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On approximation properties of the binomial power function $(1+x^q)^r$ and allied functions

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ABSTRACT. This note concerns approximation properties of scattered translates of a fixed kernel related to the binomial power function $(1+x^q)^r$. In particular, we show that associated alternant matrices are invertible and that such functions are dense in $C[a,b]$. The techniques used may be considered non-local since they rely on interpolation centers which are chosen outside of the target domain.

CONTENTS

1. Introduction

In this paper, we consider the approximation set

$$S(\varphi, X) := \left\{ \sum_{j=1}^{N} a_j \varphi(x - x_j) \, : \, N \in \mathbb{N}, a_j \in \mathbb{R}, x_j \in X \right\},$$

where X is an appropriately chosen scattered sequence and φ is a function related to the binomial power function $(1+x^q)^r$. The most popular choice of parameters is $(q,r) = (2, 1/2)$, the Hardy multiquadric, which is due in part to its implementation properties, [5]. Its approximation properties have been well studied, see for instance [2, 3, 4, 9] among many others. These works may be classified as local, in the sense that the centers X are typically chosen to be in the domain of the target function.

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Our interest in this problem stems from [8], which shows that for the Hardy multiquadric, $S(\varphi, X)$ is dense in $C[a, b]$. The argument employed there is non-local, in that the centers are chosen to be far away from the domain of the target functions. A similar scheme was used by the second author to extend the result to $(q, r) = (2, -1)$ in [6] and $(q, r) = (2, k - 1/2)$ in [7]. Our aim is to provide a single approach which generalizes these results. We do this by introducing a straightforward admissibility criterion, which allows allied functions to enjoy the same approximation properties.

The rest of this paper is organized as follows. The next section contains various definitions and basic facts necessary to the sequel. The third section provides the main result and a short proof, while the fourth section provides two basic examples including the binomial power $(1 + x^q)^r$, which subsumes previous work on this problem. The final section is devoted to an application that extends the techniques presented to allied functions.

2. Definitions and basic facts

Throughout the sequel, we denote by $\mathbb{N}_0$ the collection of non-negative integers. We denote the space of polynomials of degree at most n by Π_n and let $\Pi := \bigcup_{n\in\mathbb{N}_0} \Pi_n$.

Definition 2.1. *A sequence of real numbers, denoted X, is said to be δ*-separated *if*

$$\inf_{\substack{x,y\in X\\ x\neq y}} |x - y| = \delta > 0$$

Note that a δ-separated sequence must be countable. This allows us to index X with the integers.

Definition 2.2. *A sequence $(x_j) \subset \mathbb{R}$ is* scattered *if it is δ-separated for some $\delta > 0$ and satisfies*

$$\lim_{j\to\pm\infty} x_j = \pm\infty.$$

Throughout the remainder of the paper we let $X = (x_j)$ be a fixed but otherwise arbitrary scattered sequence. Of use to us will be the following notion.

Definition 2.3. *$Y \subset \mathbb{R}$ is a positive (negative)* doubling sequence *if*

(1) $y_1 > 0 \quad (y_1 < 0)$*, and*
(2) $y_{j+1} \geq 2y_j \quad (y_{j+1} \leq 2y_j)$; $j \in \mathbb{N}$.

Lemma 2.4. *Every scattered sequence X contains both a positive and negative doubling subsequence. Additionally, for any $M > 0$, we can find a doubling subsequence Y such that $|y_1| > M$.*

Proof. Let $M > 0$. Since $\lim_{j\to\infty} x_j = \infty$, there exists $J \in \mathbb{Z}$ such that $x_j > M$ for $j \geq J$. Since X is δ-separated, we can find the smallest such x_j, this we call y_1. Now we can repeat this procedure for $M = 2y_1$ to produce y_2. Continuing on in this fashion produces a positive doubling subsequence $Y := (y_j)$. A negative doubling subsequence is produced in an analogous manner. $\square$

Lemma 2.5. *Suppose that X is a scattered sequence and that $(y_j) \subset X$ is a doubling subsequence, then for all $i \in \mathbb{N}$*

$$\left| \prod_{j \neq i} \left[1 - \frac{y_i}{y_j} \right]^{-1} \right| \leq 4.$$

Proof. Since $y_i / y_j > 0$ for both positive and negative doubling subsequences, it is enough to consider only positive doubling subsequences. Fix $i \in \mathbb{N}$. For $1 \leq j < i$, we have $|(1 - y_i/y_j)^{-1}| \leq 1$, hence

$$\left| \prod_{j \neq i} \left[1 - \frac{y_i}{y_j} \right]^{-1} \right| \leq \prod_{j=i+1}^{\infty} \left[1 - \frac{y_i}{y_j} \right]^{-1}.$$

To see the bound, note that for $j > i$:

$$\frac{y_i}{y_j} \leq -\ln\left(1 - \frac{y_i}{y_j}\right) \leq 2\ln(2)\frac{y_i}{y_j},$$

which follows from the convexity of the logarithm and the fact that $y_{i+1} \geq 2y_i$. Hence we have

$$\begin{aligned}
\prod_{j=1}^{\infty} \left[1 - \frac{y_i}{y_{i+j}} \right]^{-1} &= \exp\left[-\sum_{j=1}^{\infty} \ln(1 - y_i/y_{i+j}) \right] \\
&\leq \exp\left[2\ln(2) \sum_{j=1}^{\infty} y_i/y_{i+j} \right] \\
&\leq \exp\left[2\ln(2) \sum_{j=1}^{\infty} 2^{-j} \right] = 4.
\end{aligned}$$

$\square$

For a fixed φ, X, and $n \in \mathbb{N}$ we let

$$S_n(\varphi, X) := \left\{ \sum_{j=1}^{n} a_j \varphi(x - x_j) \,:\, a_j \in \mathbb{R}, x_j \in X \right\}$$

and set $S(\varphi, X) := \bigcup_{n \in \mathbb{N}} S_n(\varphi, X)$. When there is no confusion, we will drop the dependence on φ and X.

We introduce the following notion of admissibility for φ.

Definition 2.6. *Suppose that translates of φ enjoy the representation*

$$\varphi(x - y) = F(y) \sum_{k=0}^{\infty} \frac{A_k(x)}{y^k}, \tag{1}$$

where $(A_k) \subset \Pi$. *We will call a function* φ *which has such a representation* admissible *provided* $F(y)$ *is eventually non-zero.*

We will need the following results concerning alternant matrices.

The $N \times N$ Vandermonde system associated to a doubling sequence Y is $V_N \mathbf{c} = \mathbf{e}_N \in \mathbb{R}^N$, where $\mathbf{e}_N$ is the N-th standard basis vector and

$$V_N := \left[y_j^{-(i-1)} \right]_{1 \le i,j \le N}.$$

The solution may be found using Cramer's rule, namely

$$c_i = y_i^{N-1} \prod_{j \neq i} \left[1 - \frac{y_i}{y_j} \right]^{-1}. \tag{2}$$

In light of Lemma 2.5, we have

$$c_i = O(y_i^{N-1}). \tag{3}$$

For admissible φ and a doubling sequence Y, we get the related system

$$\left[F(y_j) y_j^{-i+1} \right]_{1 \le i,j \le N} \tilde{\mathbf{a}} = \mathbf{e}_N, \tag{4}$$

where F is defined in (1). Using Cramer's rule, we have

$$\tilde{a}_{N,i} = \frac{c_i}{F(y_i)}, \tag{5}$$

where c_i is defined in (2).

3. Main result

We are in position to prove efficiently our main result which allows us to derive the density of $S(\varphi, X)$ in $C[a,b]$ from the density of Π in $C[a,b]$.

Theorem 3.1. *Suppose that* X *is a scattered sequence and that* φ *is admissible. If* $(A_k : k \in \mathbb{N}_0)$ *defined in* (1) *is a basis for* Π, *then for any* $f \in C[a,b]$ *and* $\varepsilon > 0$, *there exists* $s \in S(\varphi, X)$ *such that*

$$\|f - s\|_{L_\infty} < \varepsilon.$$

Proof. In light of the Stone-Weierstrass theorem, it is enough to consider $f \in \Pi$. Using (1), we need only find $\tilde{s}_n \in S_n(\varphi, X)$ such that $\tilde{s}_n(x) = A_{n-1}(x) + O(y_1^{-1})$. For $(y_j) \subset X$, we have

$$\begin{aligned} \sum_{j=1}^{n} a_j \varphi(x - y_j) &= \sum_{j=1}^{n} a_j F(y_j) \sum_{k=0}^{\infty} \frac{A_k(x)}{y_j^k} \\ &= \sum_{k=0}^{n-1} \left(\sum_{j=1}^{n} a_j F(y_j) y_j^{-k} \right) A_k(x) + \sum_{j=1}^{n} \sum_{k=n}^{\infty} a_j F(y_j) \frac{A_k(x)}{y_j^k}. \end{aligned}$$

The first sum is a Vandermonde system, hence if $(y_k) \subset X$ is a doubling sequence and $(a_j) = (\tilde{a}_{n,j})$ is chosen as in (5) then we have

$$\sum_{j=1}^{n} \tilde{a}_{n,j}\varphi(x - y_j) = A_{n-1}(x) + O(y_1^{-1}).$$

Since (A_k) is a basis for Π and y_1 may be chosen arbitrarily large, the proof is complete. □

Applying Hölder's inequality yields the following.

Corollary 3.2. *Let $p \geq 1$ and suppose that X is a scattered sequence and that φ is admissible. If $(A_k : k \in \mathbb{N}_0)$ is a basis for Π, then for any $f \in C[a,b]$ and $\varepsilon > 0$, there exists $s \in S(\varphi, X)$ such that*

$$\|f - s\|_{L_p} < \varepsilon.$$

It may happen that $(A_k : k \geq 0)$ fails to be a basis for Π while $(A_k : k \geq K)$ is a basis for Π, in this situation the proof can be amended above by splitting the first $K + n + 1$ terms from the rest

$$\begin{aligned}\sum_{j=1}^{n} a_j\varphi(x - y_j) &= \sum_{j=1}^{n} a_jF(y_j)\sum_{k=0}^{\infty}\frac{A_k(x)}{y_j^k}\\ &= \sum_{k=0}^{K+n-1}\sum_{j=1}^{K+n} a_jF(y_j)y_j^{-k}A_k(x) + \sum_{j=1}^{n}\sum_{k=K+n}^{\infty} a_jF(y_j)\frac{A_k(x)}{y_j^k},\end{aligned}$$

now letting $a_j = \tilde{a}_{K+n-1,j}$ produces $A_{K+n-1}(x) + O(y_1^{-1})$. Since $(A_{K+n-1} : n \in \mathbb{N})$ is a basis for Π, the conclusion of Theorem 3.1 still holds. We summarize this in the following.

Theorem 3.3. *Suppose that X is a scattered sequence and that φ is admissible. If there exists $K \in \mathbb{N}_0$ such that $(A_k : k \geq K)$ is a basis for Π, then for any $f \in C[a,b]$ and $\varepsilon > 0$, there exists $s \in S(\varphi, X)$ such that*

$$\|f - s\|_{L_\infty} < \varepsilon.$$

Corollary 3.4. *Let $p \geq 1$ and suppose that X is a scattered sequence and that φ is admissible. If there exists $K \in \mathbb{N}_0$ such that $(A_k : k \geq K)$ is a basis for Π, then for any $f \in C[a,b]$ and $\varepsilon > 0$, there exists $s \in S(\varphi, X)$ such that*

$$\|f - s\|_{L_p} < \varepsilon.$$

4. Examples

Theorems 3.1 and 3.3 require that we show the sequence of polynomials (A_k) defined in (1) forms a basis for Π. Thus the bulk of the work in examples is justifying this. Throughout this section we will make use of the general binomial coefficient

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$$

and the floor and ceiling functions, which we denote by $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$, respectively. We begin with the example that motivated the treatment above.

4.1. Binomial power functions. Let $q \in \mathbb{N}$ and $r \in \mathbb{R}\backslash\{0\}$, then the binomial power function with shape parameter $c > 0$ is

$$\varphi(x) = (c + x^q)^r.$$

For simplicity, we will often let $c = 1$. We begin with the special case $q \in 2\mathbb{N}$.

Lemma 4.1. *Let $r \in \mathbb{R} \setminus \{0\}$, $q \in 2\mathbb{N}$, and for $k \in \mathbb{N}_0$ suppose A_k is defined by* (1)*. Then $A_k(x)$ is given by*

$$A_k(x) = (-1)^k \binom{qr}{k} x^k + \text{ lower order terms.}$$

Proof. For y large enough, we have

$$\begin{aligned}
\varphi(x-y) = &(c + (x-y)^q)^r \\
=&y^{qr}\left(cy^{-q} + \left(1 - \frac{x}{y}\right)^q\right)^r \\
=&y^{qr} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{r}{j} \binom{q(r-j)}{k} c^j x^k y^{-(qj+k)} \\
=&y^{qr} \sum_{j=0}^{\infty} \sum_{k=qj}^{\infty} (-1)^k \binom{r}{j} \binom{q(r-j)}{k-qj} c^j x^{k-qj} y^{-k} \\
=&y^{qr} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\lfloor k/q \rfloor} (-1)^k \binom{r}{j} \binom{q(r-j)}{k-qj} c^j x^{k-qj} \right) y^{-k}.
\end{aligned}$$

This means that

$$A_k(x) = \sum_{j=0}^{\lfloor k/q \rfloor} (-1)^k \binom{r}{j} \binom{qr-qj}{k-qj} c^j x^{k-qj}, \tag{6}$$

which is the desired result. □

Corollary 4.2. *Suppose $r \in \mathbb{R} \setminus \{0\}$ and $q \in 2\mathbb{N}$ satisfy $qr \notin \mathbb{N}$, then (A_k) is a basis for Π.*

Proof. If $qr \notin \mathbb{N}$, then the leading coefficient in (6) cannot be 0. □

Corollary 4.3. *Suppose $qr \in \mathbb{N}$. Then for A_k defined in* (1)*, we have*

$$A_k(x) = \begin{cases} (-1)^k \binom{qr}{k} x^k + \text{ lower order terms}, & 0 \le k < q\lceil r \rceil \\ \binom{r}{\lceil r \rceil} \binom{qr - q\lceil r \rceil}{k - q\lceil r \rceil} c^{\lceil r \rceil} x^{k - q\lceil r \rceil} + \text{ lower order terms}, & k \ge q\lceil r \rceil. \end{cases}$$

Hence $(A_k : k \ge q\lceil r \rceil)$ is a basis for Π.

Proof. The formula for $0 \le k < q\lceil r\rceil$, follows from the fact that $\binom{qr}{k} \neq 0$ for these k. In order to see the formula for $k \ge q\lceil r\rceil$, we note that since $r \notin \mathbb{N}_0$, $\lceil r\rceil > r$, so that the second binomial coefficient will be 0 whenever the index is less than r, whence (6) reduces to

$$A_k(x) = (-1)^k \sum_{i=\lceil r\rceil}^{\lfloor k/q\rfloor} \binom{r}{i}\binom{qr-qi}{k-qi} c^i x^{k-qi}. \tag{7}$$

□

It is natural to ask what happens if $q \in \mathbb{N}$ is odd. The argument given above is invalid for a general $r \in \mathbb{R}$ when q is odd. However, if q is odd and $qr \in \mathbb{N}$, then we more or less recover (7):

$$A_k(x) = (-1)^{k+qr} \sum_{i=\lceil r\rceil}^{\lfloor k/q\rfloor} \binom{r}{i}\binom{qr-qi}{k-qi} c^i x^{k-qi}.$$

The difference is that we must use a negative doubling sequence that is contained in our scattered sequence. Hence we have the following.

Lemma 4.4. *Suppose $r \in \mathbb{R}\setminus\{0\}$ and $q \in \mathbb{N}$ is odd. If $qr \in \mathbb{N}$, then $(A_k : k \ge q\lceil r\rceil)$ defined in* (1) *is a basis for* Π.

We summarize these results in the following.

Proposition 4.5. *Suppose that φ is a binomial power function with parameters $q \in \mathbb{N}$ and $r \in \mathbb{R}\setminus\{0\}$. Then $(A_k : k \ge K)$ defined in* (1) *is a basis for* Π *if*

(1) *$q \in 2\mathbb{N}$ and $qr \notin \mathbb{N}$ and $K = 0$, or*
(2) *$q \in 2\mathbb{N}$, $qr \in \mathbb{N}$ and $K = q\lceil r\rceil$, or*
(3) *$q \in (2\mathbb{N}-1)$, $qr \in \mathbb{N}$ and $K = q\lceil r\rceil$.*

Note that in each of the cases above, φ is admissible with $F(y) = |y|^{qr}$. We end this section by noting that this class of examples subsumes those found in earlier works. The Hardy multiquadric found in [8] is $(q,r) = (2, 1/2)$, while the examples in [6] and [7] both have $q = 2$, with $r = -1$ and $r = k - 1/2$, respectively.

4.2. Arctangent. The examples in this section are related to the binomial power functions by differentiation. We will begin with $\arctan(x)$, which satisfies

$$\arctan(x-y) = -\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{B_k(x)}{y^k},$$

hence we will let $\varphi(x) = \arctan(x) + \pi/2$. Since the derivative of φ is the Poisson kernel, we can use the polynomial sequence (A_k) defined recursively in [6] to calculate (B_k). In fact, Proposition 1 in [6] gives us

$$A_k(x) = (k+1)x^k + \text{lower order terms},$$

thus, for $k \in \mathbb{N}$ we have

$$\begin{aligned} B_k(x) &= \frac{A_{k-1}(x)}{k} \\ &= x^{k-1} + \text{ lower order terms.} \end{aligned}$$

We can use (6), to generate the polynomial if we need the lower order terms. Hence φ is admissible with $F(y) = y^{-1}$, we have

$$\varphi(x-y) = y^{-1} \sum_{k=0} \frac{B_{k+1}(x)}{y^k}.$$

Hence Corollary 4.2 provides that $(B_k : k \geq 1)$ is a basis for Π.

We can combine examples using the Cauchy product. For instance,

$$\varphi(x) = (1+x^q)^r \left(\arctan(x) + \frac{\pi}{2}\right), \tag{8}$$

leads to the following.

Lemma 4.6. *For φ in* (8)*, $q \in 2\mathbb{N}$, $r \in \mathbb{R} \setminus \{0\}$, and $qr \notin \mathbb{N}$,*

$$\varphi(x-y) =: y^{qr-1} \sum_{k=0}^{\infty} \frac{C_k(x)}{y^k},$$

we have

$$C_k(x) = (-1)^k \binom{qr-1}{k} x^k + \text{lower order terms}; \quad k \geq 1.$$

Hence φ is admissible with $F(y) = y^{qr}$ and $(C_k : k \geq 1)$ is a basis for Π.

Before proving this we need the following summation formula.

Lemma 4.7. *Suppose that $u \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{N}_0$, then*

$$\sum_{j=0}^{k} (-1)^j \binom{u}{j} = (-1)^k \binom{u-1}{k}, \tag{9}$$

where $\binom{u}{j}$ is the general binomial coefficient.

Proof. We fix $u \in \mathbb{R} \setminus \{0\}$ and induct on $k \in \mathbb{N}_0$. When $k = 0$, there is nothing to show since both sides are 1. Now suppose that the formula holds for some $k \geq 0$ and consider

$$\begin{aligned} \sum_{j=0}^{k+1} (-1)^j \binom{u}{j} &= (-1)^{k+1} \binom{u}{k+1} + \sum_{j=0}^{k} (-1)^j \binom{u}{j} \\ &= (-1)^{k+1} \binom{u}{k+1} + (-1)^k \binom{u-1}{k} \\ &= (-1)^{k+1} \binom{u-1}{k+1}, \end{aligned}$$

which is the desired formula. □

Proof of Lemma 4.6. All we need to do is to use the series representation for each piece of the product. This yields

$$C_k(x) = \sum_{j=0}^{k}(-1)^j\binom{qr}{j}x^k + \text{ lower order terms.}$$

Now (9) provides the desired result. □

5. An extended example

Due to the relative simplicity of the previous example, it is natural to investigate $\ln(1 + x^2)$ since is also linked to the binomial power functions via differentiation. Unlike the previous example, however, this requires an updated version of the system (4) in order to prove a result similar to Theorem 3.1.

Lemma 5.1. *Let $\varphi(x) := x^{-1}\ln(1 + x^2)$. Then $\varphi(x - y)$ may be expanded in a series*

$$\varphi(x-y) = \ln|y|\sum_{j=1}^{\infty}\frac{A_j(x)}{y^j} + \sum_{k=2}^{\infty}\frac{B_k(x)}{y^k},$$

where (A_j) and (B_k) are given by the formulas

$$A_j(x) = -2x^{j-1}; \quad j \geq 1, \quad \textit{and}$$
$$B_k(x) = \left(\sum_{n=1}^{k-1}\frac{2}{n}\right)x^{k-1} + \textit{lower order terms}; \quad k \geq 2.$$

Proof. The main tool here is the Cauchy product as well as the results of [6]. We have

$$\begin{aligned}\frac{d}{dy}\ln(1+(x-y)^2) &= \frac{-2(x-y)}{1+(x-y)^2}\\ &= -2(x-y)\sum_{j=0}^{\infty}\frac{\tilde{A}_j(x)}{y^{j+2}},\end{aligned}$$

where $\tilde{A}_j$ are the polynomials corresponding to the Poisson kernel. We find these polynomials defined recursively in [6], where they are called (A_j). We note, however, that they may also be computed directly from (6). Regrouping yields

$$\begin{aligned}\frac{d}{dy}\ln(1+(x-y)^2) &= \frac{2\tilde{A}_0(x)}{y} + \sum_{j=2}^{\infty}\frac{2\tilde{A}_{j-1}(x) - 2x\tilde{A}_{j-2}(x)}{y^j}\\ &= \frac{2}{y} + \sum_{j=2}^{\infty}\frac{C_j(x)}{y^j},\end{aligned}$$

where $C_j(x) = 2x^{j-1}+$ lower order terms. Now integrating yields

$$\ln(1 + (x - y)^2) = 2 \ln |y| - \sum_{j=1}^{\infty} \frac{C_{j+1}(x)}{j y^j}$$

and since

$$(x - y)^{-1} = -y^{-1} \sum_{j=0}^{\infty} \frac{x^j}{y^j},$$

the Cauchy product can be employed here. This produces

$$\begin{aligned}\varphi(x - y) &= \ln |y| \sum_{j=1}^{\infty} \frac{-2x^{j-1}}{y^j} + \sum_{j,k=1}^{\infty} \frac{x^{j-1} C_{k+1}}{k y^{j+k}} \\ &= \ln |y| \sum_{j=1}^{\infty} \frac{-2x^{j-1}}{y^j} + \sum_{k=2}^{\infty} \frac{B_k(x)}{y^k},\end{aligned}$$

where

$$B_k(x) = \left(\sum_{j=1}^{k-1} \frac{2}{n} \right) x^{k-1} + \text{ lower order terms,}$$

since it is clear above that $A_j(x) = -2x^{j-1}$, the proof is complete. $\square$

In this example, φ is *not* admissible with a single function $F(y)$. Attempting to isolate A_N as in the proof of Theorem 3.1 leads us to a more general $(2N - 1) \times (2N - 1)$ alternant system

$$\begin{bmatrix} y_1^{-2} & y_2^{-2} & \cdots & y_{2N-1}^{-2} \\ \vdots & \vdots & & \vdots \\ y_1^{-N} & y_2^{-N} & \cdots & y_{2N-1}^{-N} \\ y_1^{-1} \ln(y_1) & y_2^{-1} \ln(y_2) & \cdots & y_{2N-1}^{-1} \ln(y_{2N-1}) \\ \vdots & \vdots & & \vdots \\ y_N^{-1} \ln(y_1) & y_2^{-N} \ln(y_2) & \cdots & y_{2N-1}^{-N} \ln(y_{2N-1}) \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \tilde{a}_{2N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{10}$$

where $N \geq 2$. We will be able to recover A_N provided we can show two things. The first is that this system always has a solution for a suitably chosen doubling sequence $Y = (y_j)$. The second piece of information we need is the growth rate of the $\tilde{a}_i$. The solution of this system is related to the rational interpolation problem for logarithmic data samples. Solvability of the rational interpolation problem is characterized by the invertibility of a certain Löwner matrix [1]. We find it more convenient, however, to solve these problems by appealing to properties of the logarithm and its derivatives. In order to do this, we will need the following general framework associated to alternant matrices.

Suppose we have a set of continuous functions $G := \{g_1, g_2, \dots, g_N\}$, where for $1 \leq j \leq N$, $g_j : I \to \mathbb{R}$ for some interval $I \subset \mathbb{R}$. Define $\mathcal{G} := \operatorname{span}\{g_1, \dots, g_N\}$ and for $f \in \mathcal{G}$, let $f^\sharp$ denote the number of roots that f has on I, and $\mathcal{G}^\sharp = \sup_{f \in \mathcal{G} \setminus \{\mathbf{0}\}} f^\sharp$. Our first result is straightforward.

Lemma 5.2. *Let $N \in \mathbb{N}$ and suppose that G satisfies $\mathcal{G}^\sharp < N$ and $X = (x_i : 1 \leq i \leq N) \subset I$ consists of N distinct points. Then the alternant matrix*

$$A(G, X) := \left[g_j(x_i)\right]_{1 \leq i, j \leq N}$$

is invertible.

Proof. Consider the product $A(G,X)\mathbf{a}$ in the variable $\mathbf{a}$. This results in the vector $\mathbf{v} \in \mathbb{R}^N$, whose i-th component is given by $f(x_i)$, where $f \in \mathcal{G}$. Now suppose that $A(G,X)$ is non-invertible. Then the homogeneous system $A(G,X)\mathbf{a} = \mathbf{0}$ has a non trivial solution $\mathbf{a}_0$, which leads to $f_0 \in \mathcal{G}$ that has N roots on I. This contradicts the fact that $\mathcal{G}^\sharp < N$, which shows that $A(G,X)$ must be invertible. □

The following may help us calculate $\mathcal{G}^\sharp$.

Lemma 5.3. *Suppose that $f \in C^1(I)$ and that f' has N distinct roots in I. Then f has at most $N+1$ roots in I.*

Proof. We partition I into $N+1$ subintervals with the roots of f' as endpoints. Since $f \in C^1(I)$, f is monotone on each subinterval, so that there are at most $N+1$ roots of f. □

In light of (10), we wish to show that

$$G := \{x^{-2}, \dots, x^{-N}, x^{-1}\ln(x), \dots, x^{-N}\ln(x)\}$$

satisfies $\mathcal{G}^\sharp < 2N-1$. In fact, we will show that the set $\mathcal{H} = \{p(x) + q(x)\ln(x) : p \in \Pi_{N-2}, q \in \Pi_{N-1}\}$ satisfies $\mathcal{H}^\sharp < 2N-1$, which implies the bound for $\mathcal{G}^\sharp$. We make use of the following derivative formulas, which may be easily verified via induction:

$$D^k\left(x^k \ln(x)\right) = k!\ln(x) + C_k; \quad k \in \mathbb{N} \tag{11}$$

for some positive constant C_k, and

$$D^{k+1}\left(x^k \ln(x)\right) = k! x^{-1}; \quad k \in \mathbb{N}_0. \tag{12}$$

Lemma 5.4. *Let $N \geq 2$ and suppose that $p \in \Pi_{N-1}$, with*

$$p(x) = \sum_{k=0}^{N-1} a_k x^k.$$

Then

$$D^N\left(p(x)\ln(x)\right) = x^{-N} \sum_{j=0}^{N-1} (-1)^{N-1+j} c_j a_j x^j,$$

for some positive constants c_j.

Proof. We induct on $N \geq 2$. Two applications of the product rule yields the base case:

$$((ax+b)\ln(x))'' = \frac{ax-b}{x^2}.$$

Now we assume that the conclusion holds for all k with $2 \le k \le N$. Consider $p \in \Pi_N$. We have

$$p(x)\ln(x) = \big(a_N x^N + q(x)\big)\ln(x),$$

so that

$$\begin{aligned}
D^{N+1}\left(p(x)\ln(x)\right) &= a_N D^N\left(x^N \ln(x)\right) + D^{N+1}\left(q(x)\ln(x)\right)\\
&= a_N D^{N+1}\left(x^N \ln(x)\right) + D\left(x^{-N}\sum_{j=0}^{N-1}(-1)^{N-1+j}c_j a_j x^j\right)\\
&= N! a_N x^{-1} + \sum_{j=0}^{N-1}(-1)^{N-1+j}c_j(j-N)a_j x^{j-N-1}\\
&= N! a_N x^{-1} + \sum_{j=0}^{N-1}(-1)^{N+j}c_j(N-j)a_j x^{j-N-1}\\
&= x^{-N-1}\left(N! a_N x^N + \sum_{j=0}^{N-1}(-1)^{N+j}c_j(N-j)a_j x^{j-N-1}\right)\\
&= x^{-N-1}\sum_{j=0}^{N}(-1)^{N+j}\tilde{c}_j a_j x^j.
\end{aligned}$$

We have used (12) in the third line. The result follows from the fact that $c_j > 0$ and $N - j > 0$, so that $\tilde{c}_j > 0$. □

Suppose that $f(x) = p(x) + \sum_{j=0}^{N-1} a_j x^j \ln x \in \mathcal{H}$. In order to count the roots of f, we first count the roots of $D^N f$, then repeatedly use Lemma 5.3. A preliminary bound may be found by using (12) and Descartes's rule of signs, which shows that $D^N f$ has at most $N-1$ roots *only* when all of the coefficients share the same sign. Assuming this is true, Lemma 5.4 allows us to conclude that f has at most $2N-1$ roots. In order to use Lemma 5.2, we must improve this bound.

Lemma 5.4 suggests that overall number of roots decreases when we introduce more sign changes in the coefficients, hence to improve our preliminary bound, we need only consider when there are 0 or 1 sign changes among the coefficients (a_j). Assuming all (a_j) are positive, we have, using (11),

$$
\begin{aligned}
D^{N-1}f(x) =& D^{N-1}\left(\ln x \sum_{j=0}^{N-1} a_j x^j\right)\\
=&(N-1)!a_{N-1}\ln x + C_{N-1} + x^{-N+1}\sum_{j=0}^{N-2}(-1)^{N+j}c_j a_j x^j\\
=&x^{-N+1}\left(x^{N-1}((N-1)!a_{N-1}\ln x + C_{N-1}) + \sum_{j=0}^{N-2}(-1)^{N+j}c_j a_j x^j\right)\\
=:& \; x^{-N+1}g(x).
\end{aligned}
$$

Again from (11), we can see that $D^{N-1}g(x) > 0$ for $x \geq 1$, so $D^{N-1}f$ has at most $N-1$ roots, hence f has at most $2N-2$ roots in this case. The same would be true if we took all of the coefficients negative.

Next, suppose that there is a sign change in the coefficients (a_j), then using Lemma 5.4 and Descartes's rule of signs again provide at most $N-2$ roots for $D^N f$, so Lemma 5.3 shows that f has at most $2N-2$ roots. Thus $\mathcal{H}^\sharp \leq 2N-2$. Hence Lemma 5.2 shows that for $N \geq 2$, we can solve the system (10) to isolate $A_N(x)$.

Now we establish a bound for the growth rate of the solution components $\tilde{a}_i$. Using Cramer's rule, we find an upper bound for the cofactor A_i in the numerator and a lower bound for the determinant of A. For a doubling sequence (y_j), we have for positive constants α, β, and γ

$$\alpha y_i^{-(N^2+N-1)} \leq |\det A| \leq \beta y_i^{-(N^2+N-1)} (\ln y_i)^N$$

and

$$|\det A_i| \leq \gamma y_i^{-(N^2-1)} (\ln y_i)^{N-1},$$

so that

$$|\tilde{a}_i| \leq C y_i^N (\ln y_i)^{N-1}. \tag{13}$$

If there is additional structure, for instance $X = \mathbb{Z}$, then we can get a sharper bound, but for our purposes, this is not necessary. Since (A_k) form a basis for Π, we have all of the necessary tools to prove a version of Theorem 3.1, which we write as a proposition.

Proposition 5.5. *Suppose that X is a scattered sequence and that $\varphi(x) = x^{-1}\ln(1+x^2)$. For any $f \in C[a,b]$ and $\varepsilon > 0$, there exists $s \in S(\varphi, X)$ such that*

$$\|f - s\|_{L_\infty} < \varepsilon.$$

Proof. The proof is nearly identical to the one given for Theorem 3.1 provided that $N \geq 2$. We note that since we first choose a polynomial p using the Stone-Weierstrass theorem, we can fix $N = \deg(p)$. Now just as before, we recover

$(A_j : 1 \leq j \leq N)$. We may recover A_1 with the 1×1 matrix, which produces an error term that is $O(y_1^{-1})$. For $j \geq 2$, the corresponding error term is $O(y_1^{-1} \ln(y_1)^j)$ rather than $O(y_1^{-1})$. This means that there exists (a_j) such that

$$\begin{aligned} \sum_{j=1}^{N} a_j \varphi(x - y_j) - p(x) &= O\left(y_1^{-1}\right) + O\left(\sum_{j=2}^{N} \frac{\ln(y_1)^j}{y_1}\right) \\ &= O\left(\frac{\ln(y_1)^N}{y_1}\right). \end{aligned}$$

Hence we can now choose y_1 so large that the error term is as small as we like. $\square$

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