New York Journal of Mathematics

New York J. Math. **30** (2024) 593–624.

Small angle limits of negatively curved Kähler–Einstein metrics with crossing edge singularities

Yuxiang Ji

ABSTRACT. Let (X, D) be a log smooth log canonical pair such that $K_X + D$ is ample. Extending a theorem of Guenancia and building on his techniques, we show that negatively curved Kähler–Einstein crossing edge metrics converge to Kähler–Einstein mixed cusp and edge metrics smoothly away from the divisor when some of the cone angles converge to 0. We further show that near the divisor such normalized Kähler–Einstein crossing edge metrics converge to a mixed cylinder and edge metric in the pointed Gromov–Hausdorff sense when some of the cone angles converge to 0 at (possibly) different speeds.

CONTENTS

1. Introduction

1.1. The small angle world. Let X be a compact Kähler manifold of dimension n and $D \subset X$ be a smooth hypersurface. A Kähler edge metric on X with angle $2\pi\beta$ $(0 < \beta \leq 1)$ along D is a Kähler metric on $X \setminus D$ that is quasi-isometric to the model edge metric at D:

$$\omega_{\text{cone}} = \frac{\beta^2\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^{n} \sqrt{-1}dz_i \wedge d\bar{z}_i,$$

Received June 23, 2022.

2010 *Mathematics Subject Classification.* 32Q20, 53C21.

Key words and phrases. Kähler–Einstein edge metrics, Poincaré singularities, holomorphic bisectional curvature.

Research supported in part by the NSF grant DMS-1906370 and the Michael Brin Graduate Fellowship at the University of Maryland.

ISSN 1076-9803/2024

where $z_1, \dots, z_n$ are holomorphic coordinates and D is locally given by $\{z_1 = 0\}$. Tian generalized Calabi's conjecture to Kähler–Einstein edge metrics and studied the applications of negatively curved Kähler–Einstein edge metric to algebraic geometry by letting the cone angle tend to 2π [17]. Donaldson proposed using Kähler edge metrics to study the existence problem of smooth Kähler–Einstein metrics of positive curvature on X by deforming the cone angle to 2π [4]. Since then much research has gone into understanding the large angle limits (when $\beta \to 1$) of Kähler (–Einstein) edge metrics in relation to the Yau–Tian–Donaldson conjecture. Cheltsov–Rubinstein initiated the program of studying Kähler–Einstein edge metrics in another extreme where the cone angle goes to zero [3]. One topic of their program is to understand the limit, when such exists, of Kähler–Einstein edge metrics as the cone angle tends to 0. This paper is following that program. We prove that on a log smooth log canonical pair (X, D), i.e., X is a compact Kähler manifold and $D = \sum_{i=1}^r (1-\beta_i)D_i$ is a divisor with simple normal crossing support such that $\beta_i \in [0, 1)$ for all i, assuming that $K_X + \sum_{i=1}^r D_i$ is ample, then the negatively curved Kähler–Einstein crossing edge metrics converge to the Kähler–Einstein mixed cusp and edge metric when some of the cone angles tend to 0. We further study the asymptotic behavior of the Kähler–Einstein crossing edge metrics near the divisor and show the rescaled Kähler–Einstein crossing edge metrics converge to mixed cylinder and edge metrics on $(\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$ when some of the cone angles tend to 0. A beautiful theorem of Guenancia related the Kähler–Einstein edge metric to the Kähler–Einstein cusp metric in the smooth case [8], which confirmed a conjecture made by Mazzeo [12]. Our paper is a generalization of Guenancia's results to the snc case. An added interesting feature of our work is the possibility that multiple angles converge to zero at (possibly) different rates. Such kind of small angle limits, where some angles tend to zero but others do not could be interesting when studying moduli spaces of Kähler–Einstein metrics since these limits could correspond to approaching various strata in the moduli space.

1.2. Guenancia's convergence result. Let $\mathbb{D}^*$ be the punctured unit disc in $\mathbb{C}$. The first observation is that

$$\omega_{\eta,\mathbb{D}^*} := \frac{\eta^2\sqrt{-1}dz \wedge d\bar{z}}{|z|^{2(1-\eta)}(1-|z|^{2\eta})^2}, \quad z \in \mathbb{D}^*, \eta \in (0,1),$$

is a Kähler edge metric with cone angle $2\pi\eta$ at 0 and it has constant Ricci curvature -2. When η tends to 0, $\omega_{\eta,\mathbb{D}^*}$ converges pointwise (see (5) for the detail) to the following cusp metric (also called a Poincaré metric) on $\mathbb{D}^*$:

$$\omega_{P,\mathbb{D}^*} := \frac{\sqrt{-1}dz \wedge d\bar{z}}{|z|^2\left(\log|z|^2\right)^2}. \tag{1}$$

In higher dimensions, we consider the pair (X, D) where X is a compact Kähler manifold of dimension n and D is a smooth divisor such that $K_X + D$ is ample. By Kobayashi [11, Theorem 1] or Tian–Yau [18, Theorem 2.1] with complements by Wu [19], there exists a unique complete Kähler–Einstein metric ω_0 on $X \setminus D$ with cusp singularity along D such that $\operatorname{Ric}\omega_0 = -\omega_0$.

Definition 1.1. ω_0 is said to have cusp singularities along D if whenever D is locally given by $\{z_1 = 0\}$, there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\mathrm{cusp}} \le \omega_0 \le C\omega_{\mathrm{cusp}},$$

where ω_{cusp} is the model cusp metric:

$$\omega_{\mathrm{cusp}} := \frac{\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^2 \log^2 |z_1|^2} + \sum_{i=2}^{n} \sqrt{-1}dz_i \wedge d\bar{z}_i.$$

Since ampleness is an open condition, there exists some β_0 such that for $0 < \beta < \beta_0$, $K_X + (1-\beta)D$ is also ample. Thus, by Campana–Guenancia–Păun [9, Theorem A] and Jeffres–Mazzeo–Rubinstein [10, Theorem 2], there exists a unique negatively curved Kähler–Einstein edge metric ω_β for each such small $\beta \in (0, \beta_0]$. The family of metrics $\{\omega_\beta\}_{0\le\beta<\beta_0}$ can be seen as currents on X satisfying the twisted Kähler–Einstein equation:

$$\operatorname{Ric}\omega_\beta = -\omega_\beta + (1-\beta)[D], \quad 0 \le \beta < \beta_0.$$

As a generalization of the observation discussed in the beginning of this section, Guenancia related these two metrics as follows:

Theorem 1.2. [8, Theorem A and B] *Let ω_0 be defined as in Definition* 1.1. *$\{\omega_\beta\}_{0<\beta<\beta_0}$ converge to ω_0 in both the weak topology of currents and the $C^\infty_{\mathrm{loc}}(X \setminus D)$-topology as $\beta \to 0$. Moreover, for $\beta \in (0, 1/2]$, there exists a constant $C > 0$ independent of β such that on any coordinate chart U where D is given by $\{z_1 = 0\}$, the Kähler–Einstein edge metric ω_β satisfies*

$$C^{-1}\omega_{\beta,\mathrm{mod}} \le \omega_\beta \le C\omega_{\beta,\mathrm{mod}}, \tag{2}$$

where

$$\omega_{\beta,\mathrm{mod}} := \frac{\beta^2\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}(1-|z_1|^{2\beta})^2} + \sum_{i=2}^{n} \sqrt{-1}dz_i \wedge d\bar{z}_i.$$

The relation between the convergence result in Theorem 1.2 and (2) is that the weak convergence from $(\omega_\beta)_{0<\beta<\beta_0}$ to ω_0 can be recovered from (2) by using Lebesgue's Dominated Convergence Theorem.

As an application of Theorem 1.2, Guenancia studied the asymptotic behavior of ω_β near D as $\beta \to 0$. Fix a point $p \in D$, let U_β denote the punctured metric ball $B_{\omega_\beta}(p, 1)$ of radius 1 centered at p with respect to the metric ω_β. Then after renormalization by β^{-2}, there exists a subsequence of the metric spaces $(U_\beta, \frac{1}{\beta^2}\omega_\beta)$ converging to $(\mathbb{C}^* \times \mathbb{C}^{n-1}, \omega_{\mathrm{cyl}})$ in the pointed Gromov–Hausdorff sense, where ω_{cyl} is a so-called cylindrical metric:

Definition 1.3. Let $\pi : \mathbb{C}^n \to \mathbb{C}^* \times \mathbb{C}^{n-1}$ be the universal cover of $\mathbb{C}^* \times \mathbb{C}^{n-1}$ given by $\pi(z_1 \dots, z_n) = (e^{z_1}, z_2, \dots, z_n)$. A Kähler metric ω_{cyl} on $\mathbb{C}^* \times \mathbb{C}^{n-1}$ is called cylindrical if $\pi^*\omega$ is isometric to the usual Euclidean metric on $\mathbb{C}^n$ up to a complex linear transformation.

Theorem 1.4. [8, Theorem C] *Let $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to* 0*. Then, up to extracting a subsequence, there exists a cylindrical metric ω_{cyl} on $\mathbb{C}^*\times\mathbb{C}^{n-1}$ such that the metric spaces $(U_{\beta_k}, \beta_k^{-2}\omega_{\beta_k})$ converge in pointed Gromov-Hausdorff topology to $(\mathbb{C}^*\times\mathbb{C}^{n-1}, \omega_{\mathrm{cyl}})$ when k tends to $+\infty$.*

1.3. The main results. A natural problem is to generalize Theorems 1.2 and 1.4 to the snc case when all or some of the cone angles tend to 0. This possibility is mentioned in [8] but there is no detailed proof given. In this paper, we generalize Theorems 1.2 and 1.4 to the snc setting.

From now on, let (X,ω) be an n-dimensional compact Kähler manifold with a smooth Kähler metric ω. Fix a divisor $D_\beta := \sum_{i=1}^{r}(1-\beta_i)D_i$, where $\beta_i \in (0,1)$ for $i = 1,\dots,r$. Assume each D_i is smooth and irreducible. We further assume D_β has simple normal crossing support, i.e., for any $p \in \operatorname{supp}(D_\beta)$ lying in the intersection of exactly m components $D_1,\dots,D_m$, there exists a coordinate chart $(U,\{z_i\}_{i=1}^n)$ containing p such that $D_j|_U = \{z_j = 0\}$ for $j = 1,\dots,m$, $m \le n$. Suppose $K_X+\sum_{i=1}^{r} D_i$ is ample. Let s_i denote a defining section of D_i and $h_i = |\cdot|_{h_i}$ be a smooth hermitian metric on L_{D_i}, which is the line bundle induced by D_i. We normalize h_i such that $\log|s_i|^2_{h_i} + 1 < 0$ for each i. Denote

$$\beta := (\beta_1,\dots,\beta_r) \in (0,1)^r.$$

The following result is well known (see [14, §4] for a survey) .

Theorem 1.5. [9, 10, 13] (Solution of the Calabi–Tian conjecture in the negative regime) *There exists a unique Kähler–Einstein crossing edge metric with negative curvature, denoted by $\omega_{\phi_\beta} = \omega + \sqrt{-1}\partial\bar\partial\phi_\beta$ on X with cone angle $2\pi\beta_i$ along D_i for each i. In another word, ω_{ϕ_β} satisfies the Kähler–Einstein edge equation*

$$\operatorname{Ric}\omega_{\phi_\beta} - [D_\beta] = -\omega_{\phi_\beta}.$$

Analogously to [8], let us introduce a reference metric,

$$\Omega_\beta := \omega - \sum_{i=1}^{r}\sqrt{-1}\partial\bar\partial\log\left[\frac{1-|s_i|^{2\beta_i}_{h_i}}{\beta_i}\right]^2. \tag{3}$$

Our first result is as follows.

Theorem 1.6. *Let ω_{ϕ_β} be given by Theorem* 1.5*. Let Ω_β be given by* (3)*. There exists a uniform constant $C > 0$, independent of $\beta \in (0,\frac{1}{2}]^r$, such that*

$$C^{-1}\Omega_\beta \le \omega_{\phi_\beta} \le C\Omega_\beta.$$

The key point of Theorem 1.6 is that the constant C is uniform with respect to small β_i, $i = 1, \ldots, r$. According to Theorem 1.6 and Lebesgue's Dominated Convergence Theorem, we obtain the weak convergence from ω_{ϕ_β} to the Kähler–Einstein mixed cusp and edge metric ω_0 constructed in [7] as some of the cone angles tend to 0. In particular, when $\beta \to 0 \in [0,1)^r$, the limiting metric of such ω_{ϕ_β} is the unique Kähler–Einstein cusp metric on $(X, \sum_{i=1}^{r} D_i)$ constructed in [11, 18, 19]. More precisely, the following result is shown in section 2.3.

Theorem 1.7. *The Kähler–Einstein crossing edge metric ω_{ϕ_β} converges to a Kähler–Einstein mixed cusp and edge metric on (X, D_β) globally in a weak sense and locally in a strong sense when some of the cone angles tend to* 0*. In particular, ω_{ϕ_β} converges to the Kähler–Einstein cusp metric on $(X, \sum_{i=1}^{r} D_i)$ in the above sense when $\beta \to 0 \in [0,1)^r$.*

Remark 1.8. In Theorem 1.7, we assume $K_X + \sum_{i=1}^{r} D_i$ to be ample to ensure the existence of a limiting Kähler–Einstein metric by the work of Kobayashi [11] and Tian–Yau–Wu [18, 19]. An interesting open problem is to study the convergence of ω_{ϕ_β} when we only assume the ampleness of $K_X + D_\beta$ for $0 < \beta_i \ll 1$, $i = 1, \ldots, r$.

Theorem 1.6 and Theorem 1.7 generalize Guenancia's Theorem 1.2 from the smooth case to the snc case.

As an application of Theorem 1.6, we study the asymptotic behavior of the Kähler–Einstein crossing edge metric ω_{ϕ_β} near D_β when the smallest cone angle approaches 0, with possibly other cone angles also converging to 0.

To state the result, without loss of generality, we assume for $\beta = (\beta_1, \ldots, \beta_r)$ there holds $\beta_1 \le \beta_2 \le \cdots \le \beta_r$. Fix a point $p \in D_\beta$. Choose a coordinate chart $(U, \{z_i\}_{i=1}^n)$ containing p such that $D_j|_U = \{z_j = 0\}$ for $j = 1, \ldots, m$, $m \le n$. Consider a small neighborhood U_β about p defined by

$$\left\{ z \in (\mathbb{C}^*)^m \times \mathbb{C}^{n-m} \,:\, |z_1| < e^{-\frac{1}{2\beta_1}}, |z_j| < \left(\frac{\beta_1}{\beta_j}\right)^{\frac{1}{\beta_j}}, j = 2, \ldots, m, |z_\ell| < 1, \ell = m+1, \ldots, n \right\}.$$

We show that after normalization by factor β_1^{-2}, a subsequence of metrics ω_{ϕ_β} converges to a mixed cylinder and edge metric on $(\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$ (see Definition 3.1 for more details. We note that our Definition 3.1 is in a weaker sense of quasi-isometry comparing to Definition 1.3.) as β_1 tends to 0. The limiting metric has cylindrical part along the component D_1 where the cone angle β_1 approaches 0 while has conical singularities along other components. More precisely, the third result of this paper is as follows:

Theorem 1.9. *Let $\{\beta_{1,k}\}_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to* 0. *Assume further that $\{\beta_{i,k}\}_{k\in\mathbb{N}}$ does not converge to 0 for each $i = 2, \dots, r$ and all $\beta_{i,k} \in (0, \frac{1}{2}]$. Let $\omega_{\phi_{\beta_k}}$ be the (negatively curved) Kähler–Einstein crossing edge metric on $(X, D_k = \sum_{i=1}^{r}(1-\beta_{i,k})D_i)$. Then there exists a subsequence of the metric spaces $(U_{\beta_k}, \frac{1}{\beta_{1,k}^2}\omega_{\phi_{\beta_k}})$ converging in pointed Gromov-Hausdorff topology to $((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \omega_\infty)$ where ω_∞ is a mixed cylinder and edge metric.*

Theorem 1.9 is a generalization of [8, Theorem C] which shows the convergence of Kähler–Einstein edge metrics to a cylindrical metric in the smooth case. Regarding complex dimension 1, i.e., in the Riemann surface case, but in the positive curvature regime, Rubinstein–Zhang showed that the (American) football equipped with the Ricci soliton metric converges to the cone-cigar soliton on $\mathbb{R}_+$ as two cone angles converge to 0 at a different speed and to a flat cylindrical metric as two cone angles converge to 0 at a comparable speed [16, Theorem 1.1-1.3]. In [16], the S^1-symmetry of the metric plays an important role in the proof. In higher dimensions, we generalize Theorem 1.9 to allow more than one cone angles to tend to 0 and study the limit behavior of metrics under this joint degeneration of cone angles. The result is as follows.

Theorem 1.10. *Let $\{\beta_{1,k}\}_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to* 0. *Assume further that for any $i \in \{2, \dots, r\}$ such that $\{\beta_{i,k}\}_{k\in\mathbb{N}}$ also converges to* 0, *there holds $\lim_{k\to\infty} \frac{\beta_{1,k}}{\beta_{i,k}} \in [0,1]$ and all $\beta_{i,k} \in (0, \frac{1}{2}]$. Let $\omega_{\phi_{\beta_k}}$ be the (negatively curved) Kähler–Einstein crossing edge metric on $(X, D_k = \sum_{i=1}^{r}(1-\beta_{i,k})D_i)$. Then there exists a subsequence of the metric spaces $(U_{\beta_k}, \frac{1}{\beta_{1,k}^2}\omega_{\phi_{\beta_k}})$ converging in pointed Gromov-Hausdorff topology to $((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \omega_\infty)$, where ω_∞ is a mixed cylinder and edge metric with cylindrical part along components whose cone angles converge to* 0 *and conical part along other components.*

In the language of [16], [16, Theorem 1.1-1.3] completely describe, in a geometric sense, the boundary behavior of the body of ample angles [15] of the pair $(S^2, N + S)$, where N and S denote the north and south poles of the Riemann sphere respectively. In higher dimensions, given a pair $(X, \widetilde{D} = \sum_{i=1}^{r} D_i)$, Theorem 1.10 is still not a satisfactory description of the boundary of the body of ample angles of $(X, \widetilde{D})$ in the negative curvature regime. Part of the reason is that different subsequences may converge to different mixed cylinder and edge metrics. A complete characterization of the moduli space of such $(X, \widetilde{D})$ endowed with Kähler–Einstein crossing edge metrics in the sense of [14] is still open. However, Theorem 1.10 is interesting in its own right from an analytical point of view.

1.4. Main ingredients of the proofs. We first recall the key ingredient in the proof of Theorem 1.2 is the boundedness of the holomorphic bisectional curvature of the model metric $\omega_{\beta,\mathrm{mod}}$, which makes it possible to use the Chern–Lu inequality to obtain the Laplacian estimates, cf. [8, Theorem 3.2]. Therefore, one way to prove corresponding results of Theorem 1.2 in the snc setting is to first extend [8, Theorem 3.2] to the snc setting, i.e., prove boundedness of holomorphic bisectional curvature of the model metric Ω_β (see (3) for details). This is done in Lemma 2.4 by adapting arguments in [10, Lemma 2.3] and making use of the fact that we only need to deal with small β_i's. Then the proof of Theorem 1.6 uses a modified maximum principle argument and Chern–Lu's inequality to give respectively the C^0 and Laplacian estiamte. A useful fact in the proof is the observation that Ω_β shares the same crossing edge singularities as ω_{ϕ_β} (see Claim 2.8 for the detail).

An important observation is that the reference metric Ω_β has the property of converging to a Kähler metric with mixed cusp and edge singularities when some of the cone angles tend to 0. This observation, combined with the content of Theorem 1.6, give us the result of Theorem 1.7 as a corollary. As another consequence of Theorem 1.6, Theorem 1.9 and Theorem 1.10 treat the limit behavior of the Kähler–Einstein crossing edge metric ω_{ϕ_β} near the divisor D_β when some of the cone angles approach 0. After fixing a point in the divisor D_β, we first rescale the reference metric to obtain its convergence to a mixed cylinder and edge metric (see Definition 3.1) as the smallest cone angle tends to 0 in a small neighborhood of D_β. To obtain the pointed Gromov–Hausdorff convergence of the rescaled Kähler–Einstein crossing edge metric ω_{ϕ_β} near the divisor, we actually show a stronger local smooth convergence result. We use Theorem 1.6 and the limit behavior of Ω_β mentioned above to obtain C^0-estimates of rescaled ω_{ϕ_β}. By a standard use of Evans–Krylov theory and Arzelà–Ascoli Theorem, we obtain the C^∞_{loc}-convergence of the rescaled ω_{ϕ_β} as some of the cone angles tend to 0.

2. Small angle limits of the Kähler–Einstein crossing edge metrics

Let $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ be the punctured unit disc in $\mathbb{C}$. The first observation is, for $\eta \in (0,1)$, the following Kähler metric

$$\begin{aligned}\omega_{\eta,\mathbb{D}^*} :&= -\sqrt{-1}\partial\bar\partial \log(1-|z|^{2\eta}) \\ &= \frac{\sqrt{-1}\eta^2|z|^{2\eta-2}}{(1-|z|^{2\eta})^2} dz\wedge d\bar z\end{aligned}$$

has negative constant curvature and cone singularity with cone angle $2\pi\eta$ at $0 \in \mathbb{C}$. Indeed, direct calculation using the Poincaré–Lelong formula [6] yields

$$\mathrm{Ric}\,\omega_{\eta,\mathbb{D}^*} = 2\pi(1-\eta)\delta_0 - 2\omega_{\eta,\mathbb{D}^*}, \tag{4}$$

where δ_0 denotes the Dirac measure at 0.

For a fixed $z \in \mathbb{D}^*$,

$$\lim_{\eta\to 0} \frac{\eta^2|z|^{2\eta-2}}{(1-|z|^{2\eta})^2} = \frac{1}{|z|^2(\log|z|^2)^2}. \tag{5}$$

Thus, $\omega_{\eta,\mathbb{D}^*}$ converges uniformly to the Poincaré metric $\omega_{P,\mathbb{D}^*}$ defined in (1) for any compact $K \Subset \mathbb{D}^*$ when η tends to 0. Note that

$$\operatorname{Ric}\omega_{P,\mathbb{D}^*} = 2\pi\delta_0 - 2\omega_{P,\mathbb{D}^*}.$$

Thus, $\omega_{P,\mathbb{D}^*}$ is a Kähler–Einstein cusp metric on $\mathbb{D}^*$ with cusp singularity at 0. Next we introduce a reference metric that generalizes $\omega_{\eta,\mathbb{D}^*}$ to higher dimensional manifolds.

2.1. The reference metric. From now on, let (X, D_β) be an n-dimensional compact Kähler manifold with an $\mathbb{R}$-divisor $D_\beta = \sum_{i=1}^r (1-\beta_i)D_i$ such that $K_X + \sum_{i=1}^r D_i$ is ample, where $\beta_i \in (0,1)$ for $i = 1,\dots,r$. Given this assumption, $K_X + D_\beta$ is ample for small $\beta_1,\dots,\beta_r$ since ampleness is an open condition. Assume each D_i is smooth and irreducible. We further assume D_β has simple normal crossing support, i.e., for any $p \in \operatorname{supp}(D_\beta)$ lying in the intersection of exactly m divisors $D_1,\dots,D_m$, $m \le n$, there exists a coordinate chart $(U, \{z_i\}_{i=1}^n)$ containing p such that $D_j|_U = \{z_j = 0\}$ for $j = 1,\dots,m$. Let s_i denote a defining holomorphic section of D_i and $h_i = |\cdot|_{h_i}$ be a smooth hermitian metric on L_{D_i}, which is the line bundle induced by D_i. Let θ_i denote the curvature form of each (L_{D_i}, h_i). We normalize h_i such that $\log|s_i|^2_{h_i} + 1 < 0$ for each i. Let ω be a fixed smooth Kähler metric with $[\omega] = c_1\left(K_X + \sum_{i=1}^r D_i\right)$. Below we denote $D := \sum_{i=1}^r D_i$ and $\beta := (\beta_1,\dots,\beta_r)$.

Define the reference metric:

$$\Omega_\beta := \omega - \sum_{i=1}^r \sqrt{-1}\partial\bar\partial \log\left[\frac{1-|s_i|^{2\beta_i}_{h_i}}{\beta_i}\right]^2. \tag{6}$$

Remark 2.1. The appearance of β_i in the denominator of the log term in the potential function does not affect the definition of the reference metric. We use this convention, following [8], since the potential function in (6) defined in such a way will be shown to converge weakly to a potential function for some Kähler cusp metric. See Lemma 2.10 for details.

Ω_β can be seen as a generalization of $\omega_{\eta,\mathbb{D}^*}$ to higher dimensional manifolds. First, let us recall [8, Lemma 3.1].

Lemma 2.2. *Ω_β is a Kähler edge form with cone angle $2\pi\beta_i$ along D_i for $i = 1,\dots,r$. More precisely,*

$$\Omega_\beta = \omega + 2\cdot\sum_{i=1}^r\left(\sqrt{-1}\frac{\beta_i^2}{|s_i|^{2-2\beta_i}_{h_i}(1-|s_i|^{2\beta_i}_{h_i})^2}\langle D^{1,0}s_i, D^{1,0}s_i\rangle - \frac{\beta_i|s_i|^{2\beta_i}_{h_i}}{1-|s_i|^{2\beta_i}_{h_i}}\theta_i\right), \tag{7}$$

where $D^{1,0}$ is the $(1,0)$-part of the Chern connection of (L_{D_i}, h_i) for each i. Up to rescaling $\{h_i\}_{i=1,\dots,r}$, there holds $\Omega_\beta \geq \frac{1}{2}\omega$.

Proof. A concise proof for the case $r = 1$ is given in [8, Lemma 3.1]. For the reader's convenience, we give a detailed proof here. In fact, it suffices to show (7) when $r = 1$. Hence, below we suppose $r = 1$ and drop the subscript i for simplicity.

If we set

$$f(x) = -\log\left(\frac{1-x^\beta}{\beta}\right)^2,$$
$$\phi = |s|_h^2 = h \cdot |s|^2,$$

then $\Omega_\beta = \omega + \sqrt{-1}\partial\bar{\partial} f \circ \phi$. Recall there holds

$$\sqrt{-1}\partial\bar{\partial} f \circ \phi = \sqrt{-1}(f''(\phi)\partial\phi \wedge \bar{\partial}\phi + f'(\phi)\partial\bar{\partial}\phi).$$

We calculate

$$f' = \frac{2\beta x^{\beta-1}}{1-x^\beta},$$
$$f'' = \frac{-2\beta x^{\beta-2}}{1-x^\beta} + \frac{2\beta^2 x^{\beta-2}}{(1-x^\beta)^2}.$$

Then

$$\Omega_\beta = \omega + \sqrt{-1} \cdot \frac{2\beta |s|_h^{2\beta-2}}{1-|s|_h^{2\beta}} \partial\bar{\partial}|s|_h^2 + \sqrt{-1} \cdot \left(\frac{2\beta^2 |s|_h^{2\beta-4}}{(1-|s|_h^{2\beta})^2} - \frac{2\beta |s|_h^{2\beta-4}}{1-|s|_h^{2\beta}}\right)\partial|s|_h^2 \wedge \bar{\partial}|s|_h^2.$$

Note

$$\partial\bar{\partial}|s|_h^2 = \bar{s}\partial s \wedge \bar{\partial} h + |s|^2\partial\bar{\partial} h + h\partial s \wedge \bar{\partial}\bar{s} + s\partial h \wedge \bar{\partial}\bar{s}, \tag{8}$$

$$\partial|s|_h^2 \wedge \bar{\partial}|s|_h^2 = |s|^4 \partial h \wedge \bar{\partial} h + sh|s|^2 \partial h \wedge \bar{\partial}\bar{s} + \bar{s}h|s|^2 \partial s \wedge \bar{\partial} h + |s|^2 h^2 \partial s \wedge \bar{\partial}\bar{s}, \tag{9}$$

and the fact

$$\theta = -\sqrt{-1}\partial\bar{\partial}\log h = \sqrt{-1}\left(\frac{\partial h \wedge \bar{\partial} h}{h^2} - \frac{\partial\bar{\partial} h}{h}\right) \tag{10}$$

$$\langle D^{1,0}s, D^{1,0}s\rangle = \langle \partial s \cdot e + \frac{\partial h}{h} s \cdot e, \partial s \cdot e + \frac{\partial h}{h} s \cdot e\rangle \tag{11}$$

$$= h\partial s \wedge \bar{\partial}\bar{s} + \bar{s}\partial s \wedge \bar{\partial} h + s\partial h \wedge \bar{\partial}\bar{s} + \frac{|s|^2}{h}\partial h \wedge \bar{\partial} h. \tag{12}$$

We calculate

$$\begin{aligned}\Omega_\beta &= \omega + \sqrt{-1}\frac{2\beta|s|_h^{2\beta-2}}{1-|s|_h^{2\beta}}\partial\bar\partial|s|_h^2 + \sqrt{-1}\left(\frac{2\beta^2|s|_h^{2\beta-4}}{(1-|s|_h^{2\beta})^2} - \frac{2\beta|s|_h^{2\beta-4}}{1-|s|_h^{2\beta}}\right)\partial|s|_h^2\wedge\bar\partial|s|_h^2 \\
&= \omega + \left(\sqrt{-1}\frac{2\beta|s|_h^{2\beta-2}}{1-|s|_h^{2\beta}}|s|^2\partial\bar\partial h - \sqrt{-1}\frac{2\beta|s|_h^{2\beta-4}}{1-|s|_h^{2\beta}}|s|^4\partial h\wedge\bar\partial h\right) \\
&+ \sqrt{-1}\frac{2\beta|s|_h^{2\beta-2}}{1-|s|_h^{2\beta}}(\bar s\partial s\wedge\bar\partial h + h\partial s\wedge\bar\partial\bar s + s\partial h\wedge\bar\partial\bar s) \\
&+ \sqrt{-1}\left(\frac{2\beta^2|s|_h^{2\beta-4}}{(1-|s|_h^{2\beta})^2} - \frac{2\beta|s|_h^{2\beta-4}}{1-|s|_h^{2\beta}}\right)(sh|s|^2\partial h\wedge\bar\partial\bar s + \bar s h|s|^2\partial s\wedge\bar\partial h + |s|^2h^2\partial s\wedge\bar\partial\bar s) \\
&+ \sqrt{-1}\frac{2\beta^2|s|_h^{2\beta-4}}{(1-|s|_h^{2\beta})^2}|s|^4\partial h\wedge\bar\partial h \\
&= \omega - \frac{2\beta|s|_h^{2\beta}}{1-|s|_h^{2\beta}}\sqrt{-1}(\frac{\partial h\wedge\bar\partial h}{h^2} - \frac{\partial\bar\partial h}{h}) \\
&+ \sqrt{-1}\frac{2\beta^2|s|_h^{2\beta-2}}{(1-|s|_h^{2\beta})^2}(h\partial s\wedge\bar\partial\bar s + \bar s\partial s\wedge\bar\partial h + s\partial h\wedge\bar\partial\bar s + \frac{|s|^2}{h}\partial h\wedge\bar\partial h) \\
&= \omega + 2\cdot\sqrt{-1}\frac{\beta^2|s|_h^{2\beta-2}}{(1-|s|_h^{2\beta})^2}\langle D^{1,0}s, D^{1,0}s\rangle - 2\cdot\frac{\beta|s|_h^{2\beta}}{1-|s|_h^{2\beta}}\theta,\end{aligned}$$

which is what we need. Since

$$\sqrt{-1}\frac{\beta_i^2}{|s_i|_{h_i}^{2-2\beta_i}(1-|s_i|_h^{2\beta_i})^2}\langle D^{1,0}s_i, D^{1,0}s_i\rangle$$

contributes as a non-negative $(1,1)$-form for each i, we will show that up to rescaling h_i, $\dfrac{\beta_i|s_i|_{h_i}^{2\beta_i}}{1-|s_i|_{h_i}^{2\beta_i}}$ can be made arbitrarily small to conclude that $\Omega_\beta \geq \frac{1}{2}\omega$. To see this, consider the function $f_{\beta_i}(t) := \dfrac{\beta_i t^{\beta_i}}{1-t^{\beta_i}}$. The function $f_{\beta_i}(t)$ is increasing in $(0,1)$ and satisfies $f_{\beta_i}(0) = 0$. Hence for any $\delta > 0$, $\exists t_\delta \in (0,1)$, such that for $t \in (0, t_\delta]$, $f_{\beta_i}(t) \leq \delta$ for each $i = 1, \ldots, r$. Now take $\delta = \dfrac{1}{4r\cdot\sup_{X,i}||\theta_i||_\omega}$ and rescale each h_i such that $|s_i|_{h_i}^2 \leq t_\delta$. Then

$$2\cdot\sum_{i=1}^r -\frac{\beta_i|s_i|_{h_i}^{2\beta_i}}{1-|s_i|_{h_i}^{2\beta_i}}\theta_i \geq -\frac{1}{2}\omega$$

and therefore $\Omega_\beta \geq \frac{1}{2}\omega$. □

When $r = 1$ and $\beta = \beta_1 \in (0, \frac{1}{2}]$, the following result of Guenancia states that the reference metric Ω_β has uniformly bounded holomorphic bisectional curvatures on $X \setminus D_\beta$.

Lemma 2.3. [8, Theorem 3.2] *When $r = 1$, there exists a constant $C > 0$ depending only on X such that for all $\beta \in (0, \frac{1}{2}]$, the holomorphic bisectional curvature of Ω_β is bounded by C.*

We generalize this result to the SNC case by adapting arguments from [10, Lemma 2.3].

Lemma 2.4. *There exists a constant $C > 0$ depending only on X such that for all β where $\beta_i \in (0, 1/2]$ for each $i = 1, \ldots, r$, the holomorphic bisectional curvature of Ω_β is uniformly bounded by C on $X \setminus D$.*

Proof. We prove the lemma following [10, Lemma 2.3]. When $r = 1$, this gives another proof for our Lemma 2.3. To deal with the more complicated $r > 1$ case, we need to treat non-diagonal terms in the metric tensor of Ω_β carefully. Since the idea of the proof is the same for general $r > 1$, we assume $r = 2$ for simplicity.

Step 1: Estimate the metric tensor.

Fix a point $p \in X \setminus D$. We can find local holomorphic coordinates such that $s_1 = z_1$, $s_2 = z_2$ and the hermitian metric h_k on L_{D_k} is given by $h_k = e^{-\phi_k}$ with $\phi_k(p) = 0$ and $d\phi_k(p) = 0$ for $k = 1, 2$. In these local coordinates, write

$$\omega = \sqrt{-1}\tilde{g}_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$
$$\theta^k = \sqrt{-1}\theta^k_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $\tilde{g}_{i\bar{j}}$ and $\theta^k_{i\bar{j}}$ are smooth functions of the coordinate z and $k = 1, 2$. Moreover, for $k = 1, 2$, we have

$$\begin{aligned}
\langle D^{1,0}s_k, D^{1,0}s_k\rangle &= \langle dz_k \cdot e_k - z_k\partial\phi_k \cdot e_k, dz_k \cdot e_k - z_k\partial\phi_k \cdot e_k\rangle \\
&= e^{-\phi_k}\left(1 - \bar{z}_k\frac{\partial\phi_k}{\partial\bar{z}_k} - z_k\frac{\partial\phi_k}{\partial z_k} + |z_k|^2\frac{\partial\phi_k}{\partial z_k}\frac{\partial\phi_k}{\partial\bar{z}_k}\right) dz^k \wedge d\bar{z}^k \\
&+ \sum_{i\neq k}^{n} e^{-\phi_k}\left(-\bar{z}_k\frac{\partial\phi_k}{\partial\bar{z}_i} + |z_k|^2\frac{\partial\phi_k}{\partial z_k}\frac{\partial\phi_k}{\partial\bar{z}_i}\right) dz^k \wedge d\bar{z}^i \\
&+ \sum_{j\neq k}^{n} e^{-\phi_k}\left(-z_k\frac{\partial\phi_k}{\partial z^j} + |z_k|^2\frac{\partial\phi_k}{\partial z_j}\frac{\partial\phi_k}{\partial\bar{z}_k}\right) dz^j \wedge d\bar{z}^k \\
&+ \sum_{i,j\neq k}^{n} e^{-\phi_k}|z_k|^2\frac{\partial\phi_k}{\partial z_i}\frac{\partial\phi_k}{\partial\bar{z}_j} dz^i \wedge d\bar{z}^j.
\end{aligned}$$

For the sake of brevity, we introduce the following notations for $k=1,2$:

$$\begin{aligned} a_k &= -\bar z_k \frac{\partial\phi_k}{\partial\bar z_k} - z_k\frac{\partial\phi_k}{\partial z_k} + |z_k|^2\frac{\partial\phi_k}{\partial z_k}\frac{\partial\phi_k}{\partial\bar z_k},\\ b_j^k &= \frac{\partial\phi_k}{\partial z^j} + \bar z_k\frac{\partial\phi_k}{\partial z_j}\frac{\partial\phi_k}{\partial\bar z_k}, \quad j\neq k,\\ c_{i\bar j}^k &= \frac{\partial\phi_k}{\partial z_i}\frac{\partial\phi_k}{\partial\bar z_j}, \quad i,j\neq k. \end{aligned}$$

Then a_k, b_j^k and $c_{i\bar j}^k$ are smooth and vanish at p. Writing $\Omega_\beta = \sqrt{-1}g_{i\bar j}dz^i\wedge d\bar z^j$, by (7) we have

$$\begin{aligned} g &= \tilde g_{i\bar j} - 2\cdot\sum_{k=1}^2 \frac{\beta_k|s_k|_{h_k}^{2\beta_k}}{1-|s_k|_{h_k}^{2\beta_k}}\theta_{i\bar j}^k + 2\cdot\sum_{k=1}^2\frac{\beta_k^2|s_k|_{h_k}^{2\beta_k-2}}{(1-|s_k|_{h_k}^{2\beta_k})^2}\langle D^{1,0}s_k, D^{1,0}s_k\rangle\\ &= \tilde g_{i\bar j} - 2\cdot\sum_{k=1}^2\frac{\beta_k e^{-\beta_k\phi_k}|z_k|^{2\beta_k}}{1-|z_k|^{2\beta_k}\cdot e^{-\beta_k\phi_k}}\theta_{i\bar j}^k + 2\cdot\sum_{k=1}^2\frac{\beta_k^2 e^{-(\beta_k-1)\phi_k}|z_k|^{2\beta_k-2}}{(1-|z_k|^{2\beta_k}e^{-\beta_k\phi_k})^2}\langle D^{1,0}s_k, D^{1,0}s_k\rangle. \end{aligned}$$

For each component, we have

$$\begin{aligned} g_{1\bar 1} &= \tilde g_{1\bar 1} - 2\cdot\sum_{k=1}^2\frac{\beta_k e^{-\beta_k\phi_k}|z_k|^{2\beta_k}}{1-|z_k|^{2\beta_k}\cdot e^{-\beta_k\phi_k}}\theta_{1\bar 1}^k + 2\cdot\frac{\beta_1^2e^{-(\beta_1-1)\phi_1}|z_1|^{2\beta_1-2}}{(1-|z_1|^{2\beta_1}e^{-\beta_1\phi_1})^2}e^{-\phi_1}(1+a_1)\\ &\quad + 2\cdot\frac{\beta_2^2e^{-(\beta_2-1)\phi_2}|z_2|^{2\beta_2-2}}{(1-|z_2|^{2\beta_2}e^{-\beta_2\phi_2})^2}e^{-\phi_2}|z_2|^2c_{1\bar 1}^2,\\ g_{1\bar 2} &= \tilde g_{1\bar 2} - 2\cdot\sum_{k=1}^2\frac{\beta_k e^{-\beta_k\phi_k}|z_k|^{2\beta_k}}{1-|z_k|^{2\beta_k}\cdot e^{-\beta_k\phi_k}}\theta_{1\bar 2}^k + 2\cdot\frac{\beta_1^2e^{-(\beta_1-1)\phi_1}|z_1|^{2\beta_1-2}}{(1-|z_1|^{2\beta_1}e^{-\beta_1\phi_1})^2}e^{-\phi_1}\bar z_1\bar b_2^1\\ &\quad + 2\cdot\frac{\beta_2^2e^{-(\beta_2-1)\phi_2}|z_2|^{2\beta_2-2}}{(1-|z_2|^{2\beta_2}e^{-\beta_2\phi_2})^2}e^{-\phi_2}z_2 b_1^2,\\ g_{i\bar j} &= \tilde g_{i\bar j} - 2\cdot\sum_{k=1}^2\frac{\beta_k e^{-\beta_k\phi_k}|z_k|^{2\beta_k}}{1-|z_k|^{2\beta_k}\cdot e^{-\beta_k\phi_k}}\theta_{i\bar j}^k\\ &\quad + 2\cdot\sum_{k=1}^2\frac{\beta_k^2e^{-(\beta_k-1)\phi_k}|z_k|^{2\beta_k-2}}{(1-|z_k|^{2\beta_k}e^{-\beta_k\phi_k})^2}e^{-\phi_k}|z_k|^2c_{i\bar j}^k, \quad i,j\geq 3. \end{aligned}$$

Note that $g_{2\bar 2}$ (respectively $g_{2\bar 1}$) can be treated similarly as $g_{1\bar 1}$ (respectively $g_{1\bar 2}$). The first observation is that the term $\beta_k/(1-|z_k|^{2\beta_k}\cdot e^{-\beta_k\phi_k})$ (and also its square) which appears in each $g_{i\bar j}$ will not blow up as $\beta_k\to 0$ for $k=1,2$. Indeed, the function $x\mapsto \beta/(1-x^\beta)$ is uniformly bounded for all small β under the assumption $x<e^{-1}$. Since we always assume $|s_k|_{h_k}^2 < e^{-1}$ for $k=1,2$, we only need to consider a point p that is near D (i.e., $|z_1|$ and $|z_2|$ are small) and show the holomorphic bisectional curvature at p is uniformly bounded in β. To achieve this, we consider a change of coordinate $\xi_k = z_k^{\beta_k}/\beta_k$ for $k=1,2$. Such ξ_k is multi-valued. Thus, we need to choose a single-valued branch of the Riemann surface associated to $z_k \mapsto z_k^{\beta_k}$. More specifically, whenever we work with ξ_k, denoting the polar coordinates of ξ_k by (r_k, θ_k), we always assume $\theta_k \in [0, 2\pi\beta_k)$. Under this assumption, ξ_k always take values in the space $\{(r,\theta) \in \mathbb{C} : \theta \in [0,\pi)\}$ for $k = 1,2$ for varying β_k as we have $\beta_k \in (0, 1/2]$. Moreover, when we consider the inverse map of the change of coordinates $z_k = (\beta_k \xi_k)^{1/\beta_k}$, we assume $\xi_k \in \{(r,\theta) \in \mathbb{C} : \theta \in [0, 2\pi\beta_k)\}$ and hence $z_k \in \{(r,\theta) \in \mathbb{C} : \theta \in [0, 2\pi)\}$ for varying β_k. In the new coordinates, we have

$$
\begin{aligned}
dz_1 \wedge d\bar{z}_1 &= |\beta_1 \xi_1|^{\frac{2}{\beta_1}-2} d\xi_1 \wedge d\bar{\xi}_1,\\
dz_1 \wedge d\bar{z}_2 &= \beta_1^{\frac{1}{\beta_1}-1} \beta_2^{\frac{1}{\beta_2}-1} \xi_1^{\frac{1}{\beta_1}-1} \bar{\xi}_2^{\frac{1}{\beta_2}-1} d\xi_1 \wedge d\bar{\xi}_2.
\end{aligned}
$$

From now on, we make the change of coordinates summarized as above:

$$
\begin{aligned}
\xi_k &= \frac{z_k^{\beta_k}}{\beta_k}, \quad k = 1,2\\
\xi_\ell &= z_\ell, \quad \ell = 3, \dots, n.
\end{aligned}
$$

We record the components of Ω_β, denoted by $\Omega_\beta = \sqrt{-1} h_{i\bar{j}} d\xi^i \wedge d\xi^j$, in the new coordinates:

$$
\begin{aligned}
h_{1\bar{1}} =& |\beta_1\xi_1|^{\frac{2}{\beta_1}-2} \tilde{g}_{1\bar{1}} - 2 \cdot \frac{\beta_1^{\frac{2}{\beta_1}+1} e^{-\beta_1\phi_1} |\xi_1|^{\frac{2}{\beta_1}}}{1 - e^{-\beta_1\phi_1}|\beta_1\xi_1|^2} \theta^1_{1\bar{1}} - 2 \cdot \frac{\beta_2^3 e^{-\beta_2\phi_2} |\xi_2|^2 |\beta_1\xi_1|^{\frac{2}{\beta_1}-2}}{1 - e^{-\beta_2\phi_2}|\beta_2\xi_2|^2} \theta^2_{1\bar{1}}\\
&+ 2 \cdot \frac{\beta_1^2 e^{-\beta_1\phi_1}}{(1 - e^{-\beta_1\phi_1}|\beta_1\xi_1|^2)^2}(1 + a_1) + 2 \cdot \frac{\beta_2^4 |\xi_2|^2 |\beta_1\xi_1|^{\frac{2}{\beta_1}-2}}{(1 - e^{-\beta_2\phi_2}|\beta_2\xi_2|^2)^2} c^2_{1\bar{1}},\\
h_{1\bar{2}} =& \left(\beta_1^{\frac{1}{\beta_1}-1} \beta_2^{\frac{1}{\beta_2}-1} \xi_1^{\frac{1}{\beta_1}-1} \bar{\xi}_2^{\frac{1}{\beta_2}-1}\right) \tilde{g}_{1\bar{2}} - 2 \cdot \frac{e^{-\beta_1\phi_1}\bar{\xi}_1}{1 - e^{-\beta_1\phi_1}|\beta_1\xi_1|^2} \left(\beta_1^{\frac{1}{\beta_1}+2} \beta_2^{\frac{1}{\beta_2}-1} \xi_1^{\frac{1}{\beta_1}} \bar{\xi}_2^{\frac{1}{\beta_2}-1}\right) \theta^1_{1\bar{2}}\\
&- 2 \cdot \frac{e^{-\beta_2\phi_2}\xi_2}{1 - e^{-\beta_2\phi_2}|\beta_2\xi_2|^2} \left(\beta_1^{\frac{1}{\beta_1}-1} \beta_2^{\frac{1}{\beta_2}+2} \xi_1^{\frac{1}{\beta_1}-1} \bar{\xi}_2^{\frac{1}{\beta_2}}\right) \theta^2_{1\bar{2}} + 2 \cdot \frac{\beta_1^3 e^{-\beta_1\phi_1} \bar{\xi}_1 \beta_2^{\frac{1}{\beta_2}-1} \bar{\xi}_2^{\frac{1}{\beta_2}-1}}{(1 - e^{-\beta_1\phi_1}|\beta_1\xi_1|^2)^2} \bar{b}^1_2\\
&+ 2 \cdot \frac{\beta_2^3 e^{-\beta_2\phi_2} \xi_2 \beta_1^{\frac{1}{\beta_1}-1} \xi_1^{\frac{1}{\beta_1}-1}}{(1 - e^{-\beta_2\phi_2}|\beta_2\xi_2|^2)^2} b^2_1,\\
h_{1\bar{j}} =& \beta_1^{\frac{1}{\beta_1}-1} \xi_1^{\frac{1}{\beta_1}-1} \tilde{g}_{1\bar{j}} - 2 \cdot \frac{\beta_1^{\frac{1}{\beta_1}+2} e^{-\beta_1\phi_1} \bar{\xi}_1 \xi_1^{\frac{1}{\beta_1}}}{1 - e^{-\beta_1\phi_1}|\beta_1\xi_1|^2} \theta^1_{1\bar{j}} - 2 \cdot \frac{\beta_1^{\frac{1}{\beta_1}-1} \beta_2^3 e^{-\beta_2\phi_2} |\xi_2|^2 \xi_1^{\frac{1}{\beta_1}-1}}{1 - e^{-\beta_2\phi_2}|\beta_2\xi_2|^2} \theta^2_{1\bar{j}}\\
&+ 2 \cdot \frac{\beta_1^3 e^{-\beta_1\phi_1} \bar{\xi}_1}{(1 - e^{-\beta_1\phi_1}|\beta_1\xi_1|^2)^2} \bar{b}^1_j + 2 \cdot \frac{\beta_2^4 \beta_1^{\frac{1}{\beta_1}-1} e^{-\beta_2\phi_2} |\xi_2|^2 \xi_1^{\frac{1}{\beta_1}-1}}{(1 - e^{-\beta_2\phi_2}|\beta_2\xi_2|^2)^2} c^2_{1\bar{j}}, \quad j = 3, \dots, n\\
h_{i\bar{j}} =& \tilde{g}_{i\bar{j}} - 2 \sum_{k=1}^2 \frac{\beta_k e^{-\beta_k\phi_k} |\beta_k\xi_k|^2}{1 - e^{-\beta_k\phi_k}|\beta_k\xi_k|^2} \theta^k_{i\bar{j}} + 2 \sum_{k=1}^2 \frac{\beta_k^2 e^{-\beta_k\phi_k} |\beta_k\xi_k|^2}{(1 - e^{-\beta_k\phi_k}|\beta_k\xi_k|^2)^2} c^k_{i\bar{j}}, \quad i,j = 3, \dots, n.
\end{aligned}
\tag{13}
$$

Note that $h_{2\bar{i}}$ can be treated similarly as $h_{1\bar{j}}$. A first observation is that since $\beta_k \in (0, 1/2]$, both $\beta_k^{\frac{2}{\beta_k}-2}$ and $|\xi_k|^{\frac{2}{\beta_k}-2}$ are uniformly bounded. Regarding the metric tensor of Ω_β, terms that involve $\tilde{g}_{i\bar{j}}$ or $\theta^k_{i\bar{j}}$ are also uniformly bounded. Moreover, after the change of coordinates, there does not exist singular term in each component $h_{i\bar{j}}$. In conclusion, $h_{i\bar{j}}$ is uniformly bounded for any i, j, i.e., by (13), in coordinates $(\xi_1, \xi_2, \xi_3 = z_3, \dots, \xi_n = z_n)$, there holds

$$h_{i\bar{j}} = O(1), \quad \text{for } i, j = 1, \dots, n.$$

In other words, we have shown that the metric tensor $h_{i\bar{j}}$ is bounded from above for all i, j. However, the metric tensor may degenerate as β_1 or β_2 tends to 0. According to (13), we have for rather small β_1 and β_2 the following asymptotic behaviors of each metric tensor:

$$\begin{aligned} &h_{1\bar{1}} \sim \beta_1^2, \quad h_{2\bar{2}} \sim \beta_2^2, \\ &h_{1\bar{2}}, h_{2\bar{1}} \sim \beta_1^3\beta_2^3|\xi_1||\xi_2|, \\ &h_{1\bar{j}} \sim \beta_1^3|\xi_1|, \quad j = 3, \dots, n \\ &h_{i\bar{j}} = O(1), \quad i, j = 3, \dots, n. \end{aligned} \tag{14}$$

Step 2: Estimate the inverse of the metric tensor

Recall the curvature tensor is given by

$$R_{i\bar{j}k\bar{\ell}} = -h_{i\bar{j},k\bar{\ell}} + h^{s\bar{t}}h_{i\bar{t},k}h_{s\bar{j},\bar{\ell}}. \tag{15}$$

To estimate the holomorphic bisectional curvature, we take two unit vectors (w.r.t. the metric $h_{i\bar{j}}$) $u^i\frac{\partial}{\partial\xi_i}$ and $v^j\frac{\partial}{\partial\xi_j}$. Then by (14) there holds

$$\begin{aligned} &u^1, v^1 = O(\beta_1^{-1}), \quad u^2, v^2 = O(\beta_2^{-1}), \\ &u^i, v^i = O(1), \quad i = 3, \dots, n. \end{aligned} \tag{16}$$

To finish the proof we need to bound $R_{i\bar{j}k\bar{\ell}}u^i\bar{u}^jv^k\bar{v}^\ell$. We first consider $R_{i\bar{j}k\bar{\ell}}$. We need to analyze $h^{i\bar{j}}$, $h_{i\bar{j},k}$ and $h_{i\bar{j},k\bar{\ell}}$.

We first treat $h^{i\bar{j}}$. By (13) and (14), one finds that $\det\{h_{i\bar{j}}\}$ is uniformly bounded and tends to 0 as β_1 or β_2 tends to 0. More precisely, by (14) there holds

$$\det\{h_{i\bar{j}}\} \sim \beta_1^2\beta_2^2. \tag{17}$$

Since $h^{i\bar{j}} = C_{j\bar{i}}/\det\{h_{i\bar{j}}\}$, where $C_{j\bar{i}}$ is the ji-th cofactor of the matrix $\{h_{i\bar{j}}\}$, we deduce from (14) and (17) that

$$\begin{aligned} h^{1\bar{j}}, h^{i\bar{1}} &\sim \beta_1^{-2}, \quad i, j \neq 2 \\ h^{1\bar{2}} &\sim \beta_1^{-2}\beta_2^{-2}, \\ h^{i\bar{j}} &= O(1), \quad i, j \neq 1, 2, \end{aligned} \tag{18}$$

while $h^{2\bar{j}}$ and $h^{j\bar{2}}$ can be treated in the same way. Roughly speaking, $h^{i\bar{j}}$ are bounded for fixed β near p but may tend to infinity with respect to β as described above. However, by (13) and (14), for any j, $h_{1\bar{j}}$ (respectively $h_{2\bar{j}}$) degenerate at the rate of at least β_1^2 (respectively β_2^2), when β_1 (respectively β_2) is small, and taking derivatives may only cause the terms $h_{i\bar{j},k}$ and $h_{i\bar{j},k\bar{\ell}}$ to converge to 0 at a faster speed in β. Thus, the singularity in $h^{i\bar{j}}$ does not cause a problem when we consider the curvature tensor (15), where the $h^{i\bar{j}}$ terms are multiplied by corresponding $h_{i\bar{t},k}$ or $h_{s\bar{j},\ell}$. We explain this in detail later in Step 4.

Step 3: Estimate derivatives of the metric tensor

Now we turn to show that $h_{i\bar{j},k}$ and $h_{i\bar{j},k\bar{\ell}}$ are uniformly bounded and find their dependence on β_1 and β_2. For $k, \ell \in \{3, \dots, n\}$, as taking derivative w.r.t. $\partial/\partial z_3 = \partial/\partial \xi_3, \dots, \partial/\partial z_n = \partial/\partial \xi_n$ will not contribute extra singular terms, the uniform boundedness of $h_{i\bar{j}}$ implies that this also holds for such $h_{i\bar{j},k}$ and $h_{i\bar{j},k\bar{\ell}}$. It remains to deal with $h_{i\bar{j},k}$ and $h_{i\bar{j},k\bar{\ell}}$ for $k, \ell \in \{1, 2\}$.

By the exact formula of each $h_{i\bar{j}}$ in (13), we find that the exponent of the terms $\xi_1, \bar{\xi}_1, \xi_2, \bar{\xi}_2$ is at least 1 when β_1 and β_2 are rather small, and indeed both the term $h_{i\bar{j}}$ and their first or second order derivatives are smooth in $\xi_1, \bar{\xi}_1, \xi_2, \bar{\xi}_2$. In summary, taking derivatives of the metric tensor does not cause singularities in $\xi_1, \bar{\xi}_1, \xi_2, \bar{\xi}_2$. We only need to derive the asymptotic behavior of the derivatives with respect to β. To achieve this, it is enough to deal with the following term T from (13). The reason is that any other terms in (13) have higher order dependence in β_1 and β_2, before and after taking the derivatives. So we consider taking derivatives of the following term

$$\frac{\beta_1^2 e^{-\beta_1\phi_1}}{\left(1-\beta_1^2|\xi_1|^2 e^{-\beta_1\phi_1}\right)^2} =: T \tag{19}$$

w.r.t. $\partial/\partial\xi_1$ and $\partial/\partial\xi_2$, since this is the most singular term appearing in the metric tensor. And by symmetry of indices we only need to consider $\partial/\partial\xi_1(T)$, $\partial^2/\partial\xi_1\partial\bar{\xi}_2(T)$ and $\partial^2/\partial\xi_1\partial\bar{\xi}_1(T)$. In the coordinates $(\xi_1, \xi_2, \xi_3 = z_3, \dots, \xi_n = z_n)$,

$$\begin{aligned}
\frac{\partial}{\partial\xi_1} = \frac{\partial z_1}{\partial\xi_1}\frac{\partial}{\partial z_1} &= \beta_1^{1/\beta_1-1}\xi_1^{1/\beta_1-1}\frac{\partial}{\partial z_1},\\
\frac{\partial^2}{\partial\xi_1\partial\bar{\xi}_2} &= \frac{\partial\bar{z}_2}{\partial\bar{\xi}_2}\frac{\partial z_1}{\partial\xi_1}\frac{\partial^2}{\partial z_1\partial\bar{z}_2},\\
\frac{\partial^2}{\partial\xi_1\partial\bar{\xi}_1} &= \frac{\partial\bar{z}_1}{\partial\bar{\xi}_1}\frac{\partial z_1}{\partial\xi_1}\frac{\partial^2}{\partial z_1\partial\bar{z}_1}.
\end{aligned}$$

Thus, we have

$$\frac{\partial}{\partial\xi_1}(T) = \frac{\partial}{\partial\xi_1}\frac{\beta_1^2 e^{-\beta_1\phi_1}}{\left(1-\beta_1^2|\xi_1|^2 e^{-\beta_1\phi_1}\right)^2}$$

$$
\begin{aligned}
&= \frac{\beta_1^{1/\beta_1+1}\xi_1^{1/\beta_1-1}\frac{\partial}{\partial z_1}(e^{-\beta_1\phi_1})}{\left(1-\beta_1^2|\xi_1|^2e^{-\beta_1\phi_1}\right)^2} + \frac{2\beta_1^{1/\beta_1+3}e^{-\beta_1\phi_1}\bar{\xi}_1\xi_1^{1/\beta_1}\frac{\partial}{\partial z_1}(e^{-\beta_1\phi_1})}{\left(1-\beta_1^2|\xi_1|^2e^{-\beta_1\phi_1}\right)^3}\\
&\quad + \frac{2\beta_1^4e^{-2\beta_1\phi_1}\bar{\xi}_1}{\left(1-\beta_1^2|\xi_1|^2e^{-\beta_1\phi_1}\right)^3}\\
&= O(\beta_1^4|\xi_1|) = O(1),
\end{aligned}
$$

since $\beta_1 \in (0, 1/2]$ and ϕ_1 is smooth w.r.t. $z_1, \dots, z_n$. Similarly, we get

$$
\frac{\partial^2}{\partial\xi_1\partial\bar{\xi}_2}(T) = O(\beta_1^4\beta_2^{1/\beta_2-1}|\xi_1||\xi_2|^{1/\beta_2-1}) = O(1),
$$

since we only need to address $\partial/\partial\xi_2(e^{-\beta_1\phi_1})$ when taking derivative of $\partial/\partial\xi_1(T)$ w.r.t. $\partial/\partial\bar{\xi}_2$. Finally, direct calculations yield

$$
\begin{aligned}
\frac{\partial^2}{\partial\xi_1\partial\bar{\xi}_1}T &= \frac{\beta_1^{2/\beta_1}|\xi_1|^{2/\beta_1-2}\frac{\partial^2}{\partial z_1\partial\bar{z}_1}e^{-\beta_1\phi_1}}{\left(1-\beta_1^2|\xi_1|^2e^{-\beta_1\phi_1}\right)^2}\\
&\quad + \frac{2\beta_1^{1/\beta_1+3}\xi_1^{1/\beta_1-1}\frac{\partial}{\partial z_1}e^{-\beta_1\phi_1}(\xi_1e^{-\beta_1\phi_1} + |\xi_1|^2\beta_1^{1/\beta_1-1}\bar{\xi}_1^{1/\beta_1-1}\frac{\partial}{\partial z_1}e^{-\beta_1\phi_1})}{\left(1-\beta_1^2|\xi_1|^2e^{-\beta_1\phi_1}\right)^3}\\
&\quad + \frac{2\beta_1^4\left(\frac{\partial}{\partial\bar{\xi}_1}e^{-\beta_1\phi_1}\frac{\partial}{\partial\xi_1}(|\xi_1|^2e^{-\beta_1\phi_1}) + e^{-\beta_1\phi_1}\frac{\partial^2}{\partial\xi_1\partial\bar{\xi}_1}|\xi_1|^2e^{-\beta_1\phi_1}\right)}{\left(1-\beta_1^2|\xi_1|^2e^{-\beta_1\phi_1}\right)^3}\\
&\quad + \frac{6\beta_1^6e^{-\beta_1\phi_1}\frac{\partial}{\partial\xi_1}(|\xi_1|^2e^{-\beta_1\phi_1})\frac{\partial}{\partial\bar{\xi}_1}(|\xi_1|^2e^{-\beta_1\phi_1})}{\left(1-\beta_1^2|\xi_1|^2e^{-\beta_1\phi_1}\right)^4}\\
&= O(\beta_1^4).
\end{aligned}
$$

In conclusion, we have shown

$$
\begin{aligned}
h_{i\bar{j},k} &= O(1),\\
h_{i\bar{j},k\bar{\ell}} &= O(1),
\end{aligned}
$$

for fixed β and any $i, j, k, \ell = 1, \dots, n$. In other words, the derivative of the metric tensor is bounded for fixed β. Moreover, by taking derivatives of (13), we find that the derivatives of the metric tensor degenerate (w.r.t. β_1 and β_2) at the rate shown below:

$$
\begin{aligned}
&h_{1\bar{1},1}, \quad h_{1\bar{1},1\bar{\ell}} = O(\beta_1^4), \quad \ell = 1, \dots, n \\
&h_{1\bar{1},2}, \quad h_{1\bar{1},2\bar{i}} = O(\beta_1^3\beta_2^3), \quad i = 1, \dots, n, \\
&h_{1\bar{1},k\bar{\ell}} = O(\beta_1^3), \quad k \neq 1, 2, \ell = 1, \dots, n \\
&h_{1\bar{2},1} = O(\beta_1^4\beta_2^4), \\
&h_{1\bar{2},k} = O(\beta_1^3\beta_2^3), \quad k \neq 1, \\
&h_{1\bar{2},1\bar{\ell}} = O(\beta_1^4\beta_2^4), \quad \ell = 1, \dots, n, \\
&h_{1\bar{2},k\bar{\ell}} = O(\beta_1^3\beta_2^3), \quad k \neq 1, \ell = 1, \dots, n \\
&h_{1\bar{j},k} = O(\beta_1^3), \quad j \neq 1, 2, k \neq 2, \\
&h_{1\bar{j},2} = O(\beta_1^3\beta_2^3), \quad j \neq 1, 2 \\
&h_{1\bar{j},k\bar{\ell}} = O(\beta_1^3), j \neq 1, 2, k \neq 2, \ell \neq 2, \\
&h_{1\bar{j},k\bar{\ell}} = O(\beta_1^3\beta_2^3), \quad j \neq 1, 2, k = 2 \text{ or } \ell = 2, \\
&h_{i\bar{j},1} = O(\beta_1^{1/\beta_1 - 1}), \quad i, j \neq 1, 2 \\
&h_{i\bar{j},2} = O(\beta_2^{1/\beta_2 - 1}), \quad i, j \neq 1, 2 \\
&h_{i\bar{j},1\bar{1}} = O(\beta_1^{2/\beta_1 - 2}), \quad i, j \neq 1, 2 \\
&h_{i\bar{j},2\bar{2}} = O(\beta_2^{2/\beta_2 - 2}), \quad i, j \neq 1, 2 \\
&h_{i\bar{j},1\bar{2}}, h_{i\bar{j},2\bar{1}} = O(\beta_1^{1/\beta_1 - 1}\beta_2^{1/\beta_2 - 1}), \quad i, j \neq 1, 2 \\
&h_{i\bar{j},k}, h_{i\bar{j},k\bar{\ell}} = O(1), \quad i, j, k, \ell \neq 1, 2.
\end{aligned}
\tag{20}
$$

Step 4: Estimate the sum $R_{i\bar{j}k\bar{\ell}}u^i\bar{u}^jv^k\bar{v}^\ell$.

We have shown the derivatives of $h_{i\bar{j}}$ are bounded. To show that $R_{i\bar{j}k\bar{\ell}}u^i\bar{u}^jv^k\bar{v}^\ell$ is bounded, we consider

$$
R_{i\bar{j}k\bar{\ell}}u^i\bar{u}^jv^k\bar{v}^\ell = (-h_{i\bar{j},k\bar{\ell}} + h^{s\bar{t}}h_{i\bar{t},k}h_{s\bar{j},\ell})u^i\bar{u}^jv^k\bar{v}^\ell.
$$

We consider three different cases.

When none of i, j, k, ℓ, s, t **is** 1 **or** 2**:** then the sum is uniformly bounded because none of the term blow up w.r.t. β_1 or β_2.

When $s, t \neq 1, 2$ **and** i, j, k, ℓ **may take values from** 1 **or** 2**:** then we found from (20) that the common factors of powers of β_1 or β_2 in $h_{i\bar{j},k}$ and $h_{i\bar{j},k\bar{\ell}}$ compensate for the degeneracy of u^1, u^2 and v^1, v^2.

When s **or** $t = 1, 2$**:** then in the worst case, where $s = 1, t = 2$, we find from (20) that all the derivatives that have 1 or 2 in the subscript have at least a degeneracy rate of β_1^3 or β_2^3. However, $h^{s\bar{t}}$ blows up at the rate of $\beta_1^{-2}\beta_2^{-2}$. Combining this fact with (16) we find that the common factors of β_1 and β_2 in the derivatives can still compensate for the degeneracy of $h^{s\bar{t}}$ and u, v.

Then we conclude that when β satisfies that $\beta_k \in (0, 1/2]$ for each k, the holomorphic bisectional curvature of Ω_β is uniformly bounded in β. □

Remark 2.5. As pointed out to the author by H. Guenancia, J. Sturm's trick (see [14, p. 62]) can also be applied to simplify the proof of the curvature bounds in Lemma 2.3. Our proof of Lemma 2.4 deals with the general case where $r > 1$.

2.2. A priori estimates. By Theorem 1.5, there is a unique Kähler–Einstein crossing edge metric on X with cone angle $2\pi\beta_i$ along each D_i, denoted by $\omega_{\phi_\beta} = \omega - \sum_{i=1}^r \beta_i\theta_i + \sqrt{-1}\partial\bar{\partial}\phi_\beta$, such that

$$\begin{cases} \omega_{\phi_\beta}^n = \dfrac{e^{f+\phi_\beta}\omega^n}{\Pi_{i=1}^r |s_i|_{h_i}^{2(1-\beta_i)}}, \\ \omega_{\phi_\beta} = \omega - \displaystyle\sum_{i=1}^r \beta_i\theta_i + \sqrt{-1}\partial\bar{\partial}\phi_\beta > 0, \end{cases} \tag{21}$$

where $f \in C^\infty(X)$. In this section, we establish a Laplacian estimate for ω_{ϕ_β} with respect to the reference metric Ω_β by proving the following result.

Theorem 2.6. *For $\beta = (\beta_1, \dots, \beta_r) \in (0, \frac{1}{2}]^r$, there exists a constant $C > 0$ (independent of $\beta_1, \dots, \beta_r$) such that*

$$C^{-1}\Omega_\beta \le \omega_{\phi_\beta} \le C\Omega_\beta, \tag{22}$$

on $X \setminus \operatorname{supp}(D)$.

Define

$$\psi_\beta := -\sum_{i=1}^r \log\left[\frac{1 - |s_i|_{h_i}^{2\beta_i}}{\beta_i}\right]^2,$$

then

$$\Omega_\beta = \omega + \sqrt{-1}\partial\bar{\partial}\psi_\beta.$$

Proof of Theorem 2.6. We divide the proof into two steps. First, we deduce the C^0-estimate for potential functions ϕ_β and ψ_β by using a modified maximum principle. Then we derive the Laplacian estimate by applying Chern–Lu's inequality to the identity map $(X, \omega_{\phi_\beta}) \to (X, \Omega_\beta)$.

Remark 2.7. When $r = 1$, the proof of (22) is already given in [8, Proposition 4.2]. The main difference for the case $r > 1$ is that ω_{ϕ_β} and Ω_β admit crossing edge singularities. However, thanks to Lemma 2.4, we are able to follow the arguments in [8] and treat all the components at once.

Step 1: C^0-estimate: Comparing ϕ_β with ψ_β.

We first compare the potential functions of ω_{ϕ_β} and Ω_β. Let $\tilde{\phi}_\beta := \phi_\beta - \psi_\beta$, then we get

$$\omega_{\phi_\beta}^n = \frac{e^{\phi_\beta+f}\omega^n}{\prod_{i=1}^r |s_i|_{h_i}^{2(1-\beta_i)}}, \tag{23}$$

$$\Rightarrow(\Omega_\beta - \sum_{i=1}^{r} \beta_i \theta_i + \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_\beta)^n = e^{\tilde{\phi}_\beta + F_\beta}\Omega_\beta^n, \tag{24}$$

where $F_\beta = \psi_\beta + f + \log\left(\frac{\omega^n}{\prod_{i=1}^r |s_i|_{h_i}^{2(1-\beta_i)} \cdot \Omega_\beta^n}\right)$. Then we claim that

Claim 2.8. For some uniform constant $C > 0$,

$$||F_\beta||_{C^0(X\setminus D)} \leq C. \tag{25}$$

Proof. First note that f is smooth on X by construction, hence it is bounded as X is compact. Therefore, it suffices to show

$$\begin{aligned} F_\beta - f &= \psi_\beta + \log\left(\frac{\omega^n}{\prod_{i=1}^r |s_i|_{h_i}^{2(1-\beta_i)} \cdot \Omega_\beta^n}\right) \\ &= \log\left(\frac{\prod_{i=1}^r \beta_i^2 \cdot \omega^n}{\prod_{i=1}^r |s_i|_{h_i}^{2(1-\beta_i)}(1-|s_i|_{h_i}^{2\beta_i})^2 \cdot \Omega_\beta^n}\right) \end{aligned}$$

is bounded. To prove the claim, it is equivalent to showing

$$\Omega_\beta^n = \frac{\prod_{i=1}^r \beta_i^2}{\prod_{i=1}^r |s_i|_{h_i}^{2(1-\beta_i)}(1-|s_i|_{h_i}^{2\beta_i})^2} e^{O(1)}\omega^n$$

near the divisor. This amounts to saying that Ω_β^n has a pole of order $\prod_{i=1}^r |s_i|_{h_i}^{2(1-\beta_i)}$. Without loss of generality, we can assume $r = 1$ and thus below we drop the i in the subscript for simplicity. Let $p \in M \setminus D$ near D. Let e be a local holomorphic frame for L_D, and $(z_1, \dots, z_n)$ be a local holomorphic coordinate chart such that $s = z_1 e$. Let $h = e^{-\phi}$ be a smooth hermitian metric on $\mathcal{O}_X(D)$ and θ the curvature form of (L_D, h). Denote

$$\begin{aligned} \omega &= \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j, \\ \theta &= \sqrt{-1}\phi_{i\bar{j}} dz_i \wedge d\bar{z}_j. \end{aligned}$$

Recall the expression (7) of Ω_β. We calculate

$$\begin{aligned} \langle D^{1,0}s, D^{1,0}s\rangle &= e^{-\phi}(dz_1 + z_1\frac{\partial\phi}{\partial z_k}dz_k) \wedge (d\bar{z}_1 + \bar{z}_1\frac{\partial\phi}{\partial\bar{z}_k}d\bar{z}_k) \\ &= e^{-\phi}[(1 + \bar{z}_1\frac{\partial\phi}{\partial\bar{z}_k} + z_1\frac{\partial\phi}{\partial z_1} + |z_1|^2\frac{\partial\phi}{\partial z_1}\frac{\partial\phi}{\partial\bar{z}_1})dz_1 \wedge d\bar{z}_1 \\ &+ \sum_{k=2}^{n}(\bar{z}_1\frac{\partial\phi}{\partial\bar{z}_k} + |z_1|^2\frac{\partial\phi}{\partial z_1}\frac{\partial\phi}{\partial\bar{z}_k})dz_1 \wedge d\bar{z}_k \\ &+ \sum_{k=2}^{n}(z_1\frac{\partial\phi}{\partial z_k} + |z_1|^2\frac{\partial\phi}{\partial z_k}\frac{\partial\phi}{\partial\bar{z}_1})dz_k \wedge d\bar{z}_1 \end{aligned}$$

$$+\sum_{k,l=2}^{n}|z_1|^2\frac{\partial\phi}{\partial z_k}\frac{\partial\phi}{\partial\bar z_l}dz_k\wedge d\bar z_l].$$

Hence,

$$\begin{aligned}
&\frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2}\langle D^{1,0}s,D^{1,0}s\rangle\\
&=(\frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2}+O(1))dz_1\wedge d\bar z_1+\sum_{k=2}^{n}(\frac{\beta^2|s|_h^{2\beta}}{z_1(1-|s|_h^{2\beta})^2}+O(1))dz_1\wedge d\bar z_k\\
&+\sum_{k=2}^{n}(\frac{\beta^2|s|_h^{2\beta}}{\bar z_1(1-|s|_h^{2\beta})^2}+O(1))dz_k\wedge d\bar z_1+\sum_{k,l=2}^{n}\frac{\beta^2|s|_h^{2\beta}}{(1-|s|_h^{2\beta})^2}dz_k\wedge d\bar z_l\\
&=(\frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2}+O(1))dz_1\wedge d\bar z_1+\sum_{k=2}^{n}(\frac{\beta^2|s|_h^{2\beta}}{z_1(1-|s|_h^{2\beta})^2}+O(1))dz_1\wedge d\bar z_k\\
&+\sum_{k=2}^{n}(\frac{\beta^2|s|_h^{2\beta}}{\bar z_1(1-|s|_h^{2\beta})^2}+O(1))dz_k\wedge d\bar z_1+\sum_{k,l=2}^{n}O(1)dz_k\wedge d\bar z_l.
\end{aligned}$$

Let

$$(A_{ij})_{i,j=1}^n=(g_{i\bar j})_{i,j=1}^n-\frac{\beta|s|_h^{2\beta}}{1-|s|_h^{2\beta}}(\phi_{i\bar j})_{i,j=1}^n+\frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2}\langle D^{1,0}s,D^{1,0}s\rangle.$$

Write $(A_{ij})_{i,j=1}^n$ as a block matrix

$$(A_{ij})_{i,j=1}^n=\begin{bmatrix}A_{11} & \vec A_r\\ \vec A_c & (A_{ij})_{i,j=2}^n\end{bmatrix}, \tag{26}$$

then

$$\begin{aligned}
A_{11}&=g_{1\bar 1}-\frac{\beta|s|_h^{2\beta}}{1-|s|_h^{2\beta}}\phi_{1\bar 1}+\frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2}+O(|s|_h^{2\beta-1})+O(|s|_h^{2\beta})O(1),\\
A_{1j}&=g_{1\bar j}-\frac{\beta|s|_h^{2\beta}}{1-|s|_h^{2\beta}}\phi_{1\bar j}+O(|s|_h^{2\beta-1})+O(|s|_h^{2\beta})+O(1),\quad j=2,\dots,n,\\
A_{i1}&=g_{i\bar 1}-\frac{\beta|s|_h^{2\beta}}{1-|s|_h^{2\beta}}\phi_{i\bar 1}+O(|s|_h^{2\beta-1})+O(|s|_h^{2\beta})+O(1),\quad i=2,\dots,n,\\
A_{kl}&=O(|s|_h^{2\beta})+O(1),\quad k,l=2,\dots,n.
\end{aligned}$$

Recall the formula for determinant of block matrices as in (26),

$$\det(A_{ij})_{i,j=1}^n=\det(A_{ij})_{i,j=2}^n\cdot(A_{11}-\vec A_r((A_{ij})_{i,j=2}^n)^{-1}\vec A_c). \tag{27}$$

Using (27),

$$
\begin{aligned}
\frac{\Omega_\beta^n}{\omega^n} &= \frac{\det(g_{i\bar{j}} - \frac{\beta|s|_h^{2\beta}}{1-|s|_h^{2\beta}}\phi_{i\bar{j}} + \frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2}\langle D^{1,0}s, D^{1,0}s\rangle)}{\det(g_{i\bar{j}})} \\
&= \frac{\det(A_{ij})_{i,j=1}^n}{\det(g_{i\bar{j}})} \\
&= e^{O(1)} \cdot \det(A_{ij})_{i,j=2}^n \cdot (A_{11} - \vec{A}_r((A_{ij})_{i,j=2}^n)^{-1}\vec{A}_c) \\
&= e^{O(1)} \cdot \Bigg(\frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2} + O(|s|_h^{4\beta-2}) + O(|s|_h^{2\beta-1}) + O(|s|_h^{4\beta-1}) \\
&\quad + O(|s|_h^{2\beta}) + O(|s|_h^{4\beta}) + O(1) \Bigg).
\end{aligned}
$$

Thus, one finds that the dominant term is $\frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2}$. In another word, we have shown $\Omega_\beta^n = e^{O(1)}\frac{\beta^2}{|s|_h^{2(1-\beta)}(1-|s|_h^{2\beta})^2}\omega^n$, which is exactly what we need. □

Lemma 2.9. *There exists a uniform constant $C > 0$ in β such that,*

$$\sup_{X\setminus D} |\tilde{\phi}_\beta| \le C,$$

when β_i are small enough for every i.

Proof. First note that for a fixed β, $\tilde{\phi}_\beta$ is bounded according to [10, 9]. We aim to derive a uniform bound for $\tilde{\phi}_\beta$ in β. Let $\chi_{\beta,\epsilon} = \tilde{\phi}_\beta + \epsilon\sum_{i=1}^r \log|s_i|_{h_i}^2$ for small $\epsilon > 0$. Since $\chi_{\beta,\epsilon}(p)$ approaches $-\infty$ when $p \to D$, $\chi_{\beta,\epsilon}$ obtains its maximum on $X \setminus D$, at say $p_{\max}$. Then

$$0 \ge \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_\beta(p_{\max}) - \epsilon\sum_{i=1}^r \theta_i(p_{\max}),$$

where θ_i is the curvature of the Chern connection on (L_{D_i}, h_i). Then at $p_{\max}$,

$$
\begin{aligned}
(\Omega_\beta - \sum_{i=1}^r \beta_i\theta_i + \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_\beta)^n &\le (\Omega_\beta + \sum_{i=1}^r(\epsilon - \beta_i)\theta_i)^n && (28)\\
&\le 2^n\Omega_{\beta_i}^n, && (29)
\end{aligned}
$$

by the fact that $\Omega_\beta \ge r(\epsilon - \beta_i)\theta_i$ for small enough ϵ and β_i, as shown in Lemma 2.2. Combining (24) and (29), at $p_{\max}$,

$$e^{\tilde{\phi}_\beta + F_\beta}(p_{\max}) \le 2^n$$

$$\begin{aligned}\Rightarrow \tilde{\phi}_\beta(p_{\max}) &\le n\log 2 - F_\beta(p_{\max})\\ &\le n\log 2 - \inf_{X\setminus D} F_\beta.\end{aligned}$$

For any $p \in X \setminus D$,

$$\begin{aligned}\tilde{\phi}_\beta(p) &= \chi_{\beta,\epsilon}(p) - \epsilon\sum_{i=1}^r \log|s_i|^2_{h_i}(p)\\ &\le \chi_{\beta,\epsilon}(p_{\max}) - \epsilon\sum_{i=1}^r \log|s_i|^2_{h_i}(p)\\ &\le n\log 2 - \inf_{X\setminus D} F_\beta + \epsilon\sum_{i=1}^r \log|s_i|^2_{h_i}(p_{\max}) - \epsilon\sum_{i=1}^r \log|s_i|^2_{h_i}(p)\\ &\le C\end{aligned}$$

for some constant $C > 0$, when letting $\epsilon \to 0$ and using (25). Similarly by considering $\tilde{\chi}_{\beta,\epsilon} := \tilde{\phi}_\beta - \epsilon\sum_{i=1}^r \log|s_i|^2_{h_i}$ achieving its minimum on $X \setminus D$, we can show a lower bound for $\tilde{\phi}_\beta$ on $X \setminus D$.

□

Step 2: The Laplacian estimates for ω_{ϕ_β} and Ω_β.

In this section, we use Chern–Lu's inequality to deduce the Laplacian estimate of ω_{ϕ_β} with respect to Ω_β.

Consider the identity map

$$\mathrm{id} : (X \setminus D, \omega_{\phi_\beta}) \to (X \setminus D, \Omega_\beta).$$

By definitions, $\mathrm{Ric}\,\omega_{\phi_\beta} = -\omega_{\phi_\beta}$. From Lemma 2.4, $|\,\mathrm{Bisec}_{\Omega_\beta}| \le C_3$ for some constant $C_3 > 0$ when $\beta_i \in (0, \frac{1}{2}]$ for every i. Then by Chern–Lu's inequality [10, Proposition 7.1] (see also [14, Proposition 7.2]),

$$\Delta_{\omega_{\phi_\beta}}(\log \mathrm{tr}_{\omega_{\phi_\beta}} \Omega_\beta) \ge -1 - 2C_3\, \mathrm{tr}_{\omega_{\phi_\beta}} \Omega_\beta. \tag{30}$$

Set for $0 < \epsilon \ll 1$,

$$H_{\beta,\epsilon} = \log \mathrm{tr}_{\omega_{\phi_\beta}} \Omega_\beta - 4(C_3+1)\tilde{\phi}_\beta + \epsilon\sum_{i=1}^r \log|s_i|^2_{h_i},$$

then

$$\begin{aligned}\Delta_{\omega_{\phi_\beta}} H_{\beta,\epsilon} &= \Delta_{\omega_{\phi_\beta}}(\log \mathrm{tr}_{\omega_{\phi_\beta}} \Omega_\beta - 4(C_3+1)\tilde{\phi}_\beta) - \epsilon\sum_{i=1}^r \mathrm{tr}_{\omega_{\phi_\beta}} \theta_i & (31)\\ &\ge \Delta_{\omega_{\phi_\beta}}(\log \mathrm{tr}_{\omega_{\phi_\beta}} \Omega_\beta) + 4(C_3+1)\left(\frac{1}{2}\mathrm{tr}_{\omega_{\phi_\beta}} \Omega_\beta - n\right) - \mathrm{tr}_{\omega_{\phi_\beta}\Omega_\beta}, & (32)\end{aligned}$$

where the last inequality is true by noting that $\theta_i \le M\Omega_\beta$ for some uniform constant $M > 0$ and assuming $\epsilon < \frac{1}{2rM}$ and $\sum_{i=1}^r \beta_i < \frac{1}{2M}$.

Combine (30) and (32),

$$\Delta_{\omega_{\phi_\beta}} H_{\beta,\epsilon} \geq \mathrm{tr}_{\omega_{i,\ell}} \Omega_{\beta_i} - C \tag{33}$$

for some uniform constant $C > 0$. $H_{\beta,\epsilon}$ achieves its maximum on $X \setminus D$, at, say $q_{\max}$. Then by (33),

$$\mathrm{tr}_{\omega_{\phi_\beta}} \Omega_\beta(q_{\max}) \leq C.$$

For any $q \in X \setminus D$,

$$\begin{aligned}
\log \mathrm{tr}_{\omega_{\phi_\beta}} \Omega_\beta(q) &= H_{\beta,\epsilon}(q) + 4(C_3+1)\tilde{\phi}_\beta(q) - \epsilon \sum_{i=1}^r \log |s_i|^2_{h_i}(q) \\
&\leq H_{\beta,\epsilon}(q_{\max}) + 4(C_3+1)\tilde{\phi}_\beta(q) - \epsilon \sum_{i=1}^r \log |s_i|^2_{h_i}(q) \\
&\leq C - 4(C_3+1)\tilde{\phi}_\beta(q_{\max}) + \epsilon \sum_{i=1}^r \log |s_i|^2_{h_i}(q_{\max}) \\
&+ 4(C_3+1)\tilde{\phi}_{i,\ell}(q) - \epsilon \sum_{i=1}^r \log |s_i|^2_{h_i}(q) \\
&\leq \text{ some uniform constant } C,
\end{aligned}$$

where the last inequality is true by Lemma 2.9 and letting $\epsilon \to 0$ for fixed q. Hence we have shown

$$\omega_{\phi_\beta} \geq C \cdot \Omega_\beta, \tag{34}$$

on $X \setminus \mathrm{supp}(D)$ as desired. Since ω_{ϕ_β} and Ω_β are equivalent on X, we obtain the estimate (22) on $X \setminus D$.

□

2.3. Global convergence of ω_{ϕ_β}. A smooth Kähler metric Ω_{PC} on $X \setminus D$ is said to have mixed cusp and edge singularities along a divisor D if whenever D is locally given by $D = \sum_{i=1}^t \{z_i = 0\} + \sum_{j=t+1}^m (1-\beta_j)\{z_j = 0\}$ with $t < m \leq n$, Ω_{PC} is quasi-isometric to the following metric:

$$\omega_{PC} := \sum_{i=1}^t \frac{\sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z_i|^2} + \sum_{j=t+1}^m \frac{\beta_j^2 \sqrt{-1} dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-\beta_j)}} + \sum_{\ell=m+1}^n \sqrt{-1} dz_\ell \wedge d\bar{z}_\ell.$$

In particular, when $t = m$, ω_{PC} has merely cusp singularities along D.
In the case $t = m$, it is well known [11, 18] that if $K_X + D$ is ample, there exists a unique Kähler–Einstein metric on $X \setminus D$ with cusp singularities along D. In general, it is shown that if $K_X + D$ is ample, there exists a unique Kähler–Einstein metric on $X \setminus D$ with mixed cusp and cone singularities along D [7,

Theorem A]. As a corollary of Theorem 2.6, we study the global weak convergence and local smooth convergence of the Kähler–Einstein crossing edge metrics ω_{ϕ_β} to a Kähler–Einstein mixed cusp and edge metric on (X, D_β) as some of the cone angles tend to 0. The first observation is the following lemma.

Lemma 2.10. *Assume $\beta_i \to 0$, for $i = 1, \dots, t < r$, and $\beta_j \to d_j \in (0,1)$ for $j = t+1, \dots, r$, then Ω_β weakly converges to some Kähler mixed cusp and edge metric Ω_{PC}. Moreover, Ω_β converges to Ω_{PC} in $C^\infty_{\mathrm{loc}}(X \setminus \mathrm{supp}(D_\beta))$.*

Proof. Recall the definition of Ω_β. Note that $\log\left[(1-|s_i|^{2\beta_i}_{h_i})/\beta_i\right]^2$ converges to $\log\log^2 |s_i|^2_{h_i}$ in $L^1(X,\omega)$ and $C^\infty_{\mathrm{loc}}(X \setminus \mathrm{supp}(D_\beta))$ as $\beta_i \to 0$ for each $i = 1, \dots, t$. Thus Ω_β converges to

$$\Omega_{PC} := \omega - \sum_{i=1}^{t} \sqrt{-1}\partial\bar\partial \log\log^2 |s_i|^2_{h_i} - \sum_{j=t+1}^{r} \sqrt{-1}\partial\bar\partial \log\left[\frac{1-|s_j|^{2d_j}_{h_j}}{d_j}\right]^2$$

in $C^\infty_{\mathrm{loc}}(X \setminus \mathrm{supp}(D_\beta))$ sense and weakly in the sense of currents. It remains to show that Ω_{PC} has mixed cusp and edge singularities along D_β. To see this, recall we denote by θ_i the Chern curvature form of (L_{D_i}, h_i) for each i, then by (8) and (10), we calculate that

$$\begin{aligned}
&\sum_{i=1}^{t} \sqrt{-1}\partial\bar\partial \log\log^2 |s_i|^2_{h_i} \\
&= \sum_{i=1}^{t} 2\sqrt{-1}\cdot \frac{(\partial\bar\partial |s_i|^2_{h_i})(\log|s_i|^2_{h_i})(|s_i|^2_{h_i}) - \partial(\log|s_i|^2_{h_i}|s_i|^2_{h_i})\bar\partial |s_i|^2_{h_i}}{(\log|s_i|^2_{h_i})^2(|s_i|^2_{h_i})^2} \\
&= \sum_{i=1}^{r} \frac{2\sqrt{-1}\langle D^{1,0}s_i, D^{1,0}s_i\rangle}{\log^2|s_i|^2_{h_i}|s_i|^2_{h_i}} + \frac{2}{\log|s_i|^2_{h_i}}\theta_i.
\end{aligned}$$

Thus, Ω_{PC} has cusp singularities along D_i for $i = 1, \dots, t$. The result follows. □

Theorem 2.11. *The Kähler–Einstein crossing edge metric ω_{ϕ_β} converges to the Kähler–Einstein mixed cusp and edge metric on (X, D_β) globally in a weak sense and locally in a strong sense when $\beta_i \to 0$ for $i = 1, \dots, t < r$ and $\beta_j \to d_j \in (0,1)$ for $j = t+1, \dots, r$.*

Proof. By Theorem 2.6, the family of ω_{ϕ_β} has uniformly bounded mass. Thus, the family of ω_{ϕ_β} is relatively compact in the weak topology. The same arguments in the proof of Lemma 2.9 and elliptic estimates give respectively the C^0-estimate and all higher-order estimates for the family of ω_{ϕ_β}. Therefore, any weak limit ω_0 is smooth on $X \setminus \mathrm{supp}(D_\beta)$ and this C^∞_{loc}-convergence indicates that such ω_0 is Kähler–Einstein outside D_β. Lemma 2.10 shows any such ω_0 also admits mixed cusp and edge singularities along D_β. Thus, by the uniqueness argument in this setting [7, Proposition 2.5], all such ω_0 coincides with the unique Kähler–Einstein metric on $X \setminus \operatorname{supp}(D_\beta)$ with mixed cusp and cone singularities along D_β. Hence we have shown the locally strong and globally weak convergence of ω_{ϕ_β} to a Kähler–Einstein mixed cusp and edge metric as $\beta_i \to 0$ for $i = 1, \dots, t$ and $\beta_j \to d_j$ for $j = t+1, \dots, r$. $\square$

3. Asymptotic behavior near the divisors in the small angle limit

Theorem 2.11 only gives us the smooth convergence of ω_{ϕ_β} to a Kähler–Einstein mixed cusp and edge metric away from the divisor when cone angles approach 0. In this section, we study the asymptotic behavior of ω_{ϕ_β} near D when some of the cone angles tend to 0. More precisely, consider a fixed point $p \in D_\beta$ with a holomorphic coordinate chart $(U, \{z_i\}_{i=1}^n)$ centered at p such that $D_\beta \cap U = \{z_1 \cdots z_m = 0\}$, for $m \le n$ and $D_j \cap U = \{z_j = 0\}$ for $j = 1, \dots, m$. Let β_i denote the cone angle along D_i for each i. From now on, assume $\beta_1 \le \beta_2 \le \cdots \le \beta_m$. We allow other cone angles to tend to 0, but we always assume that β_1 goes to 0 in the fastest speed, i.e., $\beta_1/\beta_i \nrightarrow +\infty$, for $i = 2, \dots, m$.

3.1. A small neighborhood of D_β. By choosing an appropriate coordinate system [2, Lemma 4.1], whenever D_β is locally given by $\{z_1 \cdots z_m = 0\}$, the reference metric Ω_β is equivalent to the following metric:

$$\omega_{\beta,\mathrm{mod}} := \sum_{i=1}^{m} \frac{\beta_i^2 \sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}(1-|z_i|^{2\beta_i})^2} + \sum_{j=m+1}^{n} \sqrt{-1} dz_j \wedge d\bar{z}_j. \tag{35}$$

Thus, Theorem 2.6 tells us on $X \setminus \operatorname{supp}(D_\beta)$, there exists a uniform constant $C > 0$ such that

$$C^{-1}\omega_{\beta,\mathrm{mod}} \le \omega_{\phi_\beta} \le C\omega_{\beta,\mathrm{mod}}. \tag{36}$$

Thanks to (36), it is enough to consider $((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \omega_{\beta,\mathrm{mod}})$ when dealing with a small neighborhood of D_β under the metric ω_{ϕ_β}. Let us fix a point $p \in D_\beta$. Let $(U, z_1, \dots, z_n)$ be a holomorphic coordinate chart centered at p, such that $U \cap D_\beta = \{z_1 \cdots z_m = 0\}$ and $U \cap D_i = \{z_i = 0\}$ for $i = 1, \dots, m$. Let $\mathbb{D} := \{|z_i| \le 1, i = 1, \dots, n\}$ be the unit polydisk. Then we claim that the distance function d_β induced by the completion of $\omega_{\beta,\mathrm{mod}}$ on $\mathbb{D}$ satisfies

$$d_\beta(0,z) \simeq \sum_{i=1}^{m} \frac{1}{2}\log\left(\frac{1+|z_i|^{\beta_i}}{1-|z_i|^{\beta_i}}\right) + \sum_{j=m+1}^{n} |z_j|, \quad z \in \mathbb{D}, \tag{37}$$

where "$\simeq$" means "is equivalent up to a constant independent of z to". Indeed, $\dfrac{1}{2}\log\left(\dfrac{1+x^{\beta_i}}{1-x^{\beta_i}}\right)$ is the primitive of $\dfrac{\beta_i}{x^{1-\beta_i}(1-x^{2\beta_i})}$, and (37) follows from this fact and (35). Summarizing the discussions above, it is enough to study the polydisk in $\mathbb{C}^n$

$$\left\{|z_i|^{\beta_i} < \frac{1-e^{-2a}}{1+e^{-2a}}, i = 1, \dots, m, z_j < a, j = m+1, \dots, n\right\}, \quad a > 0,$$

when we study a neighborhood of D_β given by the geodesic ball $B_{\omega_{\phi_\beta}}(p, a)$ centered at p of radius a with respect to the metric ω_{ϕ_β}.

3.2. The mixed cylinder and edge metric. In this section, we focus on a small neighborhood of D_β and show that in a neighborhood of D_β, a renormalization of $\omega_{\beta,\mathrm{mod}}$ locally converges to a mixed cylinder and edge metric (see Definition 3.1 below) in the C^∞-sense.

Definition 3.1. A Kähler metric $\tilde{\omega}$ on $(\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$ is called a mixed cylinder and edge metric if $\tilde{\omega}$ is quasi-isometric to the following metric:

$$\omega_{\mathrm{mix}} := \sum_{i=1}^{t} \frac{\sqrt{-1}dz_i \wedge d\bar{z}_i}{|z_i|^2} + \sum_{j=t+1}^{m} \frac{\beta_j^2 \sqrt{-1}dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-\beta_j)}} + \sum_{\ell=m+1}^{n} \sqrt{-1}dz_\ell \wedge d\bar{z}_\ell,$$

where $\beta_j \in (0,1)$ for $j = t+1, \dots, m$.

Denote by $\mathbb{D}(a_1, \dots, a_m, b)$ the set

$$\{z \in (\mathbb{C}^*)^m \times \mathbb{C}^{n-m} : |z_i| < a_i, i = 1, \dots, m, |z_j| < b, j = m+1, \dots, n\}.$$

Let

$$U_\beta := \mathbb{D}\left(e^{-\frac{1}{2\beta_1}}, \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2\beta_2}}, \dots, \left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{2\beta_m}}, 1\right).$$

From section 3.1, one realizes U_β as a neighborhood of D_β. We endow U_β with $\frac{1}{\beta_1^2}\omega_{\beta,\mathrm{mod}}$. Define a map

$$\Psi_\beta : \mathbb{D}\left(e^{\frac{1}{2\beta_1}}, \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2\beta_2}}, \dots, \left(\frac{\beta_m}{\beta_1}\right)^{\frac{1}{2\beta_m}}, \frac{1}{\beta_1}\right) \to U_\beta = \mathbb{D}\left(e^{-\frac{1}{2\beta_1}}, \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2\beta_2}}, \dots, \left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{2\beta_m}}, 1\right)$$

$$(w_1, \dots, w_m, w_{m+1}, \dots, w_n) \mapsto \left(e^{-\frac{1}{\beta_1}} w_1, \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{\beta_2}} w_2, \dots, \left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{\beta_m}} w_m, \beta_1 w_{m+1}, \dots, \beta_1 w_n\right).$$

On $\Psi_\beta^{-1}(U_\beta)$, the pull-back metric reads

$$\Psi_\beta^*(\frac{1}{\beta_1^2}\omega_{\beta,\mathrm{mod}}) = \frac{e^{-2}|w_1|^{2\beta_1}}{(1-e^{-2}|w_1|^{2\beta_1})^2} \cdot \frac{\sqrt{-1}dw_1 \wedge d\bar{w}_1}{|w_1|^2} + \sum_{i=2}^{m} \frac{\sqrt{-1}dw_i \wedge d\bar{w}_i}{|w_i|^{2(1-\beta_i)}(1 - \frac{\beta_1^2}{\beta_i^2}|w_i|^{2\beta_i})^2} \tag{38}$$

$$+ \sum_{j=m+1}^{n} \sqrt{-1}dw_j \wedge d\bar{w}_j. \tag{39}$$

Note that for $(w_1, \dots, w_m, w_{m+1}, \dots, w_n) \in (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$, $|w_1|^{2\beta_1} \to 1$ as $\beta_1 \to 0$ and $\frac{\beta_1^2}{\beta_i^2}|w_i|^{2\beta_i} \to 0$ as $\frac{\beta_1}{\beta_i} \to 0$. Moreover, for any compact subset $K \subset (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$, when β_1 is small enough, $K \subset \Psi_\beta^{-1}(U_\beta)$. Hence we have indeed shown the following result.

Lemma 3.2. *The pull-back of $\frac{1}{\beta_1^2}\omega_{\beta,\mathrm{mod}}$ by Ψ_β on any compact subset $K \subset (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$ converges to a mixed cylindrical and conical metric in $C^\infty(K)$ when $\beta_1 \to 0$ and β_i does not converge to 0 for each $i = 2, \dots, m$.*

Proof. Summarizing the discussions above, $\Psi_\beta^*(\frac{1}{\beta_1^2}\omega_{\beta,\mathrm{mod}})$ converges to

$$\frac{e^{-2}}{(1-e^{-2})^2} \cdot \frac{\sqrt{-1}dw_1 \wedge d\bar{w}_1}{|w_1|^2} + \sum_{i=2}^{m} \frac{\sqrt{-1}dw_i \wedge d\bar{w}_i}{|w_i|^{2(1-\beta_i)}} + \sum_{j=m+1}^{n} \sqrt{-1}dw_j \wedge d\bar{w}_j =: \hat{\omega},$$

which is a mixed cylinder and edge metric by Definition 3.1, in $C^\infty(K)$ as $\beta_1 \to 0$ and $\frac{\beta_1}{\beta_i} \to 0, \forall i = 2, \dots, m$. $\square$

3.3. The convergence of renormalized ω_{ϕ_β} near D_β. For a Kähler metric ξ on $\mathbb{C}^n$, let us denote $\bar{\xi} := \Psi_\beta^*\left(\frac{1}{\beta_1^2}\xi\right)$.

Theorem 3.3. *Let $\{\beta_{1,k}\}_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. Assume further that $\lim_{k\to\infty}\beta_{i,k} > 0$ for each $i = 2, \dots, r$. Assume all $\beta_{i,k} \in (0, \frac{1}{2}]$. Let $\omega_{\phi_{\beta_k}}$ be the (negatively curved) Kähler–Einstein crossing edge metric on $(X, D_k = \sum_{i=1}^r (1-\beta_{i,k})D_i)$. Then there exists a subsequence of the metric spaces $\left(U_{\beta_k}, \frac{1}{\beta_{1,k}^2}\omega_{\phi_{\beta_k}}\right)$ which converges in pointed Gromov-Hausdorff topology to $((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \bar{\omega}_\infty)$, where $\bar{\omega}_\infty$ is a mixed cylindrical and conical metric. Indeed, a subsequence of $\bar{\omega}_{\phi_{\beta_k}}$ converges in $C^\infty_{\mathrm{loc}}((\mathbb{C}^*)^m \times \mathbb{C}^{n-m})$-topology to $\bar{\omega}_\infty$.*

Proof. First note that ω_{ϕ_β} admits a potential function on U_β since $\omega_{\beta,\mathrm{mod}}$ admits one. Thus, $\bar{\omega}_{\phi_\beta}$ admits a potential function on $\Psi_\beta^{-1}(U_\beta)$, denoted by $\bar{\phi}_\beta$. The proof consists of three steps. We first deduce the C^0-estimate of $\bar{\phi}_\beta$ using Theorem 2.6. Then we derive the $C^{2,\alpha}$-estimates for $\bar{\phi}_\beta$ by the standard regularization arguments for Monge–Ampère equations. This combining with Arzelà–Ascoli Theorem gives us a cluster value of the metrics. Finally, we use that smooth convergence to conclude the pointed Gromov–Hausdorff convergence as wanted.

Step 1: C^0-estimates.

Discussions in section 3.1 indicate that there exists a uniform constant $C > 0$ (independent of $\beta \in (0, \frac{1}{2}]^r$) such that

$$C^{-1}\omega_{\beta,\mathrm{mod}} \leq \omega_{\phi_\beta} \leq C\omega_{\beta,\mathrm{mod}}$$

on U_β. Thus

$$C^{-1}\Psi_\beta^*\left(\frac{1}{\beta_1^2}\omega_{\beta,\mathrm{mod}}\right) \le \bar{\omega}_{\phi_\beta} \le C\Psi_\beta^*\left(\frac{1}{\beta_1^2}\omega_{\beta,\mathrm{mod}}\right).$$

By Lemma 3.2, $\Psi_\beta^*(\frac{1}{\beta_1^2}\omega_{\beta,\mathrm{mod}})$ converges in $C^\infty(K)$ to $\hat{\omega}$ for any compact $K \subset (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$. Hence, there exists a constant $C_K > 0$ (independent of β) such that

$$C_K^{-1}\hat{\omega} \le \bar{\omega}_{\phi_\beta} \le C_K\hat{\omega}. \tag{40}$$

By (40), $\bar{\omega}_{\phi_\beta}$ is uniformly bounded in mass on K. Then by the weak compactness of positive currents and the equivalence between this and the L^1-convergence of potential functions, we can find a normalized potential function $\bar{\phi}_\beta$ such that it has a uniform L^1_{loc} bound, hence uniform L^p_{loc} bounds, for any $p > 1$. By (40), $\Delta\bar{\phi}_\beta$ is uniformly bounded. Thus by standard elliptic regularity results, [5, Theorem 8.17], there exists a constant C independ of β such that

$$||\bar{\phi}_\beta||_{C^0(K)} \le C, \quad \text{for } C = C(K). \tag{41}$$

Step 2: Higher-order estimates and the smooth local convergence. Since ω_{ϕ_β} satisfies the Kähler–Einstein equation outside D_β, $\bar{\omega}_{\phi_\beta}$ satisfies

$$\mathrm{Ric}\,\bar{\omega}_{\phi_\beta} = -\beta_1^2\bar{\omega}_{\phi_\beta} \quad \text{on } K. \tag{42}$$

Let dV_{eucl} denote the Euclidean volume form on $(\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$. Define

$$H_\beta := \log \frac{\bar{\omega}_{\phi_\beta}^n e^{-\beta_1^2\bar{\phi}_\beta}}{dV_{\mathrm{eucl}}}.$$

H_β is pluriharmonic by (42). By the definition of H_β,

$$(\sqrt{-1}\partial\bar{\partial}\bar{\phi}_\beta)^n = e^{\beta_1^2\bar{\phi}_\beta + H_\beta} dV_{\mathrm{eucl}}. \tag{43}$$

By (40),

$$||\beta_1^2\bar{\phi}_\beta + H_\beta||_{C^0(K)} < +\infty. \tag{44}$$

Combining (41) and (44), we see $||H_\beta||_{C^0(K)} < +\infty$. Then by gradient estimates for pluriharmonic functions,

$$||H_\beta||_{C^k(K)} < C(k,K), \quad \text{where } C(k,K) \text{ only depends on } k \text{ and } K \text{ not on } \beta. \tag{45}$$

Define

$$\Phi : \phi \mapsto \log \frac{(\sqrt{-1}\partial\bar{\partial}\phi)^n e^{-\beta_1^2\phi}}{dV_{\mathrm{eucl}}}.$$

Φ is a uniform elliptic concave operator as a function of $\partial\bar{\partial}\phi$. Hence by the Evans–Krylov theory, $||\phi||_{C^{2,\alpha}(K)}$ is controlled by $||\phi||_{C^0(K')}$, $||\Delta\phi||_{C^0(K')}$, and $||\Phi(\phi)||_{C^{0,1}(K')}$ for some $K' \Supset K$. Since $\Phi(\bar{\phi}_\beta) = H_\beta$, by (45), (41) and the fact

$||\Delta\bar{\phi}_\beta||_{C^0(K')} < +\infty$, there exist some $\alpha \in (0,1)$ and a uniform constant $C > 0$ such that

$$||\bar{\phi}_\beta||_{C^{2,\alpha}(K)} \leq C.$$

By standard bootstrapping arguments, every derivative of $\bar{\phi}_\beta$ is uniformly bounded on K. Then Arzelà-Ascoli theorem indicates $(\bar{\phi}_{\beta_k})_{k\in\mathbb{N}}$ has a convergent subsequence in $C^\infty(K)$-topology. Recall (42), letting $\beta_1 \to 0$ then due to the C^∞-convergence above we get a cluster value $\bar{\omega}_\infty$ such that

$$\mathrm{Ric}\,\bar{\omega}_\infty = 0.$$

By (40), $\bar{\omega}_\infty$ is quasi-isometric to $\hat{\omega}$, and therefore is a mixed cylinder and edge metric.

Step 3: Pointed Gromov–Hausdorff convergence

It remains to show a subsequence of $(U_{\beta_k}, \frac{1}{\beta_{1,k}^2}\omega_{\phi_{\beta_k}})$ converges in pointed Gromov-Hausdorff topology to $((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \bar{\omega}_\infty)$. Fix $q \in (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$ and fix a radius $a > 0$. First note that by construction, $B_{\bar{\omega}_{\phi_{\beta_k}}}(q,a)$ is isometric to $B_{\frac{1}{\beta_{1,k}^2}\omega_{\phi_{\beta_k}}}(\Psi_{\beta_k}(q), a)$. Secondly, by letting the index $k \in \mathbb{N}$ be large enough, we have $B_{\bar{\omega}_\infty}(q, 2a) \subset \Psi_{\beta_k}^{-1}(U_{\beta_k})$. Finally, due to the local C^∞-convergence, $B_{\bar{\omega}_{\phi_{\beta_k}}}(q,a) \subset B_{\bar{\omega}_\infty}(q,2a)$ and $B_{\bar{\omega}_{\phi_{\beta_k}}}(q,a)$ converges to $B_{\bar{\omega}_\infty}(q,a)$ in the Gromov-Hausdorff topology. Therefore $(U_{\beta_k}, \frac{1}{\beta_{1,k}^2}\omega_{\phi_{\beta_k}})$ converges (up to a subsequence) in pointed Gromov-Hausdorff topology to $((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \bar{\omega}_\infty)$ by [1, Definition 8.1.1]. $\square$

If we further allow more than one cone angles converge to 0, then we have the following results by modifying the result of Lemma 3.2.

Theorem 3.4. *Let $\{\beta_{1,k}\}_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. Assume further that for any $i = 2,\dots,r$ such that $\{\beta_{i,k}\}_{k\in\mathbb{N}}$ also converges to 0, there holds $\lim_{k\to\infty}\frac{\beta_{1,k}}{\beta_{i,k}} \in [0,1]$. Assume $\beta_{i,k} \in (0,\frac{1}{2}]$ for $i = 1,2,\dots,r$ and all $k \in \mathbb{N}$. Let $\omega_{\phi_{\beta_k}}$ be the (negatively curved) Kähler–Einstein crossing edge metric on $(X, D_k = \sum_{i=1}^r (1-\beta_{i,k})D_i)$. Then there exists a subsequence of the metric spaces $(U_{\beta_k}, \frac{1}{\beta_{1,k}^2}\omega_{\phi_{\beta_k}})$ converging in pointed Gromov-Hausdorff topology to $((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \bar{\omega}_\infty)$, where $\bar{\omega}_\infty$ is a mixed cylinder and edge metric with cylindrical part along components whose cone angles converge to 0 and conical part along other components.*

Proof. Without loss of generality, assume

$$\lim_{k\to\infty}\beta_{i,k} = 0, \quad \text{for } i = 1,\dots,t, t < m, \tag{46}$$

$$\lim_{k\to\infty}\beta_{j,k} := \beta_{j,\infty} > 0, \quad \text{for } j = t+1,\dots,m. \tag{47}$$

Moreover, assume

$$\lim_{k\to\infty}\frac{\beta_{1,k}}{\beta_{\ell,k}}=c_\ell\in(0,1],\quad \text{for } \ell=1,\dots,s,s<t, \tag{48}$$

$$\lim_{k\to\infty}\frac{\beta_{1,k}}{\beta_{\ell,k}}=0,\quad \text{for } \ell=s+1,\dots,t. \tag{49}$$

Recall in Lemma 3.2, we denote by $\mathbb{D}(a_1,\dots,a_m,b)$ the set

$$\{z\in(\mathbb{C}^*)^m\times\mathbb{C}^{n-m}\,:\,|z_i|<a_i, i=1,\dots,m, |z_j|<b, j=m+1,\dots,n\},$$

and

$$\Psi_\beta\,:\,\mathbb{D}\left(e^{\frac{1}{2\beta_1}},\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2\beta_2}},\dots,\left(\frac{\beta_m}{\beta_1}\right)^{\frac{1}{2\beta_m}},\frac{1}{\beta_1}\right)\to U_\beta:=\mathbb{D}\left(e^{-\frac{1}{2\beta_1}},\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2\beta_2}},\dots,\left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{2\beta_m}},1\right)$$

$$(w_1,\dots,w_m,w_{m+1},\dots,w_n)\mapsto\left(e^{-\frac{1}{\beta_1}}w_1,\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{\beta_2}}w_2,\dots,\left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{\beta_m}}w_m,\beta_1w_{m+1},\dots,\beta_1w_n\right).$$

Now let us modify Ψ_β by defining

$$V_\beta:=\mathbb{D}\left(e^{-\frac{1}{2\beta_1}},e^{-\frac{1}{2\beta_2}},\dots,e^{-\frac{1}{2\beta_s}},\left(\frac{\beta_1}{\beta_{s+1}}\right)^{\frac{1}{2\beta_{s+1}}},\dots,\left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{2\beta_m}},1\right),$$

and

$$\Phi_\beta\,:\,\mathbb{D}\left(e^{\frac{1}{2\beta_1}},e^{\frac{1}{2\beta_2}},\dots,e^{\frac{1}{2\beta_s}},\left(\frac{\beta_{s+1}}{\beta_1}\right)^{\frac{1}{2\beta_{s+1}}},\dots,\left(\frac{\beta_m}{\beta_1}\right)^{\frac{1}{2\beta_m}},\frac{1}{\beta_1}\right)\to V_\beta,$$

$$\Phi_\beta(w_1,\dots,w_s,w_{s+1},\dots,w_m,w_{m+1},\dots,w_n)=$$

$$\left(e^{-\frac{1}{\beta_1}}w_1,e^{-\frac{1}{\beta_2}}w_2,\dots,e^{-\frac{1}{\beta_s}}w_s,\left(\frac{\beta_1}{\beta_{s+1}}\right)^{\frac{1}{\beta_{s+1}}}w_{s+1},\dots,\left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{\beta_m}}w_m,\beta_1w_{m+1},\dots,\beta_1w_n\right).$$

Then

$$\begin{aligned}\Phi_\beta^*\left(\frac{1}{\beta_1^2}\omega_{\beta,\mathrm{mod}}\right)&=\sum_{i=1}^s\frac{\beta_1^2}{\beta_i^2}\frac{e^{-2}|w_i|^{2\beta_i}}{(1-e^{-2}|w_i|^{2\beta_i})^2}\cdot\frac{\sqrt{-1}dw_i\wedge d\bar{w}_i}{|w_i|^2}\\&\quad+\sum_{j=s+1}^m\frac{\sqrt{-1}dw_j\wedge d\bar{w}_j}{|w_j|^{2(1-\beta_j)}\left(1-\frac{\beta_1^2}{\beta_j^2}|w_j|^{2\beta_j}\right)^2}+\sum_{\ell=m+1}^n\sqrt{-1}dw_\ell\wedge d\bar{w}_\ell.\end{aligned}$$

Denote

$$\beta_k:=(\beta_{1,k},\dots,\beta_{r,k}),\quad \text{for } k\in\mathbb{N}^*.$$

For a compact $K \subset (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$, there exists a large enough k such that $K \subset \Phi_{\beta_k}^{-1}(V_{\beta_k})$. By assumptions (46) and (48), $\Phi_{\beta_k}^*\left(\frac{1}{\beta_{1,k}^2}\omega_{\beta_k,\mathrm{mod}}\right)$ converges in $C^\infty(K)$ to

$$\sum_{i=1}^{s}\frac{e^{-2}}{c_i^2(1-e^{-2})}\frac{\sqrt{-1}dw_i\wedge d\bar{w}_i}{|w_i|^2}+\sum_{j=s+1}^{t}\frac{\sqrt{-1}dw_j\wedge d\bar{w}_j}{|w_j|^2}$$
$$+\sum_{\ell=t+1}^{m}\frac{\sqrt{-1}dw_\ell\wedge d\bar{w}_\ell}{|w_\ell|^{2(1-\beta_{\ell,\infty})}}+\sum_{q=m+1}^{n}\sqrt{-1}dw_q\wedge d\bar{w}_q =: \check{\omega},$$

which is a mixed cylindrical and conical metric by Definition 3.1. The cylindrical parts are along $D_i, i=1,\dots,t$, whose cone angle goes to 0. The remainder of the proof is similar to that of Theorem 3.3.

□

Acknowledgment

The author would like to thank H. Guenancia and J. Sturm for helpful conversations. The author is especially grateful to Y. A. Rubinstein for suggesting this problem and for his guidance and encouragement.

References

[1] Burago, Dmitri; Burago, Yuri; Ivanov, Sergei. A course in metric geometry. Graduate Studies in Mathematics, 33. *American Mathematical Society, Providence, RI*, 2001. xiv+415 pp. ISBN: 0-8218-2129-6 MRMR1835418, Zbl 0981.51016, doi: 10.1090/gsm/033. 621

[2] Campana, Frédéric; Guenancia, Henri; Păun, Mihai. Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields. *Ann. Sci. Éc. Norm. Supér. (4)* **46** (2013), no. 6, 879–916. MRMR3134683, Zbl 1310.32029, doi: 10.24033/asens.2205. 617

[3] Cheltsov, Ivan A; Rubinstein, Yanir A. Asymptotically log Fano varieties. *Adv. Math.* **285** (2015), 1241–1300. MRMR3406526, Zbl 1337.14033, doi: 10.1016/j.aim.2015.08.001. 594

[4] Donaldson, S. K. Kähler metrics with cone singularities along a divisor. *Essays in mathematics and its applications,* 49–79, *Springer, Heidelberg*, 2012. MRMR2975584, Zbl 1326.32039, doi: 10.1007/978-3-642-28821-0_4. 594

[5] Gilbarg, David; Trudinger, Neil S. Elliptic partial differential equations of second order. Second edition. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224. *Springer-Verlag, Berlin*, 1983. xiii+513 pp. ISBN: 3-540-13025-X MRMR0737190, Zbl 0562.35001, doi: 10.1007/978-3-642-96379-7. 620

[6] Griffiths, Phillip; Harris, Joseph. Principles of algebraic geometry. Pure and Applied Mathematics. *Wiley-Interscience [John Wiley & Sons], New York*, 1978. xii+813 pp. ISBN: 0-471-32792-1 MRMR0507725, Zbl 0408.14001, doi: 10.1002/9781118032527. 599

[7] Guenancia, Henri. Kähler–Einstein metrics with mixed Poincaré and cone singularities along a normal crossing divisor. *Ann. Inst. Fourier (Grenoble)* **64** (2014), no. 3, 1291–1330. MRMR3330171, Zbl 1310.32024, doi: 10.5802/aif.2881. 597, 616

[8] Guenancia, Henri. Kähler–Einstein metrics: from cones to cusps. *J. Reine Angew. Math.* **759** (2020), 1–27. MRMR4058174, Zbl 1469.32019, doi: 10.1515/crelle-2018-0001. 594, 595, 596, 598, 599, 600, 601, 603, 610

[9] GUENANCIA, HENRI; PĂUN, MIHAI. Conic singularities metrics with prescribed Ricci curvature: general cone angles along normal crossing divisors. *J. Differential Geom.* **103** (2016), no. 1, 15–57. MRMR3488129, Zbl 1344.53053, doi: 10.4310/jdg/1460463562. 595, 596, 613

[10] JEFFRES, THALIA; MAZZEO, RAFE; RUBINSTEIN, YANIR A. Kähler–Einstein metrics with edge singularities. *Ann. of Math. (2)* **183** (2016), no. 1, 95–176. MRMR3432582, Zbl 1337.32037, doi: 10.4007/annals.2016.183.1.3. 595, 596, 599, 603, 613, 614

[11] KOBAYASHI, RYOICHI. Kähler–Einstein metric on an open algebraic manifold. *Osaka J. Math.* **21** (1984), no. 2, 399–418. MRMR0752470, Zbl 0556.14019. 594, 597, 615

[12] MAZZEO, RAFE. Kähler–Einstein metrics singular along a smooth divisor. *Journées "Équations aux Dérivées Partielles"(Saint-Jean-de-Monts, 1999),* Exp. No. VI, 10 pp., *Univ. Nantes, Nantes*, 1999. MRMR1718970, Zbl 0990.00047, doi: 10.5802/jedp.550. 594

[13] MAZZEO, RAFE; RUBINSTEIN, YANIR A. The Ricci continuity method for the complex Monge–Ampère equation, with applications to Kähler–Einstein edge metrics. *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 13-14, 693–697. MRMR2971382, Zbl 1273.32043, doi: 10.1016/j.crma.2012.07.001. 596

[14] RUBINSTEIN, YANIR A. Smooth and singular Kähler–Einstein metrics. *Geometric and spectral analysis,* 45–138, Contemp. Math., 630, Centre Rech. Math. Proc., *Amer. Math. Soc., Providence, RI*, 2014. MRMR3328541, Zbl 1352.32009, doi: 10.1090/conm/630/12665. 596, 598, 610, 614

[15] RUBINSTEIN, YANIR A. High-dimensional convex sets arising in algebraic geometry. *Geometric aspects of functional analysis. Vol. II,* 301–323, Lecture Notes in Math., 2266, *Springer, Cham*, 2020. MRMR4175769, Zbl 1441.14033, doi: 10.1007/978-3-030-46762-3_14 598

[16] RUBINSTEIN, YANIR A; ZHANG, KEWEI. Small angle limits of Hamilton's footballs. *Bull. Lond. Math. Soc.* **52** (2020), no. 1, 189–199. MRMR4072044, Zbl 1444.53063, doi: 10.1112/blms.12319. 598

[17] TIAN, GANG. Kähler–Einstein metrics on algebraic manifolds. *Transcendental methods in algebraic geometry (Cetraro, 1994),* 143–185, Lecture Notes in Math., 1646, Fond. CIME/CIME Found. Subser., *Springer, Berlin*, 1996. MRMR1603624, Zbl 0896.32003, doi: 10.1007/BFb0094304. 594

[18] TIAN, G.; YAU, S. T. Existence of Kähler–Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. Reprint of [0915840]. *Selected works of Shing-Tung Yau. Part 1. 1971–1991. Vol. 5. Kähler geometry II,* 127–181, *Int. Press, Boston, MA*, 2019. MRMR4519042, Zbl 0682.53064, doi: 10.1142/9789812798411_0028. 594, 597, 615

[19] WU, DAMIN. Kähler–Einstein metrics of negative Ricci curvature on general quasi-projective manifolds. *Comm. Anal. Geom.* **16** (2008), no. 2, 395–435. MRMR2425471, Zbl 1151.32009, doi: 10.4310/CAG.2008.v16.n2.a4. 595, 597

(Yuxiang Ji) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20740, USA
`yxji@umd.edu`

This paper is available via `http://nyjm.albany.edu/j/2024/30-27.html`.