcal{H}(v,t)) \le f(v) - \sigma t.$$

We say that the extended real number $|df|(u)$ is the weak slope of $f$ at $u$.

It is easy to prove that the map $\{u \mapsto |df|(u)\}$ is lower semicontinuous.

**Definition 4.** Let $(X,d)$ be a metric space, $f : X \to \mathbb{R}$ a continuous function and $u \in X$. We say that $u$ is a critical point of $f$ if $|df|(u) = 0$.

**Definition 5.** Let $(X,d)$ be a metric space, $c \in \mathbb{R}$ and $f : X \to \mathbb{R}$ a continuous function. A sequence $(u_h) \subseteq X$ is said to be a *Palais–Smale sequence at level c* (*$(PS)_c$–sequence*, in short) for $f$, if $f(u_h) \to c$ and $|df|(u_h) \to 0$. We say that $f$ satisfies the Palais–Smale condition ($(PS)_c$ in short) at level $c$ if every $(PS)_c$–sequence admits a convergent subsequence.

We now come to the case when $X = W_0^{1,p}(\Omega,\mathbb{R}^N)$ and $f : W_0^{1,p}(\Omega,\mathbb{R}^N) \to \mathbb{R}$ given by (3). From (14) it follows that $f$ is well defined and continuous.

Since $M$ is metric space endowed with the metric of $W_0^{1,p}(\Omega,\mathbb{R}^N)$, the weak slope $\left|df_{|_M}\right|(u)$ and the $(PS)_c$–condition for $f_{|_M}$ may of course be defined.

**Theorem 2.** *For every $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ there exists $\lambda \in \mathbb{R}$ such that*

$$\left|df_{|_M}\right|(u) \ge \sup\left\{\nabla f(u)(v) - \lambda\nabla\Phi(u)(v) : \ v \in C_c^\infty(\Omega,\mathbb{R}^N),\ \|v\|_{1,p} \le 1\right\}.$$

*In particular, for each $(PS)_c$–sequence $(u_h)$ for $f_{|_M}$ there exists $(\lambda_h) \subseteq \mathbb{R}$ such that*

$$\lim_h \ \sup\left\{\nabla f(u_h)(v) - \lambda_h\nabla\Phi(u_h)(v) : \ v \in C_c^\infty(\Omega,\mathbb{R}^N),\ \|v\|_{1,p} \le 1\right\} = 0.$$

**Proof.** By conditions (15) and (16), for every $u \in M$ and $v \in C_c^\infty(\Omega,\mathbb{R}^N)$ there exists

$$f'(u)(v) = \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla v\,dx + \int_\Omega \nabla_s L(x,u,\nabla u)\cdot v\,dx$$

and the function $\{u \mapsto f'(u)(v)\}$ is continuous from $M$ into $\mathbb{R}$. Now, let us extend $f_{|_M}$ to the functional $f^* : W_0^{1,p}(\Omega,\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$ given by

$$f^*(u) = \begin{cases} f(u) & \text{if } \ u \in M \\ +\infty & \text{if } \ u \notin M. \end{cases} \tag{17}$$

We may assume that $|df_{|_M}|(u) < +\infty$. Consequently $|df^*|(u) = |df_{|_M}|(u)$, so that by [6, Theorem 4.13] there exists $\omega \in \partial f^*(u)$ with $|df^*|(u) \geq \|\omega\|_{-1,p'}$. Moreover, by [6, Corollary 5.4] we have

$$\partial f^*(u) \subseteq \partial f(u) + \mathbb{R}\nabla\Phi(u)\,.$$

Finally, by [6, Theorem 6.1], we get $\partial f(u) = \{\eta\}$ where

$$\langle \eta, v\rangle = \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla v\,dx + \int_\Omega \nabla_s L(x,u,\nabla u)\cdot v\,dx = \nabla f(u)(v)$$

for each $v \in C_c^\infty(\Omega,\mathbb{R}^N)$ and the proof is complete. □

By the preceding result, each critical point $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ of $f_{|_M}$ is a weak solution to the eigenvalue problem :

$$\nabla f(u) = \lambda\nabla\Phi(u) \quad (u,\lambda) \in M\times\mathbb{R}\,.$$

## 3. The Palais–Smale condition

Recall first a very useful consequence of Brezis–Browder's Theorem [5].

**Proposition 1.** *Let $T \in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\cap W^{-1,p'}(\Omega,\mathbb{R}^N)$, $v \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ and $\eta \in L^1(\Omega)$ with $T\cdot v \geq \eta$. Then $T\cdot v \in L^1(\Omega)$ and*

$$\langle T, v\rangle = \int_\Omega T\cdot v\,dx$$

**Proof.** Argue as in [13, Lemma 3] . □

As a consequence of assumption [$\mathrm{L}_1$] and convexity of $L(x,s,\cdot)$, for each $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that

$$\nabla_\xi L(x,s,\xi)\cdot\xi \geq \nu|\xi|^p - a_\varepsilon(x) - \varepsilon|s|^{p^*} \tag{18}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^N\times\mathbb{R}^{nN}$ .

We now come to one of the main results of this paper, i.e., the local compactness property for $(PS)_c$–sequences.

**Theorem 3.** *Let $(u_h)$ be a bounded sequence in $W_0^{1,p}(\Omega,\mathbb{R}^N)$ and set*

$$\langle w_h, v\rangle = \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla v\,dx + \int_\Omega \nabla_s L(x,u_h,\nabla u_h)\cdot v\,dx, \tag{19}$$

*for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$. If $(w_h)$ is strongly convergent to some $w$ in $W^{-1,p'}(\Omega,\mathbb{R}^N)$, then $(u_h)$ admits a strongly convergent subsequence in $W_0^{1,p}(\Omega,\mathbb{R}^N)$.*

**Proof.** Since $(u_h)$ is bounded in $W_0^{1,p}(\Omega,\mathbb{R}^N)$, we find a $u$ in $W_0^{1,p}(\Omega,\mathbb{R}^N)$ such that, up to subsequences,

$$\nabla u_h \rightharpoonup \nabla u \;\text{ in }\; L^p(\Omega,\mathbb{R}^N), \quad u_h \to u \;\text{ in }\; L^p(\Omega,\mathbb{R}^N), \quad u_h(x)\to u(x) \;\text{ a.e. }\; x\in\Omega.$$

By [4, Theorem 2.1], up to a subsequence, we have

$$\nabla u_h(x) \to \nabla u(x) \;\text{ a.e. }\; x \in \Omega\,.$$

Therefore, by (6) we get

$$\nabla_\xi L(x,u_h,\nabla u_h) \rightharpoonup \nabla_\xi L(x,u,\nabla u) \;\text{ in }\; L^{p'}(\Omega,\mathbb{R}^{nN}).$$

We now want to prove that we have

$$\langle w,u\rangle = \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot \nabla u\,dx + \int_\Omega \nabla_s L(x,u,\nabla u)\cdot u\,dx. \tag{20}$$

Let $\psi$ be as in [L$_4$] and test equation (19) with the following functions

$$v_h = \varphi(\sigma_1 \exp\{\sigma_1(\psi(u^1)-\psi(u_h^1))\},\dots,\sigma_N \exp\{\sigma_N(\psi(u^N)-\psi(u_h^N))\}),$$

where $\varphi\in C_c^\infty(\Omega)$, $\varphi\ge 0$ and $\sigma_i=\pm 1$ for all $i=1,\dots,N$. By direct computation we obtain for a.e. $x\in\Omega$

$$D_j v_{hi} = (\sigma_i D_j\varphi + (\psi'(u_i)D_j u_i - \psi'(u_{hi})D_j u_{hi})\varphi)\exp[\sigma_i(\psi(u_i)-\psi(u_{hi}))]$$

for each $i=1,\dots,N$ and $j=1,\dots,n$. Therefore, with the notation

$$[\exp\{\sigma(\psi(u)-\psi(u_h))\}\psi'(u_h)\nabla u_h]_{ij} := \exp\{\sigma_i(\psi(u_i)-\psi(u_{hi}))\}\psi'(u_{hi})D_j u_{hi}$$

for each $i=1,\dots,N$ and $j=1,\dots,n$, we get

$$\begin{aligned}\int_\Omega &\nabla_\xi L(x,u_h,\nabla u_h)\cdot[\sigma\nabla\varphi+\psi'(u)\nabla u\varphi]\exp\{\sigma(\psi(u)-\psi(u_h))\}\,dx\\ &-\langle w_h,\varphi\sigma\exp\{\sigma(\psi(u)-\psi(u_h))\}\rangle\\ &+\int_\Omega\Big\{\nabla_s L(x,u_h,\nabla u_h)\cdot\sigma\exp\{\sigma(\psi(u)-\psi(u_h))\}\,dx\\ &-\nabla_\xi L(x,u_h,\nabla u_h)\cdot\exp\{\sigma(\psi(u)-\psi(u_h))\}\psi'(u_h)\nabla u_h\Big\}\varphi\,dx=0\,.\end{aligned}$$

Observe that if $v=(\sigma_1\varphi,\dots,\sigma_N\varphi)$ we have

$$\lim_h \langle w_h,\varphi\sigma\exp\{\sigma(\psi(u)-\psi(u_h))\}\rangle = \langle w,v\rangle.$$

Since $u_h \rightharpoonup u$ in $W_0^{1,p}(\Omega,\mathbb{R}^N)$, we have

$$\begin{aligned}\lim_h \int_\Omega &\nabla_\xi L(x,u_h,\nabla u_h)\cdot[\sigma\nabla\varphi+\psi'(u)\nabla u\varphi]\exp\{\sigma(\psi(u)-\psi(u_h))\}\,dx\\ &=\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla v\,dx+\int_\Omega\nabla_\xi L(x,u,\nabla u)\cdot\psi'(u)\nabla u\varphi\,dx.\end{aligned}$$

Note now that by assumption (8) for each $h\in\mathbb{N}$ we have

$$\begin{aligned}&\nabla_s L(x,u_h,\nabla u_h)\cdot\sigma\exp\{\sigma(\psi(u)-\psi(u_h))\}\\ &\qquad-\nabla_\xi L(x,u_h,\nabla u_h)\cdot\exp\{\sigma(\psi(u)-\psi(u_h))\}\psi'(u_h)\nabla u_h\le 0.\end{aligned}$$

Therefore, Fatou's Lemma implies that

$$\begin{aligned}\limsup_h\Bigg\{&\int_\Omega\nabla_s L(x,u_h,\nabla u_h)\cdot\sigma\exp\{\sigma(\psi(u)-\psi(u_h))\}\varphi\,dx\\ &-\int_\Omega\nabla_\xi L(x,u_h,\nabla u_h)\cdot\exp\{\sigma(\psi(u)-\psi(u_h))\}\psi'(u_h)\nabla u_h\varphi\,dx\Bigg\}\\ &\le\int_\Omega\nabla_s L(x,u,\nabla u)\cdot v\,dx-\int_\Omega\nabla_\xi L(x,u,\nabla u)\cdot\psi'(u)\nabla u\varphi\,dx.\end{aligned}$$

Combining the previous inequalities we get

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla v\,dx+\int_\Omega\nabla_s L(x,u,\nabla u)\cdot v\,dx\ge\langle w,v\rangle$$

for each $v = (\sigma_1\varphi, \dots, \sigma_N\varphi)$, with $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$. Since we may exchange $v$ with $-v$, we obtain

$$\int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega \nabla_s L(x, u, \nabla u) \cdot v \, dx = \langle w, v \rangle. \tag{21}$$

for each $v = (\sigma_1\varphi, \dots, \sigma_N\varphi)$ with $\varphi \in C_c^\infty(\Omega)$ and $\varphi \geq 0$. Since each $v \in C_c^\infty(\Omega, \mathbb{R}^N)$ is a linear combination of such functions, taking into account Proposition 1, we obtain relation (20). The final step is to prove that $(u_h)$ goes to $u$ in $W_0^{1,p}(\Omega, \mathbb{R}^N)$. To this aim, let us first get the following inequality

$$\limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leq \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u \, dx\,. \tag{22}$$

Because of (7) Fatou's Lemma yields

$$\int_\Omega \nabla_s L(x, u, \nabla u) \cdot u \, dx \leq \liminf_h \int_\Omega \nabla_s L(x, u_h, \nabla u_h) \cdot u_h \, dx\,. \tag{23}$$

Combining this fact with (20) and taking into account that

$$\langle w_h, u_h \rangle \to \langle w, u \rangle \quad \text{as } \ h \to +\infty,$$

we deduce

$$\begin{aligned}
\limsup_h \int_\Omega & \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \\
&= \limsup_h \left[ -\int_\Omega \nabla_s L(x, u_h, \nabla u_h) \cdot u_h \, dx + \langle w_h, u_h \rangle \right] \\
&\leq \left[ -\int_\Omega \nabla_s L(x, u, \nabla u) \cdot u \, dx + \langle w, u \rangle \right] \\
&= \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u \, dx\,.
\end{aligned}$$

In particular, again by Fatou's Lemma, we have

$$\begin{aligned}
\int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u \, dx \ &\leq \ \liminf_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \\
&\leq \ \limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \\
&\leq \ \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u \, dx,
\end{aligned}$$

that is

$$\lim_h \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx = \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla u \, dx,$$

which gives convergence in $L^1(\Omega)$. Therefore, by (18) we conclude that :

$$\lim_h \int_\Omega |\nabla u_h|^p \, dx = \int_\Omega |\nabla u|^p \, dx,$$

which gives convergence of $(u_h)$ to $u$ in $W_0^{1,p}(\Omega, \mathbb{R}^N)$. $\square$

**Corollary 1.** *Let $(u_h)$ be a bounded sequence in $W_0^{1,p}(\Omega, \mathbb{R}^N)$, $(\lambda_h)$ a sequence in $\mathbb{R}$ and set for all $v \in C_c^\infty(\Omega, \mathbb{R}^N)$*

$$\langle \lambda_h w_h, v\rangle = \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_\Omega \nabla_s L(x, u_h, \nabla u_h) \cdot v \, dx \, .$$

*If $(w_h)$ converges to some $w \neq 0$ in $W^{-1,p'}(\Omega, \mathbb{R}^N)$ then $(u_h, \lambda_h)$ admits a strongly convergent subsequence in $W_0^{1,p}(\Omega, \mathbb{R}^N) \times \mathbb{R}$.*

**Proof.** By density, we can find $\eta \in C_c^\infty(\Omega, \mathbb{R}^N)$ such that

$$\lim_h \langle w_h, \eta\rangle = \langle w, \eta\rangle > 0.$$

Since of course the sequence

$$\left\{\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla \eta \, dx + \int_\Omega \nabla_s L(x, u_h, \nabla u_h) \cdot \eta \, dx\right\}$$

is bounded, $(\lambda_h)$ is also bounded and the assertion follows by Theorem 3. $\square$

In the next result we prove that $f$ satisfies $(PS)_c$–condition.

**Lemma 1.** *Let $c \in \mathbb{R}$. Then, for each $(PS)_c$–sequence $(u_h)$ for $f_{|_M}$ there exists $u \in M$ and $\lambda \in \mathbb{R}$ such that, up to subsequences, $u_h \to u$ in $W_0^{1,p}(\Omega, \mathbb{R}^N)$ and $\lambda_h \to \lambda$ in $\mathbb{R}$. In particular, we have*

$$\int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega \nabla_s L(x, u, \nabla u) \cdot v \, dx = \lambda \int_\Omega g(x, u) \cdot v \, dx$$

*for each $v \in C_c^\infty(\Omega, \mathbb{R}^N)$.*

**Proof.** Let $(u_h)$ be a $(PS)_c$–sequence for $f_{|_M}$. Since by (4) $(u_h)$ is bounded in $W_0^{1,p}(\Omega, \mathbb{R}^N)$, up to a subsequence $(u_h)$ weakly goes to a $u \in M$. Moreover, since by [$G_1$], $g$ is completely continuous as mapping from $W_0^{1,p}(\Omega, \mathbb{R}^N)$ to $W^{-1,p'}(\Omega, \mathbb{R}^N)$, up to a further subsequence, we have

$$g(x, u_h) \to g(x, u) \text{ in } W^{-1,p'}(\Omega).$$

Now, by Theorem 2, there exists a sequence $(\lambda_h) \subseteq \mathbb{R}$ with

$$\sup\Big\{\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_\Omega \nabla_s L(x, u_h, \nabla u_h) \cdot v \, dx - \lambda_h \int_\Omega g(x, u_h) \cdot v \, dx : \ v \in C_c^\infty(\Omega, \mathbb{R}^N), \ \|v\|_{1,p} \leq 1\Big\} \to 0$$

as $h \to +\infty$. Hence, by applying Corollary 1 to

$$w_h = g(x, u_h) + \Lambda_h, \qquad \Lambda_h \to 0 \text{ in } W^{-1,p'}(\Omega, \mathbb{R}^N),$$

up to subsequences $(u_h, \lambda_h)$ converges to $(u, \lambda)$ in $W_0^{1,p}(\Omega, \mathbb{R}^N) \times \mathbb{R}$. $\square$

We may now prove the main result of this paper.

**Proof of Theorem 1.** By assumption [$G_2$] and $G(x, 0) = 0$ we easily see that $0 \notin M$ and $g(x, u) \not\equiv 0$ for each $u \in M$. Since $f$ is bounded from below, there exists a $(PS)_c$–sequence $(u_h)$ for $f_{|_M}$ at the level

$$c = \inf_{u \in M} f(u) \, .$$

Indeed, let $(u_h)$ a sequence of minimizers for $f$ in $M$. Of course we have $f(u_h) \to c$. Moreover, if $|df|(u_h) \not\to 0$, we would find a $\sigma > 0$ such that $|df|(u_h) \geq \sigma$. Then by [8, Theorem 1.1.11] there exists a continuous deformation

$$\eta : M \times [0, \delta] \to M$$

for some $\delta > 0$ such that for all $t \in [0, \delta]$ and $h \in \mathbb{N}$

$$f(\eta(u_h, t)) \leq f(u_h) - \sigma t\,.$$

This easily yields the contradiction $f(\eta(u_h, t)) < c$ for sufficiently large values of $h \in \mathbb{N}$. Thus $(u_h)$ is a $(PS)_c$–sequence for $f_{|_M}$. Lemma 1 now provides a weak solution $(u, \lambda) \in M \times \mathbb{R}$ to (1). Of course $u \not\equiv 0$. □

## 4. Final remarks

We refer the reader to [2] for some concrete examples where the condition (8) is fulfilled for an integrand $L$ like (11).

**Remark 1.** Assume that there exists $R > 0$ such that

$$|s| \geq R \Longrightarrow \nabla_s L(x, s, \xi) = 0$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ and

$$|s| \leq R \Longrightarrow \sum_{k=1}^{N} |D_{s_k} L(x, s, \xi)| \leq \frac{1}{4eR} \nabla_\xi L(x, s, \xi) \cdot \xi$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Then (8) holds for a $\psi$ defined by

$$\psi(s) = \begin{cases} \frac{s}{4R} & \text{if } 0 \leq s \leq R \\ \frac{1}{4} & \text{if } s \geq R\,. \end{cases} \tag{24}$$

Indeed, (8) is implied by the following condition:

There exist $K > 0$ and an increasing bounded Lipschitz function $\psi : [0, +\infty[ \to [0, +\infty[$ with $\psi(0) = 0$, $\psi'$ non-increasing, $\psi(t) \to K$ as $t \to +\infty$ and such that

$$\sum_{k=1}^{N} |D_{s_k} L(x, s, \xi)| \leq e^{-4K} \psi'(|s|) \nabla_\xi L(x, s, \xi) \cdot \xi \tag{25}$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and for all $r, s \in \mathbb{R}^N$.

It is easy to verify that the $\psi$ defined in (24) satisfies (25).

We now exhibit an example of $L$ satisfying (8) and not of quadratic type.

**Remark 2.** Let $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ be defined by

$$L(x, s, \xi) = \frac{1}{p}(\nu + \arctan|s|^2)|\xi|^p, \qquad \nu \geq e\sqrt{N}(\sqrt{3} + \pi),$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. By following [2, Example 9.2] it is possible to show that there exist $K > 0$ and an increasing bounded Lipschitz function $\psi : [0, +\infty[ \to [0, +\infty[$ with $\psi(0) = 0$, $\psi'$ non-increasing, $\psi(s) \to K$ as $s \to +\infty$ given by

$$\psi(s) = \frac{\sqrt{N} e^{4K}}{\nu} \begin{cases} \frac{3^{3/4}}{4} s & \text{if } s \in [0, 3^{-1/4}] \\ \frac{\sqrt{3}}{4} + \int_{3^{-1/4}}^{s} \frac{\tau}{1+\tau^4}\, d\tau & \text{if } s \in [3^{-1/4}, +\infty[ \end{cases}$$

such that $L$ satisfies (25). Therefore (8) is fulfilled.

**Remark 3.** Since condition (8) does not look very nice and it is not clear how to describe the class of systems that satisfy this assumption, it seems natural to look for some classes of quasilinear systems with a more particular structure but requiring a simpler hypothesis. To this aim, consider the eigenvalue problem

$$\begin{cases} -\mathrm{div}\left(A_1(u_1)|\nabla u_1|^{p-2}\nabla u_1\right) + \frac{1}{p}A_1'(u_1)|\nabla u_1|^p = \lambda g_1(x,u) & \text{in } \Omega \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \quad\;\; \vdots \qquad\qquad\qquad \vdots & \\ -\mathrm{div}\left(A_N(u_N)|\nabla u_N|^{p-2}\nabla u_N\right) + \frac{1}{p}A_N'(u_N)|\nabla u_N|^p = \lambda g_N(x,u) & \text{in } \Omega\,. \end{cases} \tag{26}$$

In a variational setting, the weak solutions $u = (u_1, \cdots, u_N)$ of (26) are the critical points of $f|_M$ where $f : W_0^{1,p}(\Omega, \mathbb{R}^N) \to \mathbb{R}$ is given by

$$f(u) = \frac{1}{p}\sum_{k=1}^{N}\int_\Omega A_k(u_k)|\nabla u_k|^p\,dx\,.$$

and $M$ is as in (2). Consider the following assumptions ($k = 1, \dots, N$):

$[A_1]$ $A_k \in C^1(\mathbb{R})$ with $\underline{a}_k \le A_k \le \overline{a}_k$ for some $\underline{a}_k, \overline{a}_k > 0$;

$[A_2]$ $A_k'(s)s \ge 0$ for each $s \in \mathbb{R}$;

$[A_3]$ there exists a bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$A_k(s)e^{-p(\psi(t)-\psi(s))} \le A_k(t) \le A_k(s)e^{p(\psi(t)-\psi(s))} \tag{27}$$

for each $s, t \in \mathbb{R}$ with $s \le t$.

Under the previous assumptions, by exploiting the proof of Theorem 3 it is possible to see that for system (26) assumption (8) may be replaced by (27). Indeed, (27) immediately implies that

$$\forall k = 1, \dots, N,\ \ \forall s \in \mathbb{R}:\ \ |A_k'(s)| \le pA_k(s)\psi'(s)\,.$$

Of course (27) looks much simpler and more understandable. In some sense, this condition says that for each $s \in \mathbb{R}$ and $k$ fixed, $A_k(t)$ must remain within the "exponential cone" determined by

$$\left\{t \mapsto A_k(s)e^{-p(\psi(t)-\psi(s))}\right\} \ \text{ and } \ \left\{t \mapsto A_k(s)e^{p(\psi(t)-\psi(s))}\right\}$$

for each $t \ge s$.

**Remark 4.** Condition (8) in only needed when $N > 1$, since in the case $N = 1$ Theorem 3 may be substituted by [14, Theorem 3.4] where no condition like (8) is requested in order to get the compactness property of $(PS)_c$–sequences.

Secondly, we remark that in the case $N = 1$ condition (7) can be assumed only for large values of $|s|$, that is, there exists $R > 0$ such that

$$|s| \ge R \Longrightarrow D_sL(x,s,\xi)\cdot s \ge 0 \tag{28}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ (see again [14, Theorem 3.4]).

## References


[1] G. Arioli, *A note on quasilinear elliptic eigenvalue problems*, Electron. J. Diff. Eqns. (1999), no. 47, 12 pages, CMP 1 729 831.

[2] G. Arioli and F. Gazzola, *Existence and multiplicity results for quasilinear elliptic differential systems*, Comm. Partial Diff. Eqns. **25** (2000), 125–153, CMP 1 737 545.

[3] T. Bartsch, *Topological Methods for Variational Problems with Symmetries*, Lect. Notes in Math., no. 1560, Springer-Verlag, Berlin, 1993, MR 96a:58078, Zbl 789.58001.

[4] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. **19** (1992), 581–597, MR 93h:35061, Zbl 783.35020.

[5] H. Brezis and F. E. Browder, *Sur une propriété des espaces de Sobolev*, C. R. Acad. Sc. Paris **287** (1978), 113–115, MR 58 #23539, Zbl 381.46019.

[6] I. Campa and M. Degiovanni, *Subdifferential calculus and nonsmooth critical point theory*, SIAM J. Optim. in press (2000).

[7] A. Canino, *On a jumping problem for quasilinear elliptic equations*, Math. Z. **226** (1997), 193–210, MR 98j:35058.

[8] A. Canino and M. Degiovanni, *Nonsmooth critical point theory and quasilinear elliptic equations*, Topological Methods in Differential Equations and Inclusions (Montreal, 1994) (A. Granas, M. Frigon, G. Sabidussi, eds.), NATO ASI Ser., Ser. C, Math. Phys. Sci., no. 472, Kluwer, Dordrect, 1995, pp. 1–50, MR 96m:58036, Zbl 851.35038.

[9] J.N. Corvellec, M. Degiovanni and M. Marzocchi, *Deformation properties for continuous functionals and critical point theory*, Top. Meth. Nonl. Anal. **1** (1993), 151–171, MR 94c:58026, Zbl 789.58021.

[10] M. Degiovanni and M. Marzocchi, *A critical point theory for nonsmooth functionals*, Ann. Mat. Pura Appl. (4) **167** (1994), 73–100, MR 96a:58043, Zbl 828.58006.

[11] A. Ioffe and E. Schwartzman, *Metric critical point theory* I. *Morse regularity and homotopic stability of a minimum*, J. Math. Pures Appl. **75** (1996), 125–153, MR 97d:58046, Zbl 852.58018.

[12] G. Katriel, *Mountain pass theorems and global homeomorphism theorems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), 189–209, MR 95f:58022, Zbl 834.58007.

[13] M. Squassina, *Existence of multiple solutions for quasilinear diagonal elliptic systems*, Electron. J. Diff. Eqns. (1999), no. 14, 12 pages, CMP 1 692 585, Zbl 921.35052.

[14] M. Squassina, *Weak solutions to general Euler's equations via nonsmooth critical point theory*, Ann. Fac. Sci. Toulouse, in press (2000).

[15] M. Struwe, *Quasilinear elliptic eigenvalue problems*, Comment. Math. Helvetici **58** (1983), 509–527, MR 85h:58036, Zbl 531.35035.



Dipartimento di Matematica, Università degli studi di Milano, Via C. Saldini 50, I 20133 Milano, Italy.

squassin@mat.unimi.it  http://www.dmf.bs.unicatt.it/~squassin


This paper is available via http://nyjm.albany.edu:8000/j/2000/6-6.html.