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Free semigroupoid algebras from categories of paths

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ABSTRACT. Given a directed graph G, we can define a Hilbert space $\mathcal{H}_G$ with basis indexed by the path space of the graph, then represent the vertices of the graph as projections on $\mathcal{H}_G$ and the edges of the graph as partial isometries on $\mathcal{H}_G$. The weak operator topology closed algebra generated by these projections and partial isometries is called the free semigroupoid algebra for G. Kribs and Power showed that these algebras are reflexive, and that they are semisimple if and only if each path in the graph lies on a cycle. We extend the free semigroupoid algebra construction to categories of paths, which are a generalization of graphs, and provide examples of free semigroupoid algebras from categories of paths that cannot arise from graphs or higher rank graphs. We then describe conditions under which these algebras are semisimple, and we prove reflexivity for a class of examples.

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1. Acknowledgement

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2. Introduction

A directed graph is a set of vertices along with a set of edges, where each edge has a source vertex and a range vertex. Such a graph can be represented by a

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collection of operators on a Hilbert space $\mathcal{H}$; each vertex is associated to a projection, and each edge is associated to a partial isometry that maps between the subspaces corresponding to its source and range vertices. These projections and partial isometries are used to construct a C^*-algebra called the *graph C^*-algebra* of the directed graph. There are many examples of common C^*-algebras which can be realized as graph algebras, and many properties of graph algebras are determined by structural properties of the graph. C^*-algebras are self-adjoint, however, so this is not a useful construction for studying non-self-adjoint operator algebras.

Free semigroupoid algebras generated by directed graphs are a class of non-self-adjoint operator algebras introduced by Kribs and Power in 2004 [6]. The construction of these algebras from a graph is similar to the graph C^*-algebra construction in that vertices are represented by projections and edges by partial isometries. However, a free semigroupoid algebra is closed in the weak operator topology, not the norm topology, and does not include adjoints.

As in the graph C^*-algebra case, many previously-studied non-self-adjoint operator algebras can be expressed as free semigroupoid algebras for some directed graph, and many properties of the algebra correspond to properties of the graph. In fact, this relationship is in some sense stronger than the self-adjoint case; while it is possible to find two non-isomorphic graphs that produce the same graph C^*-algebra, Kribs and Power [6] showed that two free semigroupoid algebras from graphs are unitarily equivalent if and only if their corresponding graphs are isomorphic.

In addition to this isomorphism result, Kribs and Power characterized semisimplicity for free semigroupoid algebras from graphs and proved that all free semigroupoid algebras from graphs are reflexive. In another paper on the subject [7], they extended the free semigroupoid algebra construction to higher rank graphs, which are a generalization of graphs where edges have length in $\mathbb{N}^k$ and satisfy a certain factorization property. Kribs and Power then proved the same semisimplicity result, and a slightly more limited reflexivity result, for free semigroupoid algebras from higher rank graphs. See [3] for an overview and examples of C^*-algebras and free semigroupoid algebras from graphs and higher rank graphs.

There is another generalization of graphs introduced by Spielberg [13], called categories of paths, which include higher rank graphs, but also other examples without the restrictive higher rank graph factorization property. In this paper, we study free semigroupoid algebras generated by categories of paths (usually assuming a degree functor) and determine how they are similar to and how they can differ from the graph and higher rank graph cases.

In Section 3 of this paper, we look at how the free semigroupoid algebra from a category of paths is defined and show that, under the assumption of a degree functor, the same characterization of the commutant holds from the graph case. In Section 4, we provide some examples of free semigroupoid algebras that arise from this construction and which are not isomorphic to free semigroupoid algebras from graphs.

In Section 5, we study semisimplicity for free semigroupoid algebras of categories of paths with degree functors. We introduce a condition (P) on a category of paths with a degree functor. This condition has two parts: the first is similar to row-finiteness in a graph; the second is a restriction on which elements of the algebra can be nilpotent, which is similar to, but more general than, the requirement that all paths lie on a cycle. We show that the free semigroupoid algebra of a category of paths satisfying (P) is semisimple. We then employ this result to show that the single-vertex examples from Section 4 are semisimple.

Finally, in Section 6, we examine reflexivity for free semigroupoid algebras from categories of paths. We define a Double Pure Cycle Property and show that if the transpose of a category of paths with a non-degenerate degree functor satisfies this property, then the free semigroupoid algebra of the category of paths is reflexive. We also establish reflexivity for a family of single-vertex categories of paths.

3. Definition and basic properties

The following definition of a category of paths is due to Spielberg [13]. Recall that a *small category* Λ is a set of objects Λ^0 and morphisms between the objects, along with two maps: a source map $s : \Lambda \to \Lambda^0$ sending each morphism to its source, and a range map $r : \Lambda \to \Lambda^0$ sending each morphism to its range.

Definition 3.1 ([13], Definition 2.1)**.** *A small category Λ is called a category of paths if, for $\alpha, \beta, \gamma \in \Lambda$,*

- *$\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$ (left cancellation)*
- *$\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$ (right cancellation)*
- *$\alpha\beta = s(\beta)$ implies $\alpha = \beta = s(\beta)$ (no inverses)*

We call the objects of Λ vertices.

Directed graphs are an example of categories of paths. Another example is higher-rank graphs:

Example 3.2. A higher rank graph is a category of paths Λ with a degree function $d : \Lambda \to \mathbb{N}^k$ satisfying the factorization property that for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there are unique paths $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$. See [8] for the original introduction of higher rank graphs, and [10] for a good overview. For λ in a higher rank graph, we will write $|\lambda|$ to mean $|d(\lambda)|$, i.e. the sum of the components of $d(\lambda)$ in $\mathbb{N}^k$.

There are also many categories of paths which are not higher rank graphs, a few of which we will consider in Section 4.

Let Λ be a category of paths. The free semigroupoid algebra for Λ is defined analogously to the free semigroupoid algebra for a graph or higher rank graph, as in [6] and [7]. Specifically, we define a Fock space Hilbert space $\mathcal{H}_\Lambda$ with orthonormal basis $\{\xi_\mu\}_{\mu\in\Lambda}$ indexed by the elements of Λ. For $\mu, \nu \in \Lambda$, define:

$$L_\mu \xi_\nu = \begin{cases} \xi_{\mu\nu} & \text{if } s(\mu) = r(\nu) \\ 0 & \text{else} \end{cases}.$$

If $x \in \Lambda^0$ is a vertex of Λ, then L_x is a projection. Note that $\sum_{x\in\Lambda^0} L_x = I$.

Definition 3.3. *The WOT-closed algebra generated by $\{L_\mu\}_{\mu\in\Lambda}$ is called the free semigroupoid algebra for Λ and is written $\mathfrak{L}_\Lambda$.*

It is useful to have a notion of the length of a path in a category of paths. A *degree functor* on Λ is a function $\varphi : \Lambda \to \mathbb{N}^n$ such that for all $\mu, \nu \in \Lambda$ satisfying $s(\mu) = r(\nu)$:

$$\varphi(\mu\nu) = \varphi(\mu) + \varphi(\nu).$$

A degree functor can be defined into any abelian group (see [13], Section 9), but we will only consider degree functors into $\mathbb{N}^n$.

We say the degree functor is *non-degenerate* if $\varphi(\alpha) \neq 0$ when $\alpha \notin \Lambda^0$. If Λ is a category of paths with a degree functor, define the length of a path μ to be $|\mu| = |\varphi(\mu)|$, i.e., the sum of the components of $\varphi(w) \in \mathbb{N}^n$.

Remark 3.4. For a vertex x in a category of paths, $xx = x$, and thus for any degree functor φ, we have $\varphi(x) + \varphi(x) = \varphi(x)$. Therefore, each vertex has degree 0.

Definition 3.5. *For a category of paths Λ with a non-degenerate degree functor $\varphi : \Lambda \to \mathbb{N}^n$, let E_ℓ be the projection onto span$\{\xi_\mu : |\mu| = \ell\}$. Define the Cesaro sums of $A \in \mathcal{B}(\mathcal{H})$ by, for $k \in \mathbb{Z}$,*

$$\Sigma_k(A) = \sum_{j\in\mathbb{Z},\ |j|<k} \left(1 - \frac{|j|}{k}\right)\Phi_j(A),$$

where

$$\Phi_j(A) = \sum_{\ell\in\mathbb{Z},\ \ell\,\geq\,\max\{0,-j\}} E_\ell A E_{\ell+j}.$$

The Cesaro sums converge SOT to A as in Lemma 1.1 of [4] (the details of the argument are written out as Proposition 2.3.2 in [2]).

Given a category of paths Λ and $\mu \in \Lambda$, let $\tilde{\mu}$ be the path μ oriented in the opposite direction, i.e., $s(\mu) = r(\tilde{\mu})$ and $r(\mu) = s(\tilde{\mu})$. Note that if $\mu = \nu_1\nu_2$, then $\tilde{\mu} = \tilde{\nu}_2\tilde{\nu}_1$. With this, we can define a new collection of linear operators.

Definition 3.6. *Given $\mu \in \Lambda$, define the operator $R_{\tilde{\mu}}$ by*

$$R_{\tilde{\mu}} \xi_\nu = \begin{cases} \xi_{\nu\mu} & \textit{if } r(\mu) = s(\nu) \\ 0 & \textit{else} \end{cases}.$$

Let $\mathfrak{R}_\Lambda$ be the WOT-closed algebra generated by $\{R_{\tilde{\mu}}\}_{\mu\in\Lambda}$.

Let $\Lambda^t = \{\tilde{\mu} \,:\, \mu \in \Lambda\}$ be the category of paths with the same vertex set as Λ, but with all paths are oriented in the opposite direction. This is called the transpose of Λ.

The following two results are stated without proof, as they follow from the same proofs as in the graph case. In particular, Lemma 3.7 corresponds to Lemma 4.1 in [6], and Proposition 3.8 corresponds to Proposition 4.2 and Corollary 4.4 in [6].

Lemma 3.7. *Let Λ be a category of paths with a non-degenerate degree functor. The algebras $\mathfrak{L}_\Lambda$ and $\mathfrak{R}_{\Lambda^t}$ are unitarily equivalent via the map $W : \mathcal{H}_{\Lambda^t} \to \mathcal{H}_\Lambda$ given by $W\xi_{\tilde{\mu}} = \xi_\mu$.*

Proposition 3.8. *Let Λ be a category of paths with a non-degenerate degree functor. Then $\mathfrak{R}'_\Lambda = \mathfrak{L}_\Lambda$ and $\mathfrak{L}'_\Lambda = \mathfrak{R}_\Lambda$.*

Remark 3.9. As in Remark 4.3 in [6], this gives us a Fourier expansion for elements of $\mathfrak{L}_\Lambda$ as follows: let A be in $\mathfrak{L}_\Lambda$ and x a vertex. Then there are constants $\{a_w\}_{w\in\Lambda}$ such that

$$A\xi_x = AL_x\xi_x = R_x(AL_x)\xi_x = \sum_{s(w)=x} a_w\xi_w.$$

So for $\mu \in \Lambda$ with $r(\mu) = x$,

$$A\xi_\mu = AR_{\tilde{\mu}}\xi_x = R_{\tilde{\mu}}A\xi_x = \sum_{s(w)=x} a_w\xi_{w\mu}.$$

Thus, the Cesaro partial sums associated with the series $\sum_{w\in\Lambda} a_wL_w$ converge in the strong operator topology to A.

Finally, we end this section with a lemma that will be useful for Example 4.1:

Lemma 3.10. *Let Λ be a category of paths with a non-degenerate degree functor and a finite number of vertices, $|\Lambda^0| = n < \infty$. Then the number of projections in $\mathfrak{L}_\Lambda$ is 2^n.*

Proof. Let $P \in \mathfrak{L}_\Lambda$ be a non-zero projection, with Fourier expansion $P \sim \sum_{w\in\Lambda} a_wL_w$. Then for each vertex $x \in \Lambda^0$, either $P\xi_x = 0$ or

$$\xi_x = P\xi_x = \sum_{s(w)=x} a_w\xi_w.$$

In the latter case, $a_x = 1$ and for all other w such that $s(w) = x$ we have $a_w = 0$. So $P = \sum_{x\in\Lambda^0} a_xL_x$ where each a_x is either 1 or 0.

Thus, every projection on $\mathfrak{L}_\Lambda$ is a sum of projections of the form L_x for a vertex x. Since every such sum is a projection, this means Λ has exactly 2^n projections. $\square$

4. Examples

Example 4.1. Consider the category of paths Λ given by the graph

$$\begin{tikzcd} x_1 \arrow[r, bend left, "b_1"] \arrow[r, bend right, "a_1"'] & x_2 \arrow[r, bend left, "b_2"] \arrow[r, bend right, "a_2"'] & x_3 \end{tikzcd}$$

with the identifications $a_2b_1 = b_2a_1$ and $a_2a_1 = b_2b_1$, but $a_1 \neq b_1$ and $a_2 \neq b_2$ (Example 2.9, [9]). There is no degree function that makes Λ a higher rank graph. To see this, suppose Λ were a higher rank graph with degree function d and let $d(a_1) = n$ and $d(a_2) = m$. Then $a_2b_1 = b_2a_1$ implies

$$m + d(b_1) = d(b_2) + n. \tag{1}$$

Likewise, $a_2a_1 = b_2b_1$ implies

$$m + n = d(b_2) + d(b_1). \tag{2}$$

Solving for m in Equation (1) and substituting into Equation (2) gives

$$d(b_2) + n - d(b_1) + n = d(b_2) + d(b_1),$$

implying $d(b_1) = n$. Substituting this into Equation (1) gives us $d(b_2) = m$. But this contradicts the uniqueness part of the factorization property for higher rank graphs, because we have a single path $\lambda = a_2b_1 = b_2a_1$ which can be decomposed as a path of degree m concatenated with a path of degree n in two different ways.

Thus, Λ is not a higher rank graph. However, it is a category of paths, with degree functor equal to the number of edges in a path. This category of paths has three vertices (x_1, x_2, x_3), four paths of degree 1 (a_1, b_1, a_2, b_2), and two paths of degree 2 (a_2a_1, a_2b_1). The free semigroupoid algebra for Λ is the subalgebra of $M_9(\mathbb{C})$ generated by operators of the form

$$\nu_1 L_{x_1} + \nu_2 L_{x_2} + \nu_3 L_{x_3} + \alpha_1 L_{a_1} + \beta_1 L_{b_1} + \alpha_2 L_{a_2} + \beta_2 L_{b_2} + \gamma_1 L_{a_2a_1} + \gamma_2 L_{a_2b_1},$$

or, in matrix form corresponding to the ordered basis $\{\xi_{x_1}, \xi_{x_2}, \xi_{x_3}, \xi_{a_1}, \xi_{b_1}, \xi_{a_2}, \xi_{b_2}, \xi_{a_2a_1}, \xi_{a_2b_1}\}$:

$$\begin{bmatrix} \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & \nu_2 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & \nu_2 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & \nu_3 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 & \nu_3 & 0 & 0 \\ \gamma_1 & 0 & 0 & \alpha_2 & \beta_2 & 0 & 0 & \nu_3 & 0 \\ \gamma_2 & 0 & 0 & \beta_2 & \alpha_2 & 0 & 0 & 0 & \nu_3 \end{bmatrix}.$$

Proposition 4.2. *The subalgebra of $M_9(\mathbb{C})$ given by matrices of the above form cannot arise as the free semigroupoid algebra of a higher rank graph.*

Proof. Suppose Λ' is a higher rank graph such that $\mathfrak{L}_{\Lambda'}$ consists of matrices of the above form. For $\eta \in \{\nu_1, \nu_2, \nu_3, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\}$, let T_η be the operator given by setting $\eta = 1$ and all the other variables to 0. Then we can see that $\mathfrak{L}_{\Lambda'}$ has eight projections: $0, T_{\nu_1}, T_{\nu_2}, T_{\nu_3}, T_{\nu_1} + T_{\nu_2}, T_{\nu_1} + T_{\nu_3}, T_{\nu_2} + T_{\nu_3}$, and I. By Lemma 3.10, Λ' must have three vertices y_1, y_2, and y_3. Furthermore, the minimal projections must be those that correspond to projections associated to single vertices, so, without loss of generality, $T_{\nu_1} = L_{y_1}$, $T_{\nu_2} = L_{y_2}$, and $T_{\nu_3} = L_{y_3}$, and thus the first three basis vectors in this matrix form are ξ_{y_1}, ξ_{y_2}, and ξ_{y_3}.

Now for $i = 1, 2, 3$, let P_i be the projection onto $\operatorname{span}(\xi_{y_i})$. We can see from the first two columns of the matrix form that $P_3\mathfrak{L}_{\Lambda'}P_1$, $P_2\mathfrak{L}_{\Lambda'}P_1$ and $P_3\mathfrak{L}_{\Lambda'}P_2$ each have two-dimensional range. So there are exactly two paths from y_1 to y_3, two paths from y_1 to y_2, and two paths y_2 to y_3. Since the matrix is finite-dimensional, there can be no paths from y_2 to y_1 or from y_3 to y_2 or y_1. So the graph looks like

$$y_1 \rightrightarrows y_2 \rightrightarrows y_3$$

with two identifications among the paths from y_1 to y_3. As argued above, there is no degree functor that makes such a graph a higher-rank graph. Thus, the matrix cannot correspond to the free semigroupoid algebra of a higher-rank graph. □

Example 4.3. Let Λ_2 be the category with one vertex x, two edges e and f, and the identification $e^2 = f^2$:

$$e \circlearrowleft x \circlearrowright f$$

Any path in Λ_2 can be written as a concatenation of e's and f's, and since $e^2 f = f^3 = f e^2$, it follows that e^2 commutes with every other path. Thus, each path in Λ_2 can be written uniquely in the standard form $e^r (fe)^s f^t$, where $r, s \in \mathbb{N} \cup \{0\}$, $t \in \{0, 1\}$ and $e^0 = (fe)^0 = f^0 = x$.

Proposition 4.4. *The category Λ_2 described above is a category of paths.*

Proof. First, since the length of a path always increases when concatenated with e or f, the category has a degree functor equal to the length of the path and has no inverses.

To see that cancellation holds in this category, let $\alpha, \beta \in \Lambda_2$. We can write α and β in standard form, $\alpha = e^{r_1}(fe)^{s_1} f^{t_1}$ and $\beta = e^{r_2}(fe)^{s_2} f^{t_2}$.

If $e\alpha = e\beta$, then

$$e^{r_1+1}(fe)^{s_1} f^{t_1} = e^{r_2+1}(fe)^{s_2} f^{t_2},$$

so by the uniqueness of the standard form, $r_1 = r_2, s_1 = s_2$, and $t_1 = t_2$. So $\alpha = \beta$.

Now suppose $f\alpha = f\beta$. Note that in addition to the standard form where all f^2 are converted to e^2 and moved all the way to the left, we also have an alternate standard form where all e^2 are converted to f^2 and moved left, giving each path a unique form $f^r (ef)^s e^t$ where $r, s \in \mathbb{N} \cup \{0\}$, $t \in \{0, 1\}$. We can write

α and β in this form, say $\alpha = f^{r_3}(ef)^{s_3}e^{t_3}$ and $\beta = f^{r_4}(ef)^{s_4}e^{t_4}$. Then $f\alpha = f\beta$ implies

$$f^{r_3+1}(ef)^{s_3}e^{t_3} = f^{r_4+1}(ef)^{s_4}e^{t_4},$$

so by the uniqueness of the alternate standard form, $r_3 = r_4$, $s_3 = s_4$, and $t_3 = t_4$. So $\alpha = \beta$.

This proves left cancellation. A similar argument using standard forms shows right cancellation. $\square$

Note that, in a graph with edges e and f, the operators L_e and L_f always have orthogonal ranges. However, in this example, L_e and L_f do not have orthogonal ranges, since $L_e(\xi_e) = \xi_{e^2} = L_f(\xi_f)$. The path space of Λ_2 can be expressed by a tree diagram as follows:

$$\begin{tikzcd}
x \arrow[d] \arrow[dr] & & & & \\
e \arrow[d] \arrow[dr] & f \arrow[d] \arrow[dr] & & & \\
fe \arrow[d] \arrow[dr] & e^2 \arrow[d] \arrow[dr] & ef \arrow[d] \arrow[dr] & & \\
efe \arrow[d] \arrow[dr] & e^3 \arrow[d] \arrow[dr] & e^2f \arrow[d] \arrow[dr] & fef \arrow[d] \arrow[dr] & \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Notice that the path space of the *graph* with one vertex and two edges has 2^n paths of length n for each n, whereas Λ_2 has only $n + 1$ paths of length n for each n.

Define Hilbert spaces based on the rows of the tree diagram:

$$\begin{aligned}
H_0 &= \operatorname{span}\{\xi_x\}\\
H_1 &= \operatorname{span}\{\xi_e, \xi_f\}\\
H_2 &= \operatorname{span}\{\xi_{fe}, \xi_{e^2}, \xi_{ef}\}\\
H_3 &= \operatorname{span}\{\xi_{efe}, \xi_{e^3}, \xi_{e^2f}, \xi_{fef}\}\\
&\vdots
\end{aligned}$$

Each path in Λ_2 can be uniquely denoted by $p(m, k)$ where k is the length of the path and m is the "f-degree" of the path, defined as follows: $|m|$ is the number of times f appears in the standard form $e^r(fe)^sf^t$, with $m > 0$ if $t = 1$ and $m < 0$ if $t = 0$.

For example, $efefef = p(3, 6)$ and $e^6fe = p(-1, 8)$. When k is clear from context, we write $p(m)$ for brevity.

Using this notation, we can write out in a general way the orthogonal basis described above via the path diagram. First consider paths of even length $2k$. Define an ordered basis for H_{2k} by

$$\{p(-k), p(-k+1), \dots, p(-1), p(0), p(1), \dots, p(k)\}.$$

For paths of odd length $2k-1$, define an ordered basis for H_{2k-1} by

$$\{p(-k+1), \dots, p(-1), p(0), p(1), \dots, p(k)\}.$$

Let P_k be the projection onto H_k. Then $\sum_{k=0}^{\infty} P_k = I$. The following lemma describes the matrix representation of L_e and L_f with respect to this decomposition; it will be helpful in Example 5.12 when we study this free semigroupoid algebra further in order to prove that it is semisimple.

Lemma 4.5. *In the matrix decomposition described above, L_e and L_f are represented by*

$$L_e = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ J_2 & 0 & 0 & 0 & \dots \\ 0 & S_3 & 0 & 0 & \dots \\ 0 & 0 & J_4 & 0 & \dots \\ 0 & 0 & 0 & S_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}; \quad L_f = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ S_2 & 0 & 0 & 0 & \dots \\ 0 & J_3 & 0 & 0 & \dots \\ 0 & 0 & S_4 & 0 & \dots \\ 0 & 0 & 0 & J_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where J_k is the $k \times (k-1)$ matrix that is a $(k-1) \times (k-1)$ identity matrix with an extra row of 0's at the bottom (i.e., the inclusion map from H_{k-1} to H_k, sending each basis element of H_{k-1} to the corresponding basis element of H_k), and S_k is the $k \times (k-1)$ matrix that is a $(k-1) \times (k-1)$ identity matrix with an extra row of 0's at the top (i.e., the right shift map from H_{k-1} to H_k, sending each basis element in H_{k-1} to the next basis element of H_k).

Proof. First note that composing e with any path in standard form adds one to the length of the path but does not change the "f-degree" of the path:

$$(e)(e^r(fe)^n f^t) = e^{r+1}(fe)^n f^t$$

That is, $e \circ p(m,k) = p(m,k+1)$. By the way the bases for these Hilbert spaces are defined, this means that L_e acts as the right shift operator from H_{2k-1} to H_{2k}, and the inclusion map from H_{2k} to H_{2k+1}. This gives us the desired matrix representation of L_e.

For L_f, consider a basis element $p(m, 2k) \in H_{2k}$. By checking the cases when $m > 0, m < 0$, and $m = 0$, it is straightforward to show that

- if k is even, then $f \circ p(m,k) = p(m+1, k+1)$; and
- if k is odd, then $f \circ p(m,k) = p(m-1, k+1)$.

Again, by the way that the bases are defined, this means that L_f acts as the right shift operator from H_{2k} to H_{2k+1} and the inclusion map from H_{2k-1} to H_{2k}. This gives us the desired matrix representation of L_f. □

The next example is a single-vertex category of paths for which the free semigroupoid algebra contains a non-zero nilpotent element. Before introducing this example, we show that this cannot occur in the higher rank graph case. We use the following ordering of paths in a higher rank graph: given $\lambda \in \Lambda$, with degree $(n_1, n_2, \dots, n_k)$ and $\mu \in \Lambda$ with degree $(m_1, m_2, \dots, m_k)$, we say $\lambda \geq \mu$

if $(n_1, n_2, \dots, n_k) \geq (m_1, m_2, \dots, m_k)$ with respect to lexicographical ordering in $\mathbb{N}^k$.

Proposition 4.6. *If Λ is a single-vertex higher rank graph, then $\mathfrak{L}_\Lambda$ does not have a non-zero nilpotent.*

Proof. Let $T \in \mathfrak{L}_\Lambda$ be non-zero, with Fourier expansion $\sum\limits_{w \in \Lambda} \alpha_w L_w$. Let $n = \min\{|w| : \alpha_w \neq 0\}$, and let $\Gamma = \{w \in \Lambda : |w| = n, \alpha_w \neq 0\}$. Let $\gamma \in \Gamma$ be maximal with respect to lexicographic ordering. Then for any $k \in \mathbb{N}$, the expansion of T^k contains the non-zero term $\alpha_\gamma^k L_{\gamma^k}$. This term can only cancel out with other non-zero terms associated to paths of length kn, and by the minimality of n, such a path must have the form $w_1 w_2 \dots w_k$ with $|w_i| = n$ for all $i = 1, \dots, k$. However, Lemma 7.1 of [7], implies that $w_i = \gamma$ for all $i = 1, \dots, k$. So the non-zero term $\alpha_\gamma^k L_{\gamma^k}$ cannot cancel out, and T is not nilpotent. □

Example 4.7. Let Λ_3 be the category with one vertex x, three edges a, b, and c, and the following identifications:

- $a^2 = b^2 = c^2$
- $ab = bc = ca$
- $ac = cb = ba$

Using these relations, any non-vertex path can be written uniquely in the form ya^n for $y \in \{a, b, c\}$ and $n \in \mathbb{N} \cup \{0\}$.

Proposition 4.8. *The category Λ_3 described above is a category of paths.*

Proof. We will show that Λ_3 satisfies the conditions of a category of paths by means of a matrix semigroup representation.

Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and the subsemigroup S of $(\mathbb{N} \cup \{0\}, +) \oplus M_3$ generated by $(1, A), (1, B), (1, C)$, and $(0, I)$, where I is the 3×3 identity matrix. These elements satisfy:

- $(1, X)(0, I) = (0, I)(1, X) = (1, X)$ for $X \in \{A, B, C\}$
- $(1, A)^2 = (1, B)^2 = (1, C)^2 = (2, I)$
- $(1, A)(1, B) = (1, B)(1, C) = (1, C)(1, A) = (2, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix})$
- $(1, A)(1, C) = (1, C)(1, B) = (1, B)(1, C) = (2, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix})$.

Furthermore, because the matrices A, B, and C are invertible, this semigroup has left and right cancellation, and because the first coordinate of the direct sum is always positive, there are no inverses. So S is a category of paths.

Now consider the mapping $\varphi : \Lambda_3 \to S$ given by $a \mapsto (1, A)$, $b \mapsto (1, B)$, $c \mapsto (1, C)$, $x \mapsto (0, I)$, which defines a surjective semigroup homomorphism. To see that φ is injective, suppose $\varphi(ya^n) = \varphi(za^k)$, for $y, z \in \{a, b, c\}$ and $n, k \in \mathbb{N} \cup \{0\}$. Then $\varphi(y)(n, A^n) = \varphi(z)(k, A^k)$. So $n = k$ and by cancellation, $\varphi(y) = \varphi(z)$. So $y = z$, and $ya^n = za^k$.

Thus, the category Λ_3 is isomorphic to the category of paths S, implying that Λ_3 is a category of paths. □

Note that this category of paths has a non-zero nilpotent element given by $T = L_a + \omega L_b + \omega^2 L_c$, where ω is a primitive third root of unity; if we expand T^2 and use the identifications in Λ_3 to simplify, we get

$$T^2 = (1 + \omega + \omega^2)L_{a^2} + (1 + \omega + \omega^2)L_{ba} + (1 + \omega + \omega^2)L_{ca} = 0.$$

Next, we consider matrix representations for L_a, L_b, and L_c based on the Hilbert spaces $\{H_k\}_{k\geq 0}$, where $H_0 = \{\xi_x\}$, $H_1 = \{\xi_a, \xi_b, \xi_c\}$, and

$$H_k = \{\xi_{a^k}, \xi_{ba^{k-1}}, \xi_{ca^{k-1}}\}$$

for $k \geq 2$. Then $I = \sum_{k=0}^{\infty} P_k$, where P_k is the projection onto H_k.

Lemma 4.9. *In this matrix decomposition, L_a, L_b, and L_c are represented by*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ A_1 & 0 & 0 & 0 & \dots \\ 0 & A & 0 & 0 & \dots \\ 0 & 0 & A & 0 & \dots \\ 0 & 0 & 0 & A & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ B_1 & 0 & 0 & 0 & \dots \\ 0 & B & 0 & 0 & \dots \\ 0 & 0 & B & 0 & \dots \\ 0 & 0 & 0 & B & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ C_1 & 0 & 0 & 0 & \dots \\ 0 & C & 0 & 0 & \dots \\ 0 & 0 & C & 0 & \dots \\ 0 & 0 & 0 & C & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

respectively, where $A_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, *and $A, B,$ and C are defined above. (So all the blocks in the block decompositions are* 3×3*, except the 1,1-block, which is* 1×1*, the rest of the first column of blocks, which are* 3×1*, and the rest of the first row of blocks, which are* 1×3*.)*

Proof. First, note that $L_a(\xi_x) = \xi_a$, giving us A_1 in the 2,1-block. Next, for all $n \in \mathbb{N}$,

$$L_a(\xi_{a^n}) = \xi_{a^{n+1}};\ L_a(\xi_{ba^{n-1}}) = \xi_{ca^n};\ L_a(\xi_{ca^{n-1}}) = \xi_{ba^n}.$$

which gives us the matrix A in the $(n + 1, n)$-block, for all n. The calculations for L_b and L_c are similar. □

So $\mathfrak{L}_{\Lambda_3}$ is the WOT-closed algebra generated by L_a, L_b, L_c, and the identity.

As a similar example, consider the category of paths Λ_n with one vertex x, n edges $e_0, e_1, \dots, e_{n-1}$, and the identifications $e_i e_j = e_{i+\ell} e_{j+\ell}$ for all i, j, ℓ, taken mod n.

Similar to the case for Λ_3 above, this can be shown to be a category of paths using a matrix representation: Let e_k be the n-dimensional vector with 1 in the

kth coordinate and zeroes elsewhere, and let E_i be the $n \times n$ matrix with kth column equal to e_{i-k}, with all subscripts taken mod n. Then the kth row of E_i is also e_{i-k}, so E_i is a symmetric matrix for every i, and $E_iE_j = E_{i+\ell}E_{j+\ell}$ for all i, j, ℓ, taken mod n. Thus the subsemigroup of $(\mathbb{N} \cup \{0\}, +) \oplus M_n$ generated by $(0, 1)$ and $\{(1, E_i) : i = 1, \dots, n\}$ is equivalent to Λ_n, and so Λ_n satisfies cancellation and has no inverses.

Note that the relations on Λ_n imply that $e_i^2 = e_j^2$ for all i, j, and thus e_i^2 commutes with every path in Λ. Thus, this category of paths has n paths of length k for any $k \geq 2$, which can be written as $e_0^k, e_1e_0^{k-1}, e_2e_0^{k-1}, \dots, e_{n-1}e_0^{k-1}$. Additionally, it has a non-zero nilpotent

$$T = \xi_{e_0} + \omega\xi_{e_1} + \omega^2\xi_{e_2} + \cdots + \omega^{n-1}\xi_{e_{n-1}},$$

where ω is a primitive nth root of unity.

In the case where $n = 3$, this construction gives the category of paths Λ_3 described above. When $n = 2$, we get a category of paths with one vertex, two loops, and the relations $e_0^2 = e_1^2$ and $e_0e_1 = e_1e_0$, which is different than the two-loop example described in Example 4.3.

5. Semisimplicity

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *nilpotent* if $T^n = 0$. We say that T is *quasinilpotent* if the spectrum of T is 0, or, equivalently, if $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$. The *Jacobson radical* $\mathrm{rad}(\mathcal{A})$ of a Banach algebra $\mathcal{A}$ is the intersection of the kernels of all algebraically irreducible representations. It is a well-known fact (for example, Theorem 2.3.5(ii) in [11]), that the Jacobson radical of an algebra of operators is the largest quasinilpotent ideal in the algebra. An algebra $\mathcal{A}$ is called *semisimple* if $\mathrm{rad}(\mathcal{A}) = 0$.

A *cycle* in Λ is a path $\mu_1\mu_2 \dots \mu_n \notin \Lambda^0$ with $s(\mu_n) = r(\mu_1)$. Say that $\mu \in \Lambda$ *lies on a cycle* if there is some $\nu \in \Lambda$ such that $\mu\nu$ is a cycle. Let $B(\Lambda)$ be the collection of paths $\mu \in \Lambda$ which do not lie on a cycle. The set $B(\Lambda)$ is empty if and only if every path in Λ lies on a cycle. Kribs and Power showed that for a graph G, the Jacobson radical of $\mathfrak{L}_G$ is determined by these paths:

Theorem 5.1 ([6], Theorem 5.1). *Let G be a graph. Then $\mathfrak{L}_G$ is semisimple if and only if every path in G lies on a cycle. When G has finitely many vertices, $|V(G)| = M < \infty$, then the radical is nilpotent of degree at most M and is equal to the WOT-closed two-sided ideal generated by $\{L_\mu : \mu \in B(\Lambda)\}$.*

They also proved the same theorem for higher rank graphs in [7]. To obtain a similar result for categories of paths, we will use an extra assumption.

Throughout this section, the category of paths Λ is assumed to have a non-degenerate degree functor $\varphi : \Lambda \to \mathbb{N}^n$. For $w \in \Lambda$, let $|w| = |\varphi(w)|$, i.e. the sum of the components of $\varphi(w) \in \mathbb{N}^n$.

Definition 5.2. *We call a path $\mu \in \Lambda$ minimal if for $\nu, \eta \in \Lambda$, $\mu = \nu\eta$ implies $\mu = \nu$ or $\mu = \eta$.*

Definition 5.3. *Say that a category of paths Λ satisfies property (P) if:*

(i) *For each vertex $v \in \Lambda^0$, the set of minimal paths in $v\Lambda$ is finite; and*

(ii) *If $A \neq 0$ and $A = a_1 L_{w_1} + a_2 L_{w_2} + \cdots + a_k L_{w_k}$ for some $k \in \mathbb{N}$, $w_1, \dots, w_k \in \Lambda$ with $|w_1| = |w_2| = \cdots = |w_k|$, and $a_1, \dots, a_k \in \mathbb{C}$, then there is some $\mu \in \Lambda$ such that $L_\mu A$ is not nilpotent.*

If Λ is a graph or higher rank graph, then the second condition, (P)(ii), is equivalent to saying that each edge lies on a cycle, as shown in the next proposition. Notice the similarity to the proof of Proposition 4.6.

Proposition 5.4. *If Λ is a higher rank graph, then each path in Λ lies on a cycle if and only if Λ satisfies (P)(ii).*

Proof. First suppose Λ satisfies (P)(ii) and let $\nu \in \Lambda$. By (P)(ii), there is some $\mu \in \Lambda$ such that $L_\mu L_\nu$ is not nilpotent. Thus, $L_{\mu\nu}^2 = L_{\mu\nu\mu\nu}$ is not equal to 0. So $\mu\nu$ must be a cycle. Thus, every path lies on a cycle.

Now assume that every path in Λ lies on a cycle, and let $A \in \mathfrak{L}_\Lambda$ such that $A \neq 0$ and $A = a_1 L_{w_1} + a_2 L_{w_2} + \cdots + a_k L_{w_k} \in \mathfrak{L}_\Lambda$ for some $k \in \mathbb{N}$, $w_1, \dots, w_k \in \Lambda$ with $|w_1| = |w_2| = \cdots = |w_k|$, and $a_1, \dots, a_k \in \mathbb{C}$. Assume without loss of generality that $a_i \neq 0$ for $i = 1, \dots, k$. Choose μ so that μw_1 is a cycle. Let $\Gamma = \{\mu w_i : r(w_i) = s(\mu), i = 1, \dots, k\}$, and let $\gamma \in \Gamma$ be maximal in Γ with respect to lexicographic ordering, say $\gamma = \mu w_{i_0}$. Then for any $n \in \mathbb{N}$, the expansion of $(L_\mu T)^n$ contains the term $a_{w_{i_0}}^n L_{\gamma^n}$ with $a_{w_{i_0}}^n \neq 0$. By Lemma 7.1 of [7], no other path associated to a term in the expansion of $(L_\mu T)^n$ can be identified with γ^n. So the non-zero term $a_{w_{i_0}}^n L_{\gamma^n}$ cannot cancel out. So T is not nilpotent. □

Lemma 5.5. *If Λ satisfies (P)(i), then for any vertex v, there are at most finitely many paths in Λ of degree n with range v.*

Proof. Let $v \in \Lambda^0$. By (P)(i), there are only a finite number, say N_1, of minimal paths in $v\Lambda$. For each of those paths μ, there are a finite number of minimal paths in $s(\mu)\Lambda$. Let N_2 be the maximum of those finite numbers. Continue this n times, up to N_n. Then the total number of paths in Λ of degree less than or equal to n with range v is at most

$$N_1 + N_1 N_2 + \cdots + N_1 N_2 N_3 \dots N_n,$$

which is finite. □

The following theorem corresponds to Lemma 5.2 in [6].

Theorem 5.6. *If Λ satisfies (P), then $\mathfrak{L}_\Lambda$ is semisimple. In particular, for every non-zero A in $\mathfrak{L}_\Lambda$, there is a path $w \in \Lambda$ such that $L_w A$ is not quasinilpotent.*

Proof. Let $A \in \mathfrak{L}_\Lambda$, with Fourier expansion $A \sim \sum_{w \in \Lambda} a_w L_w$. Let $n = \min\{|w| : a_w \neq 0\}$.

Let $A' = \sum_{|w|=n} a_w L_w$. By condition (ii) of (P), there is some $\mu \in \Lambda$ such that $L_\mu A'$ is not nilpotent. Therefore, since only minimal-degree terms can cancel out other minimal-degree terms, the minimal-degree terms of $(L_\mu A)^k$ do not cancel out for any k. So for any k, $(L_\mu A)^k$ will have a non-zero term in its Fourier expansion of the form $b_{\nu_k} L_{\nu_k}$ where $|\nu_k| = k(n + |\mu|)$. By the minimality of n, such a path ν_k must be equal to $\mu w_k \mu w_{k-1} \dots \mu w_2 \mu w_1$ where each w_i has degree n.

Now, by Lemma 5.5, there are only finitely many paths of degree n that end at $s(\mu)$. So the following minimum is well defined:

$$a := \min\{|a_w| : |w| = n, r(w) = s(\mu), a_w \neq 0\}.$$

Then $|b_{\nu_k}| \geq a^k$. So for $k \geq 1$, we have

$$\|(L_\mu A)^k\|^{1/k} \geq \left|\left\langle (L_\mu A)^k \xi_{s(\nu_k)}, \xi_{\nu_k} \right\rangle\right|^{1/k} = |b_{\nu_k}|^{1/k} \geq (a^k)^{1/k} = a > 0.$$

Thus, $L_\mu A$ has a positive spectral radius and is not quasinilpotent. But recall the radical rad $\mathfrak{L}_\Lambda$ is equal to the largest quasinilpotent ideal in $\mathfrak{L}_\Lambda$. So A is not in the radical for $A \neq 0$. $\square$

Next, we will show a partial converse to this result, namely, that if $\mathfrak{L}_\Lambda$ is semisimple, then each path in Λ must lie on a cycle. First, the following Lemma corresponds to Lemma 5.3 from [6]:

Lemma 5.7. *The following are equivalent for $\mu \in \Lambda$:*

(i) $L_\mu \in rad\, \mathfrak{L}_\Lambda$
(ii) $\mu \in B(\Lambda)$
(iii) $(AL_\mu)^2 = 0$ *for all* $A \in \mathfrak{L}_\Lambda$
(iv) $L_w^2 = L_{w^2} = 0$ *whenever* $w \in \Lambda$ *is a path which includes* μ *(i.e., there exists* $\alpha, \beta \in \Lambda$ *such that* $w = \alpha\mu\beta$*).*

Proof. The proof for graphs also works for categories of paths, but the details for the equivalence of (iii) and (iv) are not explicitly given in Lemma 5.3 in [6], so we provide them here.

(iii) $\implies$ (iv) Assume that $(AL_\mu)^2 = 0$ for all $A \in \mathfrak{L}_\Lambda$ and let w be a path containing μ. So $w = \alpha\mu\beta$ for some $\alpha, \beta \in \Lambda$. Suppose w is a cycle. Then $s(\beta) = r(\alpha)$. Letting $A = L_{\beta\alpha}$, we have

$$L_\beta L_w^2 = L_{\beta w w} = L_{\beta\alpha\mu\beta\alpha\mu\beta} = (AL_\mu)^2 L_\beta = 0.$$

But $L_\beta L_w^2 \neq 0$ since $L_\beta L_w^2(\xi_{s(w)}) = \xi_{\beta w^2}$. This contradiction shows that w is not a cycle. So $L_{w^2} = 0$.

(iv) $\implies$ (iii) Now assume that $L_w^2 = L_{w^2} = 0$ whenever $w \in \Lambda$ is a path which includes μ. Let $\nu \in \Lambda$ such that $s(\nu) = r(\mu)$. Then $\nu\mu$ is a path containing μ, so $L_{\nu\mu}^2 = 0$ by assumption. If it were also true that $s(\mu) = r(\nu)$, then $L_{\nu\mu}^2 = L_{\nu\mu\nu\mu} \neq 0$. So it must be that $s(\mu) \neq r(\nu)$ for all $\nu \in \Lambda$ with $s(\nu) = r(\mu)$.

Now let $A \in \mathfrak{L}_\Lambda$ and let a_w be the coefficients such that $A\xi_{r(\mu)} = \sum_{s(w)=r(\mu)} a_w \xi_w$. Then

$$\begin{aligned}(AL_\mu)^2 \xi_{s(\mu)} &= AL_\mu A \xi_\mu \\ &= AL_\mu \sum_{s(w)=r(\mu)} a_w \xi_{w\mu} \\ &= A \sum_{s(w)=r(\mu)} a_w \xi_{\mu w \mu}.\end{aligned}$$

But, by the previous paragraph, $\mu w \mu$ is not a path for any w with $s(w) = r(\mu)$. So $(AL_\mu)^2 \xi_{s(\mu)} = 0$. And for any other vertex $y \neq s(\mu)$, we have

$$(AL_\mu)^2 \xi_y = (AL_\mu A) L_\mu \xi_y = 0.$$

□

Theorem 5.8. *If $\mathfrak{L}_\Lambda$ is semisimple, then every path in Λ lies on a cycle.*

Proof. Suppose that there is a path in Λ which does not lie on a cycle. Then the set $B(\Lambda)$ is nonempty, and Lemma 5.7 gives us a path $\mu \in B(\Lambda)$ such that $L_\mu \in \operatorname{rad} \mathfrak{L}_\Lambda$. Thus $\mathfrak{L}_\Lambda$ has nonzero radical and $\mathfrak{L}_\Lambda$ is not semisimple. (This does not require the assumption that Λ satisfies (P), and is the same as the graph case [6].) □

We next consider a block diagonal decomposition of $\mathfrak{L}_\Lambda$. As in [6], we say that a subset Γ of Λ is *maximally transitive* if :

(a) there are paths in both directions between every pair of vertices in Γ
(b) if $\mu \in \Gamma$, then $s(\mu)$ and $r(\mu)$ are in Γ
(c) if $\mu \in \Lambda$ such that $s(\mu)$ and $r(\mu)$ are in Γ, then $\mu \in \Gamma$
(d) Γ is maximal with respect to these properties.

Let $\{\Lambda_i\}_{i\in\mathcal{J}}$ be the maximally transitive components of Λ, and let $\{S_i\}_{i\in\mathcal{J}}$ be the projections $S_i = \sum_{x\in\Lambda_i^0} L_x$. Note that if Λ has M vertices, then $|\mathcal{J}| \le M$, since every maximally transitive component must have at least one vertex and every vertex is in exactly one maximally transitive component (though that component could be just a vertex with no paths). Thus, we have

$$I = \oplus_{i\in\mathcal{J}} S_i.$$

Note that paths in $B(\Lambda)$ are not contained in any maximally transitive components, since paths in $B(\Lambda)$ do not lie on a cycle. Therefore, $B(\Lambda) = \Lambda \setminus \cup_{i\in\mathcal{J}} \Lambda_i$.

Now we may consider the block matrix form of $\mathfrak{L}_\Lambda$ with respect to the above decomposition. Note that, for $i \neq j$, if the (i, j)-block is non-zero, then the (j, i)-block must be 0, because if there were a path from Λ_i to Λ_j and a path from Λ_j to Λ_i, it would violate the maximality of the maximally transitive components.

A graph version of the following lemma was stated but not explicitly proved in [6], so we include a proof here for the category of paths case even though the same proof would apply to graphs:

Lemma 5.9. *Let $\mathcal{J}$ be the WOT-closed two-sided ideal in $\mathfrak{L}_\Lambda$ generated by $\{L_\mu : \mu \in B(\Lambda)\}$. Then $\mathcal{J}$ is given by the off-diagonal entries of $\mathfrak{L}_\Lambda$ in the decomposition described above; that is, $\mathcal{J} = \sum_{i \neq j} S_i \mathfrak{L}_\Lambda S_j$.*

Proof. Let $A \in \mathfrak{L}_\Lambda$. For each vertex x, there exist constants $\{a_w : w \in \Lambda, s(w) = x\}$ such that

$$A\xi_x = \sum_{s(w)=x} a_w \xi_w.$$

In the block diagonal form of A described above, the coefficient a_w will be in the column block corresponding to $s(w)$ and the row block corresponding to $r(w)$.

So if $A \in \mathfrak{L}_\Lambda$ and the diagonal blocks are 0 in this decomposition, then the Fourier coefficients a_w are 0 for all $w \notin B(\Lambda)$. Thus, the Cesaro sums of A are in $\mathcal{J}$, and since they converge SOT to A, that means $A \in \mathcal{J}$.

Conversely, if $A \in \mathcal{J}$, we know A is a WOT limit of operators in span$\{L_\mu : \mu \in B(\Lambda)\}$. Note that any path $\mu \in B(\Lambda)$ has at most one endpoint in any given maximally transitive component Λ_i. Thus, the block diagonals in this matrix decomposition will be 0 for every L_μ for $\mu \in B(\Lambda)$, and hence also for A. □

The following theorem is similar to Theorem 5.1 in [6], but the proof is slightly more complicated in the category of paths case.

Theorem 5.10. *If Λ has $M < \infty$ maximally transitive components, and each maximally transitive component satisfies (P), then the radical is nilpotent of degree at most M and is equal to the WOT-closed two-sided ideal generated by $\{L_\mu : \mu \in B(\Lambda)\}$.*

Proof. Let $\mathcal{J}$ be the WOT-closed two-sided ideal in $\mathfrak{L}_\Lambda$ generated by $\{L_\mu : \mu \in B(\Lambda)\}$. We will first show that the radical contains this ideal. By Lemma 5.9, $\mathcal{J}$ is given by the off-diagonal entries of $\mathfrak{L}_\Lambda$ in the decomposition

$$I = \oplus_{i \in \mathcal{I}} S_i,$$

where S_i is the projection onto the subspace $\ell^2(\Lambda_i)$ corresponding to the maximally transitive component Λ_i.

Now, since there are M blocks in each row and column, and only one of the (i,j)- and the (j,i)-block can be non-zero for $i \neq j$, it follows that $\mathcal{J}^M = \{0\}$. Since $\mathcal{J}$ is an ideal, we have for all $X \in \mathfrak{L}_\Lambda$ and $A \in \mathcal{J}$, that $(XA)^M = 0$. Hence $\mathcal{J}$ is contained in rad $\mathfrak{L}_\Lambda$ and is nilpotent of degree at most M.

Finally, we need to show that rad $\mathfrak{L}_\Lambda$ is contained in $\mathcal{J}$. So suppose $A \in$ rad $\mathfrak{L}_\Lambda$ with Fourier expansion scalars $\{a_w\}_{w \in \Lambda}$. We will show that a coefficient a_w is non-zero only if $w \in B(\Lambda)$. Suppose by way of contradiction that there is a path ν with $a_\nu \neq 0$ and $\nu \notin B(\Lambda)$. Choose ν so that $|\nu|$ is minimal with this property. Let Λ' be the maximally transitive component of Λ that contains ν.

Let $S = \{w \in \Lambda' : |w| = |\nu|, r(w) = r(\nu)\}$. Note that this set is finite by Lemma 5.5. Let A' be the operator of terms of A corresponding to paths in S; that is, $A' = \sum_{w \in S} a_w L_w$.

Note that this means $A' \in \mathfrak{L}_{\Lambda'}$. Since we are assuming that (P) holds on Λ', there is some $\mu \in \Lambda'$ such that $L_\mu A'$ is not nilpotent. We now want to show that $L_\mu A$ has positive spectral radius.

The Fourier series of the operator $L_\mu A$ is given by $\sum_{w \in \Lambda} a_w L_{\mu w}$. Taking this to the kth power formally gives us

$$\sum_{w_i, \eta \in \Lambda} a_{w_1} a_{w_2} \dots a_{w_{k-1}} a_\eta L_{\mu w_1 \mu w_2 \dots \mu w_{k-1} \mu \eta}.$$

But in fact, we know each w_i is in Λ' because $s(\mu)$ and $r(\mu)$ are in Λ'.

Let $\mathcal{M} = \{\mu u_1 \mu u_2 \dots \mu u_{k-1} \mu u_k : u_i \in S\}$. We will show that it is impossible for all the terms associated to paths in $\mathcal{M}$ to cancel out in the product $(L_\mu A)^k$. Let $w_1, w_2, \dots, w_{k-1} \in \Lambda'$ with $a_{w_i} \neq 0$, and let $u_1, u_2, \dots, u_k \in S$ with $a_{u_i} \neq 0$. In what follows, we will determine for which paths $\eta \in \Lambda$ it is possible that $a_\eta \neq 0$ and

$$\mu w_1 \mu w_2 \dots \mu w_{k-1} \mu \eta = \mu u_1 \mu u_2 \dots \mu u_k.$$

First, suppose $|\eta| < |\nu|$. Since $|\nu|$ is minimal with the property that $a_\nu \neq 0$ and $\nu \notin B(\Lambda)$, this implies $\eta \in B(\Lambda)$. Thus, either $s(\eta) \notin \Lambda'$ or $r(\eta) \notin \Lambda'$. So since $u_k \in S$, then either $s(\eta) \neq s(u_k)$, implying

$$\mu w_1 \mu w_2 \dots \mu w_{k-1} \mu \eta \neq \mu u_1 \mu u_2 \dots \mu u_k$$

or $r(\eta) \neq s(\mu)$, implying the path on the left is undefined.

Now suppose $|\eta| > |\nu|$. Then $\mu w_1 \mu w_2 \dots \mu w_{k-1} \mu \eta$ has degree larger than $(|\mu| + |\nu|)^k$, since each w_i is in Λ', and thus by the minimality of $|\nu|$, satisfies $|w_i| \geq |\nu|$ for all i. So $\mu w_1 \mu w_2 \dots \mu w_{k-1} \mu \eta \neq \mu u_1 \mu u_2 \dots \mu u_k$.

Finally, suppose $|\eta| = |\nu|$. If $\eta \notin \Lambda'$, then, as above, either $s(\eta) \neq s(u_k)$, implying

$$\mu w_1 \mu w_2 \dots \mu w_{k-1} \mu \eta \neq \mu u_1 \mu u_2 \dots \mu u_k$$

or $r(\eta) \neq s(\mu)$, implying the path on the left is undefined. If $|\eta| = |\nu|$ and η is in Λ', then $\mu w_1 \mu w_2 \dots \mu w_{k-1} \mu \eta$ is in $\mathcal{M}$.

Therefore, only terms corresponding to paths in $\mathcal{M}$ can cancel out other terms in $\mathcal{M}$, and we know they do not all cancel out because $L_\mu A'$ is not nilpotent.

Thus, for any k, $(L_\mu A)^k$ will have a non-zero term in its Fourier expansion of the form $b_{w_k} L_{w_k}$ where w_k is the result of concatenating k paths of the form μu for $u \in S$. Let $a = \min\{|a_u| : u \in S\}$, which is well defined since S is a finite set. Then $|b_{w_k}| \geq a^k$.

So for $k \geq 1$, we have

$$\|(L_\mu A)^k\|^{1/k} \geq \left|\left\langle (L_\mu A)^k \xi_{s(w_k)}, \xi_{w_k} \right\rangle\right|^{1/k} = |b_{w_k}|^{1/k} \geq (a^k)^{1/k} = a > 0.$$

This contradicts that $A \in \operatorname{rad} \mathfrak{L}_\Lambda$, thus proving the claim. So a coefficient a_w in the Fourier expansion of A is non-zero only if $w \in B(\Lambda)$. Thus, the Cesaro sums for A are in $\mathcal{J}$, and they converge SOT to A. So $A \in \mathcal{J}$. $\square$

Before turning to examples, we give one further result on the nilpotency degree of the ideal $\mathcal{J}$. Given a category of paths Λ, let a chain of length n be a set of maximally transitive components $\{\Lambda_1, \Lambda_2, \dots, \Lambda_n\}$ of Λ with paths $w_1, w_2, \dots, w_{n-1}$ in $B(\Lambda)$ such that w_j begins in Λ_j and ends in Λ_{j+1}. If there are a finite number of maximally transitive components, then all chains are finite.

Proposition 5.11. *Let Λ be a category of paths with M maximally transitive components, where $M < \infty$. Let $\mathcal{J}$ be the WOT-closed ideal generated by $\{L_\mu : \mu \in B(\Lambda)\}$. The nilpotency degree of $\mathcal{J}$ is equal to the length of the largest chain of maximally transitive components, which is at most M.*

Proof. Let $\{\Lambda_i\}_{i\le M}$ be the maximally transitive components of Λ, and let $\{S_i\}_{i\le M}$ be the projections $S_i = \sum_{x\in\Lambda_i^0} L_x$. Then $I = \oplus_{i\le M} S_i$.

Lemma 5.9 says that the ideal $\mathcal{J}$ is given by the off-diagonal entries of $\mathfrak{L}_\Lambda$ in this decomposition. Let $B_{i,j}$ be the block in the ith row and jth column of this decomposition. Let n be the length of the largest chain of maximally transitive components. A chain of length n of maximally transitive components corresponds to a sequence of blocks $B_{j_1,j_2}, B_{j_2,j_3}, \dots, B_{j_{n-1},j_n}$ such that each $B_{j_k,j_{k+1}}$ is non-zero and all $j_1, \dots, j_n$ are distinct. Since there are no chains of length bigger than n, $\mathcal{J}^n = 0$, and $\mathcal{J}$ is nilpotent of degree less than or equal to n.

Suppose $\{\Lambda_1, \dots, \Lambda_n\}$ with paths $\{w_1, \dots, w_{n-1}\}$ is a maximum length chain. Since each component Λ_i is transitive, there are paths $\mu_i \in \Lambda_i$ for $1 < i < n$ with $s(\mu_i) = r(w_{i-1})$ and $r(\mu_i) = s(w_i)$. So $w_{n-1}\mu_{n-1} \dots \mu_3 w_2 \mu_2 w_1$ is a path in Λ. Thus,

$$A := L_{w_{n-1}\mu_{n-1}} + \cdots + L_{w_3\mu_3} + L_{w_2\mu_2} + L_{w_1}$$

is an element of $\mathcal{J}$ such that $A^{n-1}\xi_{s(w_1)} = \xi_{w_{n-1}\mu_{n-1}\dots\mu_3 w_2\mu_2 w_1} \neq 0$. So the nilpotency degree of $\mathcal{J}$ is equal to n. □

Example 5.12. Recall that Λ_2 is the category of paths with one vertex x and two edges e and f satisfying $e^2 = f^2$. The degree functor for Λ_2 is given by the length of the path. We will show that $\mathfrak{L}_{\Lambda_2}$ is semisimple by showing that Λ_2 satisfies Property (P). Clearly, Λ_2 satisfies (P)(i) since there are only three minimal paths in $x\Lambda$ (namely, x, e, and f). So we must show Λ_2 satisfies (P)(ii).

As in Example 4.3, each path in Λ can be uniquely denoted by $p(m, k)$ where k is the length of the path and m is the "f-degree" of the path: $|m|$ is the number of times f appears, with $m > 0$ if the path ends in f and $m < 0$ if the path ends in e. Using this, we can show that the following concatenation formula holds:

Lemma 5.13. *Two paths in Λ_2 are concatenated according to the following rule:*

$$p(m_1, k_1)p(m_2, k_2) = \begin{cases} p(m_1 + m_2, k_1 + k_2), & \text{if } k_2 \text{ is even} \\ p(m_2 - m_1, k_1 + k_2), & \text{if } k_2 \text{ is odd} \end{cases}$$

Example 5.14. (a) Consider concatenating $fe = p(-1, 2)$ and $e^3 f = p(1, 4)$. Using Lemma 5.13,

$$p(-1, 2)p(1, 4) = (-1 + 1, 2 + 4) = (0, 6) = e^6.$$

(b) Consider concatenating $e^3fe = p(-1,5)$ and $e^2fef = p(2,5)$. Using Lemma 5.13,

$$p(-1,5)p(2,5) = (2-(-1), 5+5) = (3,10) = e^5fefef.$$

Proof. (of Lemma 5.13)

Since the second component is the length of the path, the second component of the concatenations will clearly be the sum of the second components of the individual paths.

For the first component, note that $p(m_1,k_1)$ can be written as a sequence of e's and f's. Thus, when we apply $p(m_1,k_1)$ to $p(m_2,k_2)$, we can do the calculation by applying e and f sequentially.

Concatenating with e on the left does not change the number of times f appears in the standard representation, so $e \circ p(m_2,k_2) = p(m_2, k_2+1)$.

The effect of concatenating with f on the left depends on whether k_2 is odd or even. As in Lemma 4.5, one can check the cases when $m_2 > 0$, $m_2 < 0$, and $m_2 = 0$ to show that

- if k_2 is even, then $f \circ p(m_2,k_2) = p(m_2+1, k_2+1)$; and
- if k_2 is odd, then $f \circ p(m_2,k_2) = p(m_2-1, k_2+1)$.

Applying these repeatedly proves the lemma. □

Proposition 5.15. *The free semigroupoid algebra $\mathfrak{L}_{\Lambda_2}$ does not contain any nilpotent elements.*

Proof. Let $A \in \Lambda_2$ with $A \neq 0$. By the Fourier expansion of A, there are constants a_w such that $A \sim \sum_{w\in\Lambda} a_w L_w$. For every $k \in \mathbb{N}$, let

$$S_k = \{w \in \Lambda : |w| = k \text{ and } a_w \neq 0\}.$$

Since $A \neq 0$, there must be at least one $S_k \neq \emptyset$. Let $n = \min\{k : S_k \neq \emptyset\}$.

Suppose that $A^2 = 0$. This means $A^2\xi_x = 0$, so

$$\sum_{w\in\Lambda}\sum_{z\in\Lambda} a_w a_z \xi_{zw} = 0.$$

In particular, all terms associated to paths of length $2n$ must cancel out. By the minimality of n, any path of length $2n$ associated to a non-zero term in A^2 can only result from the product of two paths of length n associated to non-zero terms in A. We will show that it is impossible for all terms associated to paths of length $2n$ to cancel out by looking at the terms associated to paths with minimal "f-degree", as defined above Lemma 5.13.

First suppose $n = 2k$ is even. The paths of length n are

$$\{p(-k,n), p(-k+1,n), \dots, p(0,n), \dots, p(k,n)\}$$

By the concatenation rule for even-length paths in Lemma 5.13, the smallest f-degree among paths of length $2n$ is $-2k = (-k)+(-k)$, uniquely obtained from the product $p(-k,n)p(-k,n)$. Thus, the coefficient of $p(-2k,2n)$ in A^2 is $(a_{p(-k,n)})^2$. So $(a_{p(-k,n)})^2 = 0$, implying $a_{p(-k,n)} = 0$. So $p(-k,n) \notin S_n$.

Thus, only the elements

$$\{p(-k+1,n),\dots,p(0,n),\dots,p(k,n)\}$$

could have non-zero coefficients. The minimal f-degree among products of pairs of these paths is $-2k+2$, uniquely obtained as $p(-k+1)p(-k+1)$. Using the same reasoning as above, we can show that $p(-k+1,n) \notin S_n$. Continuing in this manner shows that $S_n = \emptyset$, a contradiction.

Now suppose $n = 2k+1$ is odd. The paths of length n are

$$\{p(-k,n),p(-k+1,n),\dots,p(0,n),\dots,p(k,n),p(k+1,n)\}.$$

By the concatenation rule for odd-length paths in Lemma 5.13, the minimal f-degree among products of these paths is $-2k-1$, which can be uniquely obtained from the product $p(k+1,n)p(-k,n)$. Thus, the coefficient of $p(-2k-1,2n)$ is $a_{p(k+1,n)}a_{p(-k,n)}$. So either $a_{p(k+1,n)} = 0$, or $a_{p(-k,n)} = 0$. That is, either $p(k+1,n) \notin S_n$ or $p(-k,n) \notin S_n$.

This means the non-zero terms of A associated to paths of length n are either associated to paths from the set

$$\{p(-k,n),\dots,p(0,n),\dots,p(k,n)\}$$

or from the set

$$\{p(-k+1,n),\dots,p(0,n),\dots,p(k+1,n)\}.$$

Either way, the minimal f-degree among non-zero term in the product will be $-2k$, uniquely obtained from the product of the highest f-degree term with the lowest f-degree term. Once again, either the highest or lowest f-degree term must have coefficient 0, and can be removed from the list. Proceeding in like fashion, we again obtain $S_n = \emptyset$, a contradiction.

Thus, $A^2 \neq 0$, and by induction $A^{2^k} \neq 0$ for all k. Furthermore, if $m \in \mathbb{N}$, then there is some k with $2^k > m$ and $A^{2^k} \neq 0$. So $A^m \neq 0$. Thus, A is not nilpotent. □

Corollary 5.16. *The free semigroupoid algebra $\mathfrak{L}_{\Lambda_2}$ is semisimple.*

Proof. As mentioned at the beginning of this section, Λ_2 satisfies (P)(i) because there are only three minimal paths in $x\Lambda$ (namely, x, e, and f). Also, Proposition 5.15 shows that Λ_2 satisfies (P)(ii). Thus Λ_2 satisfies (P), and so $\mathfrak{L}_{\Lambda_2}$ is semisimple by Theorem 5.6. □

Example 5.17. Recall the 3-loop example, Example 4.7, where Λ_3 is the category of paths given by the graph with one vertex x, three edges a, b, and c, and the identifications:

- $a^2 = b^2 = c^2$
- $ab = bc = ca$
- $ac = cb = ba$

We saw that $\mathfrak{L}_{\Lambda_3}$ has a non-zero nilpotent $T = L_a + \omega L_b + \omega^2 L_c$, where ω is a primitive third root of unity. We will now show that $\mathfrak{L}_{\Lambda_3}$ is nonetheless semisimple.

Proposition 5.18. *The free semigroupoid algebra $\mathfrak{L}_{\Lambda_3}$ is semisimple.*

Proof. We will show that Λ_3 satisfies (P). First note that Λ_3 satisfies (P)(i) because there are only four minimal paths in $x\Lambda$ (namely, x, a, b, and c).

To show let Λ_3 satisfies (P)(ii), let $T = \alpha_1 L_{w_1} + \alpha_2 L_{w_2} + \cdots + \alpha_n L_{w_n} \in \mathfrak{L}_{\Lambda_3}$ be non-zero, where $\alpha_i \in \mathbb{C}$ and $w_i \in \Lambda$ with $|w_1| = \cdots = |w_n|$. Since Λ_3 has only three distinct paths of any given length, we know in fact that $T = \alpha_1 L_x$ or $T = xL_{a^n} + yL_{ba^{n-1}} + zL_{ca^{n-1}}$ for $x, y, z \in \mathbb{C}$. Clearly L_x is not nilpotent, so assume $T = xL_{a^n} + yL_{ba^{n-1}} + zL_{ca^{n-1}}$ for $x, y, z \in \mathbb{C}$ and $n \geq 1$.

Assume first that n is even. We have the following multiplication table:

	a^n	ba^{n-1}	ca^{n-1}
a^n	a^{2n}	ba^{2n-1}	ca^{2n-1}
ba^{n-1}	ba^{2n-1}	ca^{2n-1}	a^{2n}
ca^{n-1}	ca^{2n-1}	a^{2n}	ba^{2n-1}

So if $T = xL_{a^n} + yL_{ba^{n-1}} + zL_{ca^{n-1}}$, then

$$T^2 = (x^2 + 2yz)L_{a^{2n}} + (2xy + z^2)L_{ba^{2n-1}} + (2xz + y^2)L_{ca^{2n-1}}.$$

Thus, $T^2 = 0$ if and only if

$$\begin{aligned} x^2 + 2yz &= 0 \\ 2xy + z^2 &= 0 \\ 2xz + y^2 &= 0, \end{aligned}$$

which implies $x = y = z = 0$.

Thus, $T^2 \neq 0$, and T^2 has the form $x' L_{a^{2n}} + y' L_{ba^{2n-1}} + z' L_{ca^{2n-1}}$, and thus is still a sum of terms with even-length paths. So the same argument applies repeatedly, showing that for all k, $T^{2^k} \neq 0$. If $T^m = 0$ for any m, then for $2^k > m$, we would have $T^{2^k} = 0$, a contradiction. So T is not nilpotent.

Now suppose again that $T = xL_{a^n} + yL_{ba^{n-1}} + zL_{ca^{n-1}}$, but now n is odd. Then $L_a T = xL_{a^{n+1}} + yL_{ca^n} + zL_{ba^n}$ is a sum of even length terms, so by the previous argument, $L_a T$ is not nilpotent. Thus, Λ_3 satisfies (P)(ii).

Therefore, Λ_3 satisfies Property (P) and is semisimple by Theorem 5.6. □

The previous argument can be generalized in the following way:

Proposition 5.19. *The free semigroupoid algebra $\mathfrak{L}_{\Lambda_n}$ is semisimple for $n \leq 8$.*

Proof. Recall that Λ_n is the category of paths from Example 4.7 with one vertex x, n edges $e_0, e_1, \ldots, e_{n-1}$, and the identifications $e_i e_j = e_{i+\ell} e_{j+\ell}$ for all i, j, ℓ, taken mod n. If k is even, then the product of two standard-form elements $e_i e_0^{k-1}$ and $e_j e_0^{k-1}$ is

$$\begin{aligned} e_i e_0^{k-1} e_j e_0^{k-1} &= e_i e_0 e_j e_0^{2k-3} \\ &= e_i e_{n-j} e_0^{2k-2} \\ &= e_{i+j} e_0^{2k-1}. \end{aligned}$$

Thus, given the element $T = \sum_{i=0}^{n-1} \alpha_i L_{e_i} L_{e_0^{k-1}}$ for $\alpha_i \in \mathbb{C}$ and k even, we have

$$T^2 = \sum_{\ell=0}^{n-1} \sum_{i+j=\ell} \alpha_i \alpha_j L_{e_{i+j}} L_{e_0^{2k-1}}.$$

So $T^2 = 0$ if the system of equations given by

$$\left\{ \sum_{i=0}^{n-1} \alpha_i \alpha_{\ell - i} = 0 \,:\, \ell = 0, 1, \dots, n-1;\ \text{subscripts taken mod } n \right\}$$

has only the trivial solution. When this is the case, the same argument for the $n = 3$ case shows that Λ_n is semisimple. One can verify with computational software that this is true for at least $n \leq 8$. □

6. Reflexivity

A subspace M of a Hilbert space $\mathcal{H}$ is *invariant* for an operator $A \in \mathcal{B}(\mathcal{H})$ if $A(M) \subseteq M$. For a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, the set of all subspaces that are invariant for all operators in $\mathcal{A}$ forms a lattice, written Lat($\mathcal{A}$). The set of all operators in $\mathcal{B}(\mathcal{H})$ for which all subspaces in Lat($\mathcal{A}$) are invariant forms an algebra, written Alg Lat($\mathcal{A}$). It is immediate that $\mathcal{A} \subseteq$ Alg Lat($\mathcal{A}$). When the opposite containment holds, $\mathcal{A}$ is called *reflexive*. See [3] for an overview of reflexivity in operator algebras.

In this section, we will first prove some general results for reflexivity which are based on those in Kribs and Power's papers [6], [7]. We then prove reflexivity for the family of single-vertex categories of paths from Example 4.7.

The following definition is an adjustment of the Double Pure Cycle Property for higher rank graphs, defined in Section 6 of [7].

Definition 6.1. *Say that a vertex x in a category of paths Λ has double pure cycles if there exist cycles $\lambda_1 \neq \lambda_2$ at x such that $\lambda_1\mu_1 \neq \lambda_2\mu_2$ for all $\mu_1, \mu_2 \in \Lambda$. Then Λ satisfies the Double Pure Cycle Property for Categories of Paths if for every $w \in \Lambda^0$, there exists $\lambda_w \in \Lambda$ such that $s(\lambda_w) = w$ and $r(\lambda_w)$ has double pure cycles.*

Remark 6.2. A higher rank graph that satisfies the Double Pure Cycle Property from Section 6 of [7] also satisfies this version, including any single-vertex graph with two or more edges and any single-vertex higher-rank graph with at least two edges of the same color.

Example 6.3. An example of a category of paths that is not a higher rank graph and satisfies this Double Pure Cycle Property is the category of paths Λ with one vertex x, three edges e, f, and g, and the identification $e^2 = f^2$. Then e and g are non-equal cycles satisfying $e\mu_1 \neq g\mu_2$ for all $\mu_1, \mu_2 \in \Lambda$.

Neither the free semigroupoid algebra from Example 4.3 nor the free semigroupoid algebras from Example 4.7 satisfy the Double Pure Cycle Property, however.

Proposition 6.4. *Suppose that Λ is a countable category of paths which satisfies the Double Pure Cycle Property. Then $\mathfrak{L}_\Lambda$ contains a pair of isometries with mutually orthogonal ranges.*

Proof. This follows by the same proof as Lemma 6.1 in [7]. The key step is showing that, for vertex v with double pure cycles $\lambda_1 \neq \lambda_2$, the operators $L_{\lambda_1^k \lambda_2}$ and $L_{\lambda_1^m \lambda_2}$ are orthogonal for $k \neq m$. That is, for all $\mu_1, \mu_2 \in \Lambda$, we must show $\lambda_1^k \lambda_2 \mu_1 \neq \lambda_1^m \lambda_2 \mu_2$. But this follows directly from the adjusted definition of double pure cycles. $\square$

As in [7], this gives us:

Theorem 6.5. *If Λ is a countable category of paths with a non-degenerate degree functor such that Λ^t satisfies the Double Pure Cycle Property, then $\mathfrak{L}_\Lambda$ is reflexive.*

Proof. Since $\mathfrak{L}_{\Lambda^t}$ is unitarily equivalent to $\mathfrak{R}_\Lambda = \mathfrak{L}'_\Lambda$ by the unitary from Lemma 3.7, we know that $\mathfrak{L}'_\Lambda$ contains a pair of isometries with mutually orthogonal ranges. Thus, by Bercovici's Hyper-Reflexivity Theorem [1], $\mathfrak{L}_\Lambda$ is reflexive. $\square$

One more result from [7] can be adjusted to the category of paths case:

Definition 6.6. *We say x is a radiating vertex if for all $\lambda \in \Lambda$, $r(\lambda) = x$ implies $s(\lambda) = x$.*

Proposition 6.7. *Suppose that Λ is a category of paths with a non-degenerate degree functor such that each radiating vertex x satisfies*

(a) *for the single-vertex category of paths Λ' consisting of x and all paths $\mu \in \Lambda$ with $s(\mu) = r(\mu) = x$, we have that $\mathfrak{L}_{\Lambda'}$ is reflexive*

(b) *if μ_1 and μ_2 are loops at x with $\mu_1 \neq \mu_2$, and w_1 and w_2 are paths with source x, then $w_1\mu_1 \neq w_2\mu_2$.*

Then $\mathfrak{L}_\Lambda$ is reflexive.

Proof. With the restrictions given here, the proof of Theorem 6.4 from [7] applies with only slight modification. $\square$

Corollary 6.8. *If Λ is a finite category of paths with a non-degenerate degree functor, then $\mathfrak{L}_\Lambda$ is reflexive.*

Proof. Since Λ is finite, Λ does not contain any loops or cycles. The semigroupoid algebra of a single vertex with no paths is $\mathbb{C}$, which is reflexive. Thus, all vertices of Λ satisfy the conditions of Proposition 6.7. $\square$

Unlike in the graph and higher rank graph cases, we do not know whether all single-vertex categories of paths have reflexive free semigroupoid algebras, even assuming a degree functor. However, one example where reflexivity holds is the family of single-vertex categories of paths described in Example 4.7. In what follows, we prove the $n = 3$ case, but the same proof generalizes to all $n \in \mathbb{N}$ (see [2]).

As in Example 4.7, let Λ_3 be the category of paths with one vertex x, three edges a, b, and c, and the identifications $a^2 = b^2 = c^2$, $ab = bc = ca$, and

$ac = cb = ba$. In order to show that $\mathfrak{L}_{\Lambda_3}$ is reflexive, we will characterize the structure of elements of $\mathfrak{L}_{\Lambda_3}$ with respect to a particular basis, then show that $T \in \operatorname{Alg\,Lat}(\mathfrak{L}_{\Lambda_3})$ has the same structure. To this end, let ω be a primitive third root of unity. Note that $\omega + \omega^2 + 1 = 0$. Then an orthogonal basis for $\mathcal{H}_{\Lambda_3}$ is $\{\xi_x\} \cup \{h_i, j_i, k_i\}_{i\in\mathbb{N}}$, where

$$\begin{aligned} h_n &= \xi_{a^n} + \xi_{ba^{n-1}} + \xi_{ca^{n-1}} \\ j_n &= \begin{cases} \xi_{a^n} + \omega\xi_{ba^{n-1}} + \omega^2\xi_{ca^{n-1}}, & \text{for } n \text{ odd} \\ \xi_{a^n} + \omega^2\xi_{ba^{n-1}} + \omega\xi_{ca^{n-1}}, & \text{for } n \text{ even} \end{cases} \\ k_n &= \begin{cases} \xi_{a^n} + \omega^2\xi_{ba^{n-1}} + \omega\xi_{a^{n-1}}, & \text{for } n \text{ odd} \\ \xi_{a^n} + \omega\xi_{ba^{n-1}} + \omega^2\xi_{a^{n-1}}, & \text{for } n \text{ even} \end{cases} \end{aligned}$$

Lemma 6.9. *For an arbitrary element* $A = tL_x + \sum_{n=1}^{\infty}\left(x_nL_{a^n} + y_nL_{ba^{n-1}} + z_nL_{ca^{n-1}}\right)$ *in* $\mathfrak{L}_{\Lambda_3}$*, the matrix form of* $A\big|_{\{\xi_x\}^\perp}$ *with respect to the basis above is:*

$$A\big|_{\{\xi_x\}^\perp} = \begin{bmatrix} tI & 0 & 0 & 0 & 0 & \dots \\ S_1 & tI & 0 & 0 & 0 & \dots \\ T_2 & T_1 & tI & 0 & 0 & \dots \\ S_3 & S_2 & S_1 & tI & 0 & \dots \\ T_4 & T_3 & T_2 & T_1 & tI & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where I *is the* 3×3 *identity matrix,*

$$S_n = \begin{bmatrix} x_n + y_n + z_n & 0 & 0 \\ 0 & x_n + \omega y_n + \omega^2 z_n & 0 \\ 0 & 0 & x_n + \omega^2 y_n + \omega z_n \end{bmatrix}$$

and

$$T_n = \begin{bmatrix} x_n + y_n + z_n & 0 & 0 \\ 0 & x_n + \omega^2 y_n + \omega z_n & 0 \\ 0 & 0 & x_n + \omega y_n + \omega^2 z_n \end{bmatrix}.$$

Proof. For $n \geq 1$, let Q_n be the projection onto paths of length n. Then, with respect to the above basis, we have

$$Q_{2n}L_aQ_{2n-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_{2n+1}L_aQ_{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_{2n}L_bQ_{2n-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad Q_{2n+1}L_bQ_{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}$$

$$Q_{2n}L_cQ_{2n-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad Q_{2n+1}L_cQ_{2n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}.$$

Furthermore, for $k \in \mathbb{N}$ and $e \in \{a, b, c\}$:

$$Q_{n+k+1} L_{ea^k} Q_n = Q_{n+k+1} L_e L_{a^k} Q_n = Q_{n+k+1} L_e Q_{n+k}.$$

Thus $Q_{2n} A Q_{2n-1} = S_n$ and $Q_{2n+1} A Q_{2n} = T_n$, with T_n and S_n as defined in the statement of the lemma. □

Remark 6.10. Notice that given any constants $\kappa, \lambda, \mu \in \mathbb{C}$, the system of equations

$$\begin{aligned} \kappa &= x + y + z \\ \lambda &= x + \omega y + \omega^2 z \\ \mu &= x + \omega^2 y + \omega z \end{aligned}$$

has a unique solution for x, y, and z. Thus, the above form of A is equivalent to saying that for all $m > n$, there exist constants $\alpha_{m,n}, \beta_{m,n}$, and $\gamma_{m,n}$ in $\mathbb{C}$ such that

$$Q_n A Q_n = \begin{bmatrix} \alpha_{n,n} & 0 & 0 \\ 0 & \alpha_{n,n} & 0 \\ 0 & 0 & \alpha_{n,n} \end{bmatrix}; \quad Q_m A Q_n = \begin{bmatrix} \alpha_{m,n} & 0 & 0 \\ 0 & \beta_{m,n} & 0 \\ 0 & 0 & \gamma_{m,n} \end{bmatrix}$$

and $\alpha_{m,n} = \alpha_{m+1,n+1}$, $\beta_{m,n} = \gamma_{m+1,n+1}$, $\gamma_{m,n} = \beta_{m+1,n+1}$. Our next goal is to show that elements of Alg Lat $\mathfrak{L}_{\Lambda_3}$ have this same form.

Lemma 6.11. *Let $T \in$ Alg Lat$(\mathfrak{L}_{\Lambda_3})$. Then the matrix form of $T\big|_{\{\xi_x\}^\perp}$ with respect to the basis above is:*

$$T\big|_{\{\xi_x\}^\perp} = \begin{bmatrix} tI & 0 & 0 & 0 & 0 & \dots \\ S_{2,1} & tI & 0 & 0 & 0 & \dots \\ S_{3,1} & S_{3,2} & tI & 0 & 0 & \dots \\ S_{4,1} & S_{4,2} & S_{4,3} & tI & 0 & \dots \\ S_{5,1} & S_{5,2} & S_{5,3} & S_{5,4} & tI & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where I is the 3×3 identity matrix and $S_{m,n} = \begin{bmatrix} \alpha_{m,n} & 0 & 0 \\ 0 & \beta_{m,n} & 0 \\ 0 & 0 & \gamma_{m,n} \end{bmatrix}$ for some constants $\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n} \in \mathbb{C}$.

Proof. Since $T \in$ Alg Lat$(\mathfrak{L}_{\Lambda_3})$, the $\mathfrak{L}_{\Lambda_3}$-invariant subspaces $\mathcal{M}_h = \overline{\text{span}}\{h_n : n \geq 1\}$, $\mathcal{M}_j = \overline{\text{span}}\{j_n : n \geq 1\}$, $\mathcal{M}_k = \overline{\text{span}}\{k_n : n \geq 1\}$ are each also invariant for T. So for $m \geq n$, there exist constants $\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n}$ such that

$$\begin{aligned} Q_m T(h_n) &= \alpha_{m,n} h_m \\ Q_m T(j_n) &= \beta_{m,n} j_m \\ Q_m T(k_n) &= \gamma_{m,n} k_m. \end{aligned}$$

Thus
$$Q_m T Q_n = \begin{bmatrix} \alpha_{m,n} & 0 & 0 \\ 0 & \beta_{m,n} & 0 \\ 0 & 0 & \gamma_{m,n} \end{bmatrix}.$$

Furthermore, the subspace $\mathcal{M}_n$ generated by $h_n + j_n + k_n$ is also $\mathfrak{L}_{\Lambda_3}$-invariant and thus invariant for T. For all $\zeta \in \mathcal{M}_n$,
$$\langle \zeta, h_n \rangle = \langle \zeta, j_n \rangle = \langle \zeta, k_n \rangle.$$
Thus
$$\langle T(h_n + j_n + k_n), h_n \rangle = \langle T(h_n + j_n + k_n), j_n \rangle = \langle T(h_n + j_n + k_n), k_n \rangle,$$
i.e., $\alpha_{n,n} = \beta_{n,n} = \gamma_{n,n}$. □

The next step is to prove that for any $T \in \text{Alg Lat}(\mathfrak{L}_{\Lambda_3})$, there is some $A \in \mathfrak{L}_{\Lambda_3}$ such that $T\big|_{\{\xi_x\}^\perp} = A\big|_{\{\xi_x\}^\perp}$. This will be shown in Lemma 6.13. However, an important piece of the proof of that lemma is the following lemma:

Lemma 6.12. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$. If $M \in Lat(\mathcal{A})$ such that $\mathcal{A}\big|_M$ is reflexive, then for all $T \in Alg\, Lat(\mathcal{A})$, there exists $A \in \mathcal{A}$ such that $T\big|_M = A\big|_M$.*

Proof. Let $T \in \text{Alg Lat}\,\mathcal{A}$, and suppose that $M_0 \subseteq M$ is an invariant subspace for $\mathcal{A}\big|_M$. This implies that M_0 is an invariant subspace for $\mathcal{A}$. Hence, $T(M_0) \subseteq M_0$. Since $M_0 \subseteq M$, this means $T\big|_M(M_0) \subseteq M_0$. So M_0 is invariant for $T\big|_M$, for all $M_0 \in \text{Lat}\,\mathcal{A}\big|_M$. Since $\mathcal{A}\big|_M$ is reflexive, this implies that $T\big|_M \in \mathcal{A}\big|_M$. Thus, there is some operator $A \in \mathcal{A}$ such that $T\big|_M = A\big|_M$. □

Lemma 6.13. *Let $T \in Alg\, Lat(\mathfrak{L}_{\Lambda_3})$. There is some $A \in \mathfrak{L}_{\Lambda_3}$ such that $T\big|_{\{\xi_x\}^\perp} = A\big|_{\{\xi_x\}^\perp}$.*

Proof. Let $T \in \text{Alg Lat}(\mathfrak{L}_\Lambda)$. Given the block matrix form for T from Lemma 6.11, we need to show for all $m \geq n$, that $\alpha_{m,n} = \alpha_{m+1,n+1}$, $\beta_{m,n} = \gamma_{m+1,n+1}$, and $\gamma_{m,n} = \beta_{m+1,n+1}$. We will first show that $\alpha_{m,n} = \alpha_{m+1,n+1}$.

Let $\mathcal{M}_h$ be the $\mathfrak{L}_{\Lambda_3}$-invariant subspace of $\mathcal{H}_{\Lambda_3}$ generated by h_1. Then $\mathcal{M}_h$ has orthogonal basis $\{h_1, h_2, h_3, \dots\}$, and L_a, L_b, and L_c all act as the unilateral shift on $\mathcal{M}_h$. So $\mathfrak{L}_{\Lambda_3}\big|_{\mathcal{M}_h}$ is isomorphic to $\mathfrak{L}_1$, the analytic Toeplitz algebra, and thus is reflexive [12]. By Lemma 6.12, there is some $A \in \mathfrak{L}_{\Lambda_3}$ such that $A\big|_{\mathcal{M}_h} = T\big|_{\mathcal{M}_h}$. Since $A \in \mathfrak{L}_{\Lambda_3}$, there are constants λ_ℓ such that
$$Q_{n+\ell} A(h_n) = \lambda_\ell h_{n+\ell} \quad \text{for all } n \geq 1, \ell \geq 0.$$
Thus,
$$Q_{n+\ell} T(h_n) = \lambda_\ell h_{n+\ell} \quad \text{for all } n \geq 1, \ell \geq 0.$$
This means ℓth diagonal of 3×3 blocks in the matrix decomposition of T all have the same $(1,1)$-entries. In particular, $\alpha_{m,n} = \alpha_{m+1,n+1}$ for all $m > n$.

Now consider the subspace of $\mathcal{H}_{\Lambda_3}$ given by

$$\mathcal{M}_1 = \Big\{ \sum_{n=0}^{\infty} \lambda_n (j_n + k_{n+1}) \,:\, \sum_{n=1}^{\infty} |\lambda_n|^2 < \infty \Big\}.$$

This space is invariant for $\mathfrak{L}_{\Lambda_3}$ because

$$L_a(j_n + k_{n+1}) = j_{n+1} + k_{n+2} \in \mathcal{M}_1,$$

and for n odd,

$$L_b(j_n + k_{n+1}) = \omega(j_{n+1} + k_{n+2}) \in \mathcal{M}_1$$

$$L_c(j_n + k_{n+1}) = \omega^2(j_{n+1} + k_{n+2}) \in \mathcal{M}_1;$$

whereas if n is even, then

$$L_b(j_n + k_{n+1}) = \omega^2(j_{n+1} + k_{n+2}) \in \mathcal{M}_1$$

$$L_c(j_n + k_{n+1}) = \omega(j_{n+1} + k_{n+2}) \in \mathcal{M}_1.$$

Thus, $\mathcal{M}_1$ is also invariant for T. Notice that for all $\zeta \in \mathcal{M}_1$, and $n \geq 1$, $\langle \zeta, j_n \rangle = \langle \zeta, k_{n+1} \rangle$. It follows that

$$\langle T(j_n + k_{n+1}), j_m \rangle = \langle T(j_n + k_{n+1}), k_{m+1} \rangle,$$

that is to say, $\beta_{m,n} = \gamma_{m+1,n+1}$.

Similarly, using the $\mathfrak{L}_{\Lambda_3}$-invariant subspace

$$\mathcal{M}_2 = \Big\{ \sum_{n=0}^{\infty} \lambda_n (k_n + j_{n+1}) \,:\, \sum_{n=1}^{\infty} |\lambda_n|^2 < \infty \Big\},$$

we can show that $\gamma_{m,n} = \beta_{m+1,n+1}$. This proves the lemma. □

The last result we need concerns the following vectors, for $0 < |\varepsilon| < 1$:

(even length terms); (odd length terms)

$$A_\varepsilon = \xi_x + \sum_{n=1}^{\infty} \varepsilon^{2n} \xi_{a^{2n}}; \quad A'_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{2n-1} \xi_{a^{2n-1}}$$

$$B_\varepsilon = \xi_x + \sum_{n=1}^{\infty} \varepsilon^{2n} \xi_{ba^{2n-1}}; \quad B'_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{2n-1} \xi_{ba^{2n-2}}$$

$$C_\varepsilon = \xi_x + \sum_{n=1}^{\infty} \varepsilon^{2n} \xi_{ca^{2n-1}}; \quad C'_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{2n-1} \xi_{ca^{2n-2}}$$

Lemma 6.14. *For $0 < |\varepsilon| < 1$, the subspace $M = span\{A_\varepsilon, B_\varepsilon, C_\varepsilon, A'_\varepsilon, B'_\varepsilon, C'_\varepsilon\}$ is invariant for Alg Lat$(\mathfrak{L}^*_{\Lambda_3})$.*

Proof. Note that

$$\begin{aligned}
L_a^*(A_\varepsilon) &= \varepsilon A'_\varepsilon & \qquad L_a^*(A'_\varepsilon) &= \varepsilon A_\varepsilon\\
L_a^*(B_\varepsilon) &= \varepsilon C'_\varepsilon & \qquad L_a^*(B'_\varepsilon) &= \varepsilon C_\varepsilon\\
L_a^*(C_\varepsilon) &= \varepsilon B'_\varepsilon & \qquad L_a^*(C'_\varepsilon) &= \varepsilon B_\varepsilon\\[1ex]
L_b^*(A_\varepsilon) &= \varepsilon B'_\varepsilon & \qquad L_b^*(A'_\varepsilon) &= \varepsilon B_\varepsilon\\
L_b^*(B_\varepsilon) &= \varepsilon A'_\varepsilon & \qquad L_b^*(B'_\varepsilon) &= \varepsilon A_\varepsilon\\
L_b^*(C_\varepsilon) &= \varepsilon C'_\varepsilon & \qquad L_b^*(C'_\varepsilon) &= \varepsilon C_\varepsilon\\[1ex]
L_c^*(A_\varepsilon) &= \varepsilon C'_\varepsilon & \qquad L_c^*(A'_\varepsilon) &= \varepsilon C_\varepsilon\\
L_c^*(B_\varepsilon) &= \varepsilon B'_\varepsilon & \qquad L_c^*(B'_\varepsilon) &= \varepsilon B_\varepsilon\\
L_c^*(C_\varepsilon) &= \varepsilon A'_\varepsilon & \qquad L_c^*(C'_\varepsilon) &= \varepsilon A_\varepsilon
\end{aligned}$$

Thus, M is invariant for L_a^*, L_b^*, and L_c^*, and hence for $\mathfrak{L}^*_{\Lambda_3}$. □

Finally we can prove that this free semigroupoid algebra is reflexive:

Theorem 6.15. *$\mathfrak{L}_{\Lambda_3}$ is reflexive.*

Proof. Let $T \in \mathfrak{L}_{\Lambda_3}$. Lemma 6.13 implies that there is some $A \in \operatorname{Alg}\operatorname{Lat}\mathfrak{L}_{\Lambda_3}$ such that $T - A$ is equal to 0 on $\{\xi_x\}^\perp$. Let $R = T - A$. Then $R \in \operatorname{Alg}\operatorname{Lat}(\mathfrak{L}_{\Lambda_3})$, $R\big|_{\{\xi_x\}^\perp} = 0$, and there are constants $\{\rho_w\}_{w\in\Lambda_3}$ such that

$$R\xi_x = \sum_{w\in\Lambda_3} \rho_w \xi_w.$$

We want to show that $\rho_w = 0$ for all $w \in \Lambda_3$.

Since R is a rank one operator, R^* is also a rank one operator, and range(R^*) is closed. Thus, by Proposition 4.6 from [5], range $R^* = (\ker R)^\perp$, so if $R \neq 0$, then range $R^* = \operatorname{span}\{\xi_x\}$. Thus, with A_ε defined as in Lemma 6.14, $R^*(A_\varepsilon) = k\xi_x$ for some k. But also, $R^* \in \operatorname{Alg}\operatorname{Lat}(\mathfrak{L}^*_{\Lambda_3})$, so by Lemma 6.14, $R^*(A_\varepsilon) = \lambda_A A_\varepsilon + \lambda_B B_\varepsilon + \lambda_C C_\varepsilon + \lambda'_A A'_\varepsilon + \lambda'_B B'_\varepsilon + \lambda'_C C'_\varepsilon$ for some constants $\lambda_A, \lambda_B, \lambda_C, \lambda'_A, \lambda'_B, \lambda'_C$. So for $w \in \Lambda_3, w \neq x$:

$$0 = \langle R^*(A_\varepsilon), \xi_w\rangle = \begin{cases} \varepsilon^{2n}\lambda_A & : w = a^{2n}\\ \varepsilon^{2n-1}\lambda'_A & : w = a^{2n-1}\\ \varepsilon^{2n}\lambda_B & : w = ba^{2n-1}\\ \varepsilon^{2n-1}\lambda'_B & : w = ba^{2n-2}\\ \varepsilon^{2n}\lambda_C & : w = ca^{2n-1}\\ \varepsilon^{2n-1}\lambda'_C & : w = ca^{2n-2} \end{cases}.$$

So $\lambda_A = \lambda_B = \lambda_C = \lambda'_A = \lambda'_B = \lambda'_C = 0$, and so $R^*(A_\varepsilon) = 0$.

Now we will find R^* explicitly. Let $\mu \in \Lambda_3$ and $h \in \mathcal{H}_{\Lambda_3}$ be arbitrary, and let $\lambda = \langle h, \xi_x\rangle$. Then

$$\langle R^*\xi_\mu, h\rangle = \langle \xi_\mu, Rh\rangle = \langle \xi_\mu, R\lambda\xi_x\rangle = \overline{\lambda}\langle \xi_\mu, R\xi_x\rangle = \overline{\lambda\rho_\mu} = \langle \overline{\rho_\mu}\xi_x, h\rangle.$$

Thus, $R^*\xi_\mu = \overline{\rho_\mu}\xi_x$, for any path $\mu \in \Lambda_3$. So, for $0 < |\varepsilon| < 1$:

$$R^*(A_\varepsilon) = R^*\big(\xi_x + \sum_{n=1}^{\infty} \varepsilon^{2n}\xi_{a^{2n}}\big)$$
$$= \big(\overline{\rho_x} + \sum_{n=1}^{\infty} \varepsilon^{2n}\overline{\rho_{a^{2n}}}\big)\xi_x$$

But we've already shown that $R^*(A_\varepsilon) = 0$. So in fact

$$\overline{\rho_x} + \sum_{n=1}^{\infty} \varepsilon^{2n}\overline{\rho_{a^{2n}}} = 0.$$

This holds for all $0 < |\varepsilon| < 1$. So we have a power series equal to 0 on the set $\mathbb{D} \setminus \{0\}$. This implies that $\rho_x = \rho_{a^{2n}} = 0$ for all n.

Similarly, by looking at R^* applied to $A'_\varepsilon, B_\varepsilon, B'_\varepsilon, C_\varepsilon$, and C'_ε, we can show that $\rho_{a^n} = \rho_{ba^{n-1}} = \rho_{ca^{n-1}} = 0$ for all $n > 0$. Thus, $R = 0$. So $T = A \in \mathfrak{L}_{\Lambda_3}$. □

The same proof can be generalized (see [2] for details) to show:

Theorem 6.16. *For all $n \in \mathbb{N}$, the free semigroupoid algebra $\mathfrak{L}_{\Lambda_n}$ defined in Example* 4.7 *is reflexive.*

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