New York Journal of Mathematics

New York J. Math. **30** (2024) 231–269.

# The method of infinite descent in stable homotopy theory II

**Hirofumi Nakai** and **Douglas C. Ravenel**

Abstract. This paper is a continuation of [Rav02] of the same title, which we will refer hereafter to as [I], which intends to clarify and expand the results in the last chapter of [Rav86] ("the green book"). In particular, we give the stable homotopy groups of $p$-local spectra $T(m)_{(1)}$ for $m > 0$. This is a part of a program to compute the $p$-components of $\pi_*(S^0)$ through dimension $2p^4(p-1)$ for $p > 2$. We will refer to the results from [I] freely as if they were in the first four sections of this paper, which begins with section 5.

Contents



## 1. Introduction

In [Rav04] the second author described a method for computing the Adams-Novikov $E_2$-term for spheres and used it to determine the stable homotopy groups through dimension 108 for $p = 3$ and 999 for $p = 5$. The latter computation was a substantial improvement over prior knowledge, and neither has been improved upon since. It is generally agreed among homotopy theorists that it is not worthwhile to try to improve our knowledge of stable homotopy groups by a few stems, but that the prospect of increasing the known range by a factor of $p$ would be worth pursuing. This possibility may be within reach now, due to a better understanding of the methods of [Rav04, Chapter 7] and improved

Received April 13, 2022.

2010 *Mathematics Subject Classification.* 55Q10, 55Q45, 55Q51, 55T05.

*Key words and phrases.* stable homotopy groups of spheres, Adams-Novikov spectral sequence, Cartan-Eilenberg spectral sequence.

ISSN 1076-9803/2024

computer technology. This paper should be regarded as laying the foundation for a program to compute $\pi_*(S^0)_{(p)}$ through roughly dimension $2p^4(p-1)$, i.e., 324 for $p = 3$ and 5,000 for $p = 5$.

It is unlikely that either author will take up this computational project any time soon. The purpose of the present paper is to document what we believe to be the most promising method of extending the computation of [Rav04, Chapter 7] in hopes that some more energetic mathematicians will use it in the future.

The paper [Rav02], which we will refer to here as [I], is published in a conference proceedings volume which is not available online. However a digital copy can be found on the second author's home page, for which a link is given in the bibliography of the present paper

**1.1. Summary of [I].** The method referred to in the title involves the connective $p$-local ring spectra $T(m)$ satisfying

$$BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*(BP)$$

and the natural map $T(m) \to BP$ which is an equivalence below dimension $|t_{m+1}|$. In particular, we have $T(0) = S^0_{(p)}$ and $T(\infty) = BP$.

For a Hopf algebroid $(A, \Gamma)$ and $\Gamma$-comodule $M$, we will often drop the first variable of Ext for short, i.e., $\mathrm{Ext}_\Gamma(A, M)$ will be denoted by $\mathrm{Ext}_\Gamma(M)$. If we define the quotient module $\Gamma(k)$ by

$$\Gamma(m+1) = BP_*(BP)/(t_1, \dots, t_m) \cong BP_*[\widehat{t}_1, \widehat{t}_2, \dots],$$

where $\widehat{t}_i = t_{m+i}$, then the pair $(BP_*, \Gamma(m+1))$ forms a Hopf algebroid, whose structure maps are inherited from $(BP_*, BP_*(BP))$. Note that $\Gamma(1) = BP_*(BP)$. By the change-of-rings isomorphism [Rav04, Theorem A1.3.12], the Adams-Novikov $E_2$-term for $T(m)$ is reduced to $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$. We will also use the notation

$$\widehat{v}_i = v_{m+i} \qquad \text{and} \qquad A(m) = \mathbb{Z}_{(p)}[v_1, \dots v_m].$$

It is not difficult to find the structure of $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$ in low dimensions. We know by Proposition 3.6 for $n = 0$, that

$$\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*) \cong A(m).$$

The group $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$ is described in Theorem 3.16. Excluding the case $m = 0$ and $p = 2$ (which is handled in [Rav04, Theorem 5.2.6]), it is the $A(m)$-module generated by the set

$$\left\{ \widehat{\widehat{\alpha}}_j := \alpha\left(\frac{\widehat{v}_1^j}{jp}\right) : j > 0 \right\},$$

where $\alpha$ is the connecting homomorphism for the short exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow BP_* & \longrightarrow & M^0 & \longrightarrow & N^1 & \longrightarrow & 0 \\ & & \| & & \| & & \\ & & p^{-1}BP_* & & BP_*/(p^\infty) & & \end{array}$$

as in (1.6). We also define

$$\widehat{\alpha}_j := \alpha\left(\frac{\widehat{v}_1^j}{p}\right) \quad \text{for} \quad j > 0, \quad \text{with} \quad \widehat{h}_{1,0} := \widehat{\alpha}_1.$$

The structure of $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$ below dimension $p^2|\widehat{v}_1|$ was determined in Theorem 4.5. We make use of the 4-term exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow BP_* \longrightarrow M^0 & \longrightarrow & M^1 & \longrightarrow & N^2 & \longrightarrow & 0 \\ & & \| & & \| & & \\ & & v_1^{-1}BP_*/(p^\infty) & & BP_*/(p^\infty, v_1^\infty), & & \end{array}$$

which leads to a double connecting homomorphism

$$\beta : \mathrm{Ext}^s_{\Gamma(m+1)}\left(N^2\right) \to \mathrm{Ext}^{s+2}_{\Gamma(m+1)}\left(BP_*\right).$$

We define

$$\widehat{\beta}_j := \beta\left(\frac{\widehat{v}_2^j}{pv_1}\right) \quad \text{for} \quad j > 0, \quad \text{with} \quad \widehat{b}_{1,0} = \widehat{\beta}_1.$$

Theorem 4.5 says that below dimension $p^2|\widehat{v}_1|$, the groups $\mathrm{Ext}^{s+2}_{\Gamma(m+1)}(BP_*)$ for $s \geq 0$ have the form

$$A(m+1)/I_2 \otimes E(\widehat{h}_{1,0}) \otimes P(\widehat{b}_{1,0}) \otimes \left\{\widehat{\beta}_j : j \geq 1\right\}.$$

where $I_n$ is the ideal $(p, v_1, \dots, v_{n-1})$ as usual. We have constructed the short exact sequence of $\Gamma(m+1)$-comodules

$$0 \longrightarrow BP_* \xrightarrow{i_1} D^0_{m+1} \xrightarrow{j_1} E^1_{m+1} \longrightarrow 0 \qquad \text{for } m \geq 0 \tag{1.1}$$

where the map $i_1$ induces an isomorphism of $\mathrm{Ext}^0$ (cf. Theorems 3.7 and 3.11), and $D^0_{m+1}$ is a weak injective $\Gamma(m+1)$-comodule. Hence we have isomorphisms

$$\mathrm{Ext}^t_{\Gamma(m+1)}(E^1_{m+1}) \cong \mathrm{Ext}^{t+1}_{\Gamma(m+1)}(BP_*) \quad \text{for } t \geq 0.$$

$D_0^{m+1}$ is the sub-$A(m)$-algebra of $p^{-1}BP_*$ generated by certain elements $\widehat{\lambda}_{m+i}$ for $i > 0$ congruent to $\widehat{v}_i/p$ modulo decomposables. To describe them we need to recall Hazewinkel's formula [Haz77] relating polynomial generators $v_i \in BP_*$ to the coefficients $\ell_i$ of the formal group law, namely

$$p\ell_i = \sum_{0 \leq j < i} \ell_j v_{i-j}^{p^j}. \tag{1.2}$$

This recursive formula expands to

$$\ell_1 = \frac{v_1}{p}, \quad \ell_2 = \frac{v_2}{p} + \frac{v_1^{p+1}}{p^2}, \quad \ell_3 = \frac{v_3}{p} + \frac{v_1 v_2^p}{p^2} + \frac{v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3}, \quad \cdots$$

We need to define reduced log coefficients $\widehat{\ell}_k$ obtained from the $\ell_{m+k}$ by subtracting the terms which are monomials in the $v_j$ for $j \leq m$. Thus for $m = 1$ we have

$$\widehat{\ell}_1 = \frac{\widehat{v}_1}{p}, \qquad \widehat{\ell}_2 = \frac{\widehat{v}_2}{p} + \frac{v_1 \widehat{v}_1^p}{p^2} + \frac{\widehat{v}_1 v_1^{p^2}}{p^2}, \qquad \cdots$$

The analog of Hazewinkel's formula for these elements is

$$p\widehat{\ell}_i = \begin{cases} 0 & \text{if } i \leq 0 \\ \displaystyle\sum_{0 \leq j < i} \ell_j \widehat{v}_{i-j}^{p^j} + \sum_{0 < j < \min(i, m+1)} \widehat{\ell}_{i-j} v_j^{p^{i-j}\omega} & \text{if } i > 0. \end{cases} \tag{1.3}$$

We use these to define our generators $\widehat{\lambda}_i$ recursively for $i > 0$ by

$$\widehat{\ell}_i = \sum_{0 \leq j < i} \ell_j \widehat{\lambda}_{i-j}^{p^j}. \tag{1.4}$$

We may also assume the existence of the short exact sequence

$$0 \longrightarrow E^1_{m+1} \xrightarrow{i_2} D^1_{m+1} \xrightarrow{j_2} E^2_{m+1} \longrightarrow 0. \tag{1.5}$$

where $D^1_{m+1}$ is weak injective: it is specifically constructed in Lemma 4.1 for $m = 0$ and $p$ odd, with the map $i_2$ inducing an isomorphism in $\mathrm{Ext}^0$. For $m > 0$, it is shown that $v_1^{-1} E^1_{m+1}$ is weak injective with

$$\mathrm{Ext}^0_{\Gamma(m+1)}(v_1^{-1} E^1_{m+1}) \cong v_1^{-1} \mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$$

thus we may regard $D^1_{m+1}$ as $v_1^{-1} E^1_{m+1}$ at worst (cf. Lemma 3.18).

It is desirable to define $D^1_{m+1}$ for $m > 0$ to make its $\mathrm{Ext}^0$ as small as possible. If we assume that the map $i_2$ induces an isomorphism in $\mathrm{Ext}^0$, then we have isomorphisms

$$\mathrm{Ext}^t_{\Gamma(m+1)}(E^2_{m+1}) \cong \mathrm{Ext}^{t+2}_{\Gamma(m+1)}(BP_*) \quad \text{for } t \geq 0.$$

We constructed such isomorphisms [1] and computed the Ext groups below dimension $p^2|v_{m+1}|$ by producing $E^2_{m+1}$ satisfying some desirable conditions and

[1]Unfortunately, $i_2$ induces an isomorphism in $\mathrm{Ext}^0$ only below dimension $p|v_{m+2}|$ for $m > 0$. See Remark 3.3.

the weak injective $D^1_{m+1}$ as the induced extension (cf. Corollary 4.3):

$$\begin{tikzcd}
0 \arrow[r] & E^1_{m+1} \arrow[r, "i_2"] \arrow[d, equal] & D^1_{m+1} \arrow[r, "j_2"] \arrow[d, hook] & E^2_{m+1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & E^1_{m+1} \arrow[r] & v_1^{-1}E^1_{m+1} \arrow[r] & E^1_{m+1}/(v_1^\infty) \arrow[r] & 0.
\end{tikzcd}$$

Since there is no Adams-Novikov differential and no nontrivial group extension in this range (except in the case $m = 0$ and $p = 2$), this also determines $\pi_*(T(m))$ in the same range. This was the goal of [I].

**1.2. Introduction to II.** To descend from $T(m+1)$ to $T(m)$, we can consider some interpolating spectra $T(m)_{(i)}$ introduced in Lemma 1.15. Each $T(m)_{(i)}$ is the $T(m)$-module spectrum satisfying

$$BP_*(T(m)_{(i)}) = BP_*(T(m))\{t^\ell_{m+1} \mid 0 \le \ell < p^i\}$$

and the natural map $T(m)_{(i)} \to T(m+1)$ is an equivalence in dimensions below $p^i|t_{m+1}|$. In particular, we have $T(m)_{(0)} = T(m)$ and $T(m)_{(\infty)} = T(m+1)$.

The Adams-Novikov $E_2$-term for $T(m)_{(i)}$ is

$$E_2^{s,*} = \mathrm{Ext}^{s,*}_{BP_*(BP)}(BP_*(T(m)_{(i)}))$$

and it is reduced to

$$\mathrm{Ext}^{s,*}_{\Gamma(m+1)}(T^{(i)}_m)$$

by Lemma 1.15, where $T^{(i)}_m$ is the $BP_*$-module generated by

$$\{t^\ell_{m+1} \mid 0 \le \ell < p^i\}.$$

Then, we have the 3-term resolution of $T^{(i)}_m$ by tensoring the short exact sequence (1.1) with $T^{(i)}_m$, and the associated spectral sequence $\{E_r^{s,t}, d_r\}_{r\ge1}$ converges to $\mathrm{Ext}^*_{\Gamma(m+1)}(T^{(i)}_m)$ with

$$E_1^{s,t} = \begin{cases} \mathrm{Ext}^0_{\Gamma(m+1)}(T^{(i)}_m \otimes_{BP_*} D^0_{m+1}) & \text{for } s = 0, \\ \mathrm{Ext}^t_{\Gamma(m+1)}(T^{(i)}_m \otimes_{BP_*} E^1_{m+1}) & \text{for } s = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

The only nontrivial differential is $d_1 : E_1^{0,0} \to E_1^{1,0}$ induced by $j_1$ (1.1), and the spectral sequence collapses from $E_2$-term. Thus we have

**Proposition 1.7.** *The Adams-Novikov $E_2$-term for $T(m)_{(i)}$ is*

$$\mathrm{Ext}^s_{\Gamma(m+1)}(T^{(i)}_m) \cong \begin{cases} \ker d_1 & \text{for } s = 0, \\ \operatorname{coker} d_1 & \text{for } s = 1, \\ \mathrm{Ext}^{s-1}_{\Gamma(m+1)}(T^{(i)}_m \otimes_{BP_*} E^1_{m+1}) & \text{for } s \ge 2. \end{cases}$$

*Note that the groups for $s = 0$ and $1$ were determined in* [Nak08, Proposition 2.5, Theorem 4.1 and §5] *(See also Proposition 2.6).*

Once we know about $T(m)_{(i+1)}$ for some $i$, we can descend the value of $i$ by using the small descent spectral sequence (Theorem 1.21), whose $E_1$-term is

$$E(h_{m+1,i}) \otimes P(b_{m+1,i}) \otimes \pi_*(T(m)_{(i+1)})$$

where $h_{m+1,i} \in E_1^{1,2p^i(p^{m+1}-1)}$ and $b_{m+1,i} \in E_1^{2,2p^{i+1}(p^{m+1}-1)}$ are permanent cycles. Note that we know $\pi_*(T(1)_{(3)})$ below dimension $p^3|t_2|$ by Theorem 4.5 without any use of spectral sequences, since the dimension is smaller than $p^2|t_3|$ and $T(1)_{(3)} = T(2)$ in that range. This allows us to compute $\pi_*(T(1))$ from the information of $\pi_*(T(1)_{(3)})$. Since $T(0)_{(4)} = T(1)$ below dimension $p^4|v_1|$, this also makes possible to have $\pi_*(S^0)$ in the same range.

In this paper we assume that $m > 0$ unless otherwise noted. The main results are the determination of the Adams-Novikov $E_2$-terms for $T(m)_{(1)}$ below dimension $p|v_{m+3}|$ in Theorem 6.14. In this range there is still no room for Adams-Novikov differentials, so the homotopy and Ext calculations coincide[2]. It is only when we pass from $T(m)_{(1)}$ to $T(m)$ that we encounter Adams-Novikov differentials below dimension $p^2|v_{m+2}|$. For $m = 0$, the first of these is the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$ of [Tod67] and [Tod68], and the relevant calculations were the subject of [Rav04, Chapter 7]. An analogous differential for $m > 0$ was also established in [Rav], and we will discuss it somewhere else in the future.

## 2. A variant of Cartan-Eilenberg spectral sequence

Assume that $M$ is a $\Gamma(m)$-comodule for some $m$. Once we know the structure of $\mathrm{Ext}^*_{\Gamma(m)}(M)$, there is an inductive step reducing the value of $m$. Set

$$A(m) = \mathbb{Z}_{(p)}[v_1, \dots, v_m] \quad \text{and} \quad G(m) = A(m)[t_m].$$

The pair $(A(m), G(m))$ is a Hopf algebroid. Then we have an extension of Hopf algebroids (cf. Proposition 1.2)

$$(A(m), G(m)) \longrightarrow (BP_*, \Gamma(m)) \longrightarrow (BP_*, \Gamma(m+1))$$

and the associated Cartan-Eilenberg spectral sequence

$$\mathrm{Ext}^*_{G(m)}(\mathrm{Ext}^*_{\Gamma(m+1)}(M)) \implies \mathrm{Ext}^*_{\Gamma(m)}(M).$$

A $\Gamma(m+1)$-comodule $M$ is naturally a $\Gamma(m+2)$-comodule, and we will denote $\mathrm{Ext}^0_{\Gamma(m+2)}(M)$ by $\overline{M}$ for short. In particular, we have

$$\overline{T}_m^{(i)} = A(m+1)\{t_{m+1}^\ell \mid 0 \le \ell < p^i\}.$$

Then the Cartan-Eilenberg $E_2$-term converging to $\mathrm{Ext}^*_{\Gamma(m+1)}(T_m^{(i)} \otimes_{BP_*} E_{m+1}^1)$ is

$$\tilde{E}_2^{s',s''} = \mathrm{Ext}^{s'}_{G(m+1)}(\mathrm{Ext}^{s''}_{\Gamma(m+2)}(T_m^{(i)} \otimes_{BP_*} E_{m+1}^1))$$

[2]For $m = 0$, the second author determined the structure of $\mathrm{Ext}^{*+2}_{\Gamma(1)}(T_0^{(1)})$ in [Rav04, Theorem 7.5.1] for $p > 2$ below dimension $(p^3 + p)|v_1|$.

$$\cong \mathrm{Ext}^{s'}_{G(m+1)}(\overline{T}^{(i)}_m \otimes_{A(m+1)} \mathrm{Ext}^{s''}_{\Gamma(m+2)}(E^1_{m+1})) \tag{2.1}$$

with differentials $\tilde{d}_r : \tilde{E}^{s',s''}_r \to \tilde{E}^{s'+r,s''-r+1}_r$. Since the case $s' = s'' = 0$ is not interesting, we will assume that $s' + s'' \geq 1$.

For simplicity, we will hereafter omit the subscript in $\otimes_{A(m+1)}$, and we will denote $\mathrm{Ext}^{s''}_{\Gamma(m+2)}(BP_*)$ by $U^{s''}_{m+1}$. Since $D^0_{m+1}$ in (1.1) is weak injective, we have isomorphisms $\mathrm{Ext}^{s''}_{\Gamma(m+2)}(E^1_{m+1}) \cong U^{s''+1}_{m+1}$ and

$$\tilde{E}^{s',s''}_2 \cong \mathrm{Ext}^{s'}_{G(m+1)}(\overline{T}^{(i)}_m \otimes U^{s''+1}_{m+1}) \qquad \text{for } s'' \geq 1. \tag{2.2}$$

Note that the structure of $U^*_{m+1}$ can be read from Theorem 4.5. This will be discussed again in Corollary 4.1.

To describe $\tilde{E}^{s',0}_2$, we need a resolution of $\overline{E}^1_{m+1} = \mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1})$. The obvious one is obtained by applying $\mathrm{Ext}^0_{\Gamma(m+2)}(-)$ to (1.5). In practice, there is a "smaller resolution".

Now we recall some notations used in [I]. For a fixed positive integer $m$, we will set $\widehat{v}_i = v_{m+i}$ and $\widehat{t}_i = t_{m+i}$, and define

$$\widehat{\beta}_{i/e_1,e_0} = \frac{\widehat{v}^i_2}{p^{e_0} v^{e_1}_1}, \qquad \widehat{\beta}_{i/e_1} = \widehat{\beta}_{i/e_1,1}, \qquad \widehat{\beta}_i = \widehat{\beta}_{i/1},$$

$$\widehat{\beta}'_{i/e_1} = \frac{1}{i}\widehat{\beta}_{i/e_1} = \frac{\widehat{v}^i_2}{ipv^{e_1}_1}, \qquad \widehat{\beta}'_i = \widehat{\beta}'_{i/1}, \qquad \text{and} \qquad \widehat{\gamma}_i = \frac{\widehat{v}^i_3}{pv_1v_2}.$$

Then we have

**Proposition 2.3.** *Let $B_{m+1}$ be the $A(m+1)$-module generated by $\widehat{\beta}'_{i/i}$ for $i > 0$. Then $B_{m+1}$ is a sub $G(m+1)$-comodule of $E^1_{m+1}/(v^\infty_1)$ and it is invariant over $\Gamma(m+2)$. Its Poincaré series is*

$$g(B_{m+1}) = g_{m+1}(t) \sum_{k \geq 0} \frac{x^{p^{k+1}}(1 - y^{p^k})}{(1 - x^{p^{k+1}})(1 - x^{p^j}_2)}$$

*where $y = t^{|v_1|}$, $x = t^{|\widehat{v}_1|}$, $x_2 = t^{|\widehat{v}_2|}$ and*

$$g_{m+1}(t) = \prod_{i=1}^{m+1} \frac{1}{1 - y_i} \qquad \textit{where } y_i = t^{|v_i|}.$$

**Proof.** This is [NR09, Theorem 2.4]. To clarify that $\widehat{\beta}'_{i/i}$ are in $E^1_{m+1}/(v^\infty_1)$, note that an element in $N^2$ lies in $E^1_{m+1}/(v^\infty_1)$ if and only if it has trivial image in

$(M^0/D^0_{m+1})/(v_1^\infty)$. This can be shown using the commutative diagram [3]

$$\begin{tikzcd}
0 \arrow[r] & E^1_{m+1} \arrow[r] \arrow[d, tail] & v_1^{-1}E^1_{m+1} \arrow[r] \arrow[d, tail] & E^1_{m+1}/(v_1^\infty) \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & N^1 \arrow[r] \arrow[d, two heads] & M^1 \arrow[r] \arrow[d, two heads] & N^2 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & M^0/D^0_{m+1} \arrow[r] & v_1^{-1}(M^0/D^0_{m+1}) \arrow[r] & (M^0/D^0_{m+1})/(v_1^\infty) \arrow[r] & 0
\end{tikzcd}$$

where $M^i$ and $N^i$ are usual chromatic comodules. Define $w \in D^0_{m+1}$ by

$$w = (1 - p^{p-1})\widehat{\lambda}_1^p - v_1^{p^{m+1}-1}\lambda_1. \tag{2.4}$$

Then we have $\widehat{v}_2 = p(\widehat{\lambda}_2 + \lambda_1 w)$ and

$$\widehat{\beta}'_{i/i} = \frac{p^i(\widehat{\lambda}_2 + \lambda_1 w)^i}{ipv_1^i} = \frac{p^{i-1}(\widehat{\lambda}_2 + \lambda_1 w)^i}{iv_1^i}$$

which is clearly in $(M^0/D^0_{m+1})/(v_1^\infty)$ as desired. $\square$

Let $W_{m+1}$ be the $G(m+1)$-comodule[4] defined by the induced extension in the following commutative diagram (cf. [NR09, (1.4)]):

$$\begin{tikzcd}
0 \arrow[r] & \overline{E}^1_{m+1} \arrow[r, "\iota"] \arrow[d, equal] & W_{m+1} \arrow[r, "\rho"] \arrow[d, hook] & B_{m+1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \overline{E}^1_{m+1} \arrow[r] & v_1^{-1}\overline{E}^1_{m+1} \arrow[r] & \overline{E}^1_{m+1}/(v_1^\infty) \arrow[r] & 0
\end{tikzcd}$$

In fact, we can describe $W_{m+1}$ explicitly. Recall that

$$\mathrm{Ext}^1_{\Gamma(m+2)}(BP_*) \cong A(m+1)\left\{\frac{\widehat{v}_2^i}{ip} \mid i > 0\right\}.$$

Applying $\mathrm{Ext}_{\Gamma(m+2)}$ to (1.1) we have the short exact sequence

$$0 \longrightarrow A(m)[\widehat{\lambda}_1]/A(m+1) \longrightarrow \overline{E}^1_{m+1} \xrightarrow{\delta} U^1_{m+1} \longrightarrow 0.$$

Then, a lift of $\widehat{v}_2^i/ip \in U^1_{m+1}$ to $\overline{E}^1_{m+1}$ is given by

$$b_i = \frac{\widehat{v}_2^i - (v_1 w)^i}{ip} \qquad \text{where } w \text{ is as in (2.4).}$$

[3] For $m = 0$ and $p > 2$, $E_1^1/(v_1^\infty)$ is isomorphic to $N^2$.

[4] For $m = 0$ and $p > 2$, we may simply set $W_1 = \mathrm{Ext}^0_{\Gamma(2)}(D_1^1)$ (cf. [Rav04, (7.2.17)]), since the map $E_1^1 \to D_1^1$ induces an isomorphism in $\mathrm{Ext}^0_{\Gamma(1)}$.

and a lift of $\widehat{\beta}'_{i/i} \in B_{m+1}$ to $W_{m+1}$ is given by

$$v_1^{-i} b_i = \sum_{0<j\le i} \binom{i-1}{j-1} \frac{(pv_1^{-1}\widehat{\lambda}_2)^j}{pj} w^{i-j}.$$

So, $W_{m+1}$ is the subcomodule of $M^1$ obtained by adjoining $v_1^{-i} b_i$ $(i > 0)$ to $\overline{E}^1_{m+1}$.

The following properties of $W_{m+1}$ can be read from [NR09, Theorem 2.4].

**Proposition 2.5.** *$W_{m+1}$ is weak injective and the map $\iota : \overline{E}^1_{m+1} \to W_{m+1}$ induces an isomorphism in $\mathrm{Ext}^0$: we have $\mathrm{Ext}^0_{G(m+1)}(W_{m+1}) \cong U^1_{m+1}$.*

Now we have a 3-term resolution of $\overline{E}^1_{m+1}$

$$0 \longrightarrow \overline{E}^1_{m+1} \xrightarrow{\iota} W_{m+1} \xrightarrow{\rho} B_{m+1} \longrightarrow 0.$$

Let $C^{*,s}$ denote the cochain complex obtained by applying $\mathrm{Ext}^s_{G(m+1)}(T_m^{(j)} \otimes -)$ to the sequence

$$\overline{D}^0_{m+1} \xrightarrow{\iota\circ(j_1)_*} W_{m+1} \xrightarrow{\rho} B_{m+1}$$

and let $H^{*,s}(C)$ be the associated cohomology group. Then we have

**Proposition 2.6.** *For $n = 0$ and $1$, $H^{n,0}(C)$ is isomorphic to the Adams-Novikov $E_2$-term $\mathrm{Ext}^n_{\Gamma(m+1)}(T_m^{(j)})$.*

**Proof.** Since $W_{m+1}$ is weak injective over $G(m+1)$, $\overline{T}_m^{(i)} \otimes W_{m+1}$ is also weak injective by Lemma 1.14 and $C^{1,s} = 0$ for $s \ge 1$. We have the commutative diagram

$$\begin{tikzcd}
 & C^{0,0} \arrow[r] \arrow[d, "(j_1)_*"] & C^{1,0} \arrow[r] \arrow[d, equal] & C^{2,0} \arrow[d, equal] & & \\
0 \arrow[r] & \tilde{E}_2^{0,0} \arrow[r, "\iota_*"] & C^{1,0} \arrow[r, "\rho_*"] & C^{2,0} \arrow[r] & \tilde{E}_2^{1,0} \arrow[r] & 0 \quad \text{(exact)}
\end{tikzcd}$$

and isomorphisms $C^{2,s-1} \cong \tilde{E}_2^{s,0}$ for $s \ge 2$. The map $(j_1)_*$ coincides with the differential $d_1 : E_1^{0,0} \to E_1^{1,0}$ of the resolution spectral sequence of (1.6), so we have

$$\begin{aligned}
H^{0,0}(C) &= \ker(j_1)_* = \ker d_1,\\
H^{1,0}(C) &= \ker \rho_* / \operatorname{im}(j_1)_* \cong \tilde{E}_2^{0,0} / \operatorname{im}(j_1)_* = \operatorname{coker} d_1.
\end{aligned}$$ □

The structure of $H^{n,0}(C)$ for $n = 0, 1$ was determined in [Nak08]. We can also read the following result from the above proof.

**Proposition 2.7.** *For the Cartan-Eilenberg spectral sequence of* (2.1) *we have*

$$\tilde{E}_2^{s',0} \cong \begin{cases} \ker \rho_* & \text{for } s' = 0, \\ \operatorname{coker} \rho_* \quad (= H^{2,0}(C)) & \text{for } s' = 1, \\ \operatorname{Ext}_{G(m+1)}^{s'-1}(\overline{T}_m^{(j)} \otimes B_{m+1}) & \text{for } s' \geq 2. \end{cases}$$

Combining this with (2.2), we have the chart of Cartan-Eilenberg $E_2$-terms as in Table 1.

TABLE 1. The Cartan-Eilenberg $E_2$-term of (2.1). Here all Ext groups are over $G(m+1)$.

| | | | | |
|---|---|---|---|---|
| | $\vdots$ | $\vdots$ | $\vdots$ | |
| $s'' = 2$ | $\operatorname{Ext}^0(\overline{T}_m^{(j)} \otimes U_{m+1}^3)$ | $\operatorname{Ext}^1(\overline{T}_m^{(j)} \otimes U_{m+1}^3)$ | $\operatorname{Ext}^2(\overline{T}_m^{(j)} \otimes U_{m+1}^3)$ | $\cdots$ |
| $s'' = 1$ | $\operatorname{Ext}^0(\overline{T}_m^{(j)} \otimes U_{m+1}^2)$ | $\operatorname{Ext}^1(\overline{T}_m^{(j)} \otimes U_{m+1}^2)$ | $\operatorname{Ext}^2(\overline{T}_m^{(j)} \otimes U_{m+1}^2)$ | $\cdots$ |
| $s'' = 0$ | $\ker \rho_*$ | $\operatorname{coker} \rho_*$ | $\operatorname{Ext}^1(\overline{T}_m^{(j)} \otimes B_{m+1})$ | $\cdots$ |
| | $s' = 0$ | $s' = 1$ | $s' = 2$ | |

Note that the case $s' = s'' = 0$ is not interesting here, as we stated before. For $\operatorname{coker} \rho_*$, we need to recall some results from the other papers. For a $G(m+1)$-comodule $M$, denote the subgroup $\bigcap_{n \geq p^j} \ker \widehat{r}_n$ of $M$ by $L_j(M)$. Then, the map

$$(c \otimes 1)\psi : L_j(M) \longrightarrow \operatorname{Ext}_{G(m+1)}^0(\overline{T}_m^{(j)} \otimes M)$$

is an isomorphism between $A(m+1)$-modules by Lemma 1.12. Thus, to obtain the structure of $\tilde{E}_2^{1,0}$, we may alternatively examine the map

$$\rho_* : L_j(W_{m+1}) \longrightarrow L_j(B_{m+1}).$$

The following can be read from [Nak08, Corollary 4.3].

**Lemma 2.8.** *The* $\operatorname{coker} \rho_*$ *is isomorphic to the quotient*

$$L_j(B_{m+1}) \Big/ \left( A(m+1)\{\widehat{\beta}'_{i/i} \mid 0 < i \leq p^{j-1}\} \right).$$

The structure of $L_j(B_{m+1})$ is determined in [NR09] for all $m$ and $j$. In particular, the following is the results for $j = 2$.

**Lemma 2.9** ([NR09, Theorem 6.1])**.** *Below dimension* $p^3|\widehat{v}_2|$, $L_2(B_{m+1})$ *is the* $A(m+1)$*-module generated by*

$$\left\{\widehat{\beta}'_{i/t} \mid i \geq 1, 0 < t \leq \min(i,p)\right\} \cup \left\{\widehat{\beta}_{ap^2+b/t} \mid p < t \leq p^2, a > 0 \text{ and } 0 \leq b < p\right\}.$$

*In particular, below dimension* $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$, *the comodule* $B_{m+1}$ *is 2-free and* $L_2(B_{m+1})$ *is the* $A(m+1)$*-module generated by*

$$\left\{\widehat{\beta}'_{i/\min(i,p)} \mid i > 0\right\} \cup \left\{\widehat{\beta}_{i/t} \mid p < t \le p^2 \le i < p^2 + p\right\}. \tag{2.10}$$

## 3. Extending the range of $E_{m+1}^2$

In Theorem 4.5 we determined the structure of $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*)$ below dimension $p^2|\widehat{v}_1|$. Here we extend this range to $p|\widehat{v}_2|$. This is the dimension where the subcomodule $E_{m+1}^2$ of $E_{m+1}^1/(v_1^\infty)$ starts to behave badly for $m > 0$.

By Lemma 4.2 the Poincaré series of $E_{m+1}^2$ below dimension $p|\widehat{v}_2|$ is at least

$$g_{m+2}(t)\left(\frac{x_1^p(1-y)}{(1-x_1^p)(1-x_2)} + \frac{x_1^{p^2}(1-y^{p+1})}{(1-x_1^{p^2})(1-x_3)}\right), \tag{3.1}$$

where

$$g_{m+2}(t) = \prod_{1\le i\le m+2} \frac{1}{1-t^{|v_i|}}, \qquad x_i = t^{|\widehat{v}_i|}, \qquad \text{and} \qquad y = t^{|v_1|}.$$

The first term corresponds to the module described in Theorem 4.5, and the second term presumably corresponds to

$$BP_*/(p, v_1)\left\{\widehat{\beta}_{p/j,p+2-j} \mid 0 < j \le p\right\}.$$

We see that

$$\widehat{\beta}_{p/j,p+2-j} = \frac{\widehat{v}_2^p}{p^{p+2-j}v_1^j} = \sum_{0\le k<j} \binom{p}{k} \frac{p^{j-2-k}}{v_1^{j-k}} \widehat{\lambda}_2^{p-k} w^k \quad \in E_{m+1}^1/(v_1^\infty)$$

(where $w$ is as in (2.4)) for $j \ge 2$, but $\widehat{\beta}_{p/1,p+1} \notin E_{m+1}^1/(v_1^\infty)$. We get around this problem by replacing $\widehat{\beta}_{p/1,p+1}$ with

$$\tilde{\widehat{\beta}}_{p/1,p+1} = \frac{\widehat{v}_2^p}{p^{p+1}v_1} - \frac{\widehat{v}_3}{pv_1^2} + \frac{v_2\widehat{v}_2^p}{pv_1^{p+2}} - \frac{v_2^{p^{m+1}}\widehat{v}_1}{p^2v_1^2} \quad \in E_{m+1}^1/(v_1^\infty).$$

Then, our extension of Theorem 4.5 for $m > 0$ is the following.

**Theorem 3.2.** *Let* $E_{m+1}^2$ *be the* $A(m+2)$*-module generated by the set*

$$\left\{\widehat{\beta}_{i/j,k} \mid i+1 \ge j+k\right\} \cup \left\{\widehat{\beta}_{p/j,p+2-j} \mid 2 \le j \le p\right\} \cup \left\{\tilde{\widehat{\beta}}_{p/1,p+1}\right\}.$$

*Below dimension* $p|\widehat{v}_2|$, *it has the Poincaré series specified in* (3.1), *it is a sub* $\Gamma(m+1)$*-comodule of* $E_{m+1}^1/(v_1^\infty)$, *and its Ext group is isomorphic to*

$$A(m+1)/I_2 \otimes E(\widehat{h}_{1,0}) \otimes P(\widehat{b}_{1,0}) \otimes \left\{\widehat{\beta}'_i, \widehat{\beta}_{p/k} \mid i \ge 1, 2 \le k \le p\right\}.$$

*In particular* $\mathrm{Ext}^0$ *maps monomorphically to* $\mathrm{Ext}^2_{\Gamma(m+1)}(BP_*)$ *in that range.*

**Proof.** Define a decreasing filtration on $BP_*/(p^\infty, v_1^\infty)$ by $\widehat{v}_2^a/p^b v_1^c \in F^n$ if and only if $a-b-c \geq n$. Then, each element of the first set belongs to $F^{-1}$ and the submodule generated by the set is a subcomodule. We also see that the reduced expansion of $\widehat{\beta}_{p/j,p+2-j}$ is in $F^{-1}$ though $\widehat{\beta}_{p/j,p+2-j}$ itself is belonging to $F^{-2}$, and the reduced expansion of $\widetilde{\widehat{\beta}}_{p/1,p+1}$ is in $F^{-2}$. Thus the module generated by the assigned set is a comodule as desired.

The Ext group can be computed similarly to the proof of Theorem 4.5. □

**Remark 3.3.** From (1.5), we have the long exact sequence:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^0(E^1_{m+1}) \arrow[r, "(i_2)_*"] & \mathrm{Ext}^0(D^1_{m+1}) \arrow[r, "(j_2)_*"] & \mathrm{Ext}^0(E^2_{m+1}) \arrow[dll, "\delta^1", rounded corners] \\
& \mathrm{Ext}^1(E^1_{m+1}) \arrow[r, "(i_2)_*"] & \cdots,
\end{tikzcd}
$$

where all Ext groups are over $\Gamma(m+1)$. As we have seen in Lemma 4.1, the map $(i_2)_*$ induces an isomorphism in $\mathrm{Ext}^0$ for $m=0$. However, for $m>0$, we have a non-trivial element

$$pv_1\widetilde{\widehat{\beta}}_{p/1,p+1} = -v_2^{p^{m+1}}\widehat{v}_1/pv_1 \in \ker\delta^1.$$

This is actually the first such element and the map $(i_2)_*$ is still isomorphic and $\mathrm{Ext}^0_{\Gamma(m+1)}(E^2_{m+1})$ is isomorphic to $\mathrm{Ext}^2_{\Gamma(m+1)}(BP_*)$ below its dimension, $p|\widehat{v}_2|$.

## 4. Quillen operations of some elements

Recall that the Quillen operation $\widehat{r}_j : M \to \Sigma^{j|\widehat{t}_1|}M$ for $G(m+1)$-comodule $M$ is defined by

$$\psi(x) = \sum_j \widehat{t}_1^j \otimes \widehat{r}_j(x) + \cdots.$$

In the following sections we will need the action of some Quillen operations on $M = U^*_{m+1}$ to compute the Cartan-Eilenberg $E_2$-terms $\widetilde{E}_2^{s',s''}$ $(s'' \geq 1)$ of Table 1.

A translation of Theorem 3.2 to the present context is the following.

**Corollary 4.1.** *Below dimension $p|\widehat{v}_3|$, we have an isomorphism*

$$U^{*+2}_{m+1} \cong E(\widehat{h}_{2,0}) \otimes P(\widehat{b}_{2,0}) \otimes U^2_{m+1}$$

*where $U^2_{m+1}$ is isomorphic to the $A(m+1)/I_2$-module generated by*

$$\left\{ \widehat{u}_{i,j} = \delta^0\delta^1\left(\frac{\widehat{v}_2^j\widehat{v}_3^i}{i!\,pv_1}\right), \widehat{u}_{p/k} = \delta^0\delta^1\left(\frac{\widehat{v}_3^p}{pv_1^k}\right) \mid 0 < i \leq p, j \geq 0, 2 \leq k \leq p \right\} \tag{4.2}$$

*and $\delta^0$ and $\delta^1$ are the connecting homomorphisms for the short exact sequences*

$$0 \to BP_* \to M^0 \to N^1 \to 0 \qquad \text{and} \qquad 0 \to N^1 \to M^1 \to N^2 \to 0$$

*respectively. The bidegrees of elements are* $|\widehat{h}_{2,0}| = (1, |\widehat{t}_2|)$ *and* $|\widehat{b}_{2,0}| = (2, |\widehat{t}_2^p|)$.

In particular, we have

$$U_{m+1}^{2a+\varepsilon} \cong \widehat{b}_{2,0}^{a-1} \otimes \widehat{h}_{2,0}^{\varepsilon} \otimes U_{m+1}^2 \qquad \text{for } a \geq 1 \text{ and } \varepsilon = 0, 1.$$

So, it is sufficient to know the Quillen operations on $U_{m+1}^2$. Instead, we here compute the Quillen operation on $\operatorname{Ext}^0_{\Gamma(m+2)}(E^1_{m+1}/(v_1^\infty))$ after pulling back elements of (4.2) by the composition of connecting homomorphisms:

$$\operatorname{Ext}^0_{\Gamma(m+2)}(E^1_{m+1}/(v_1^\infty)) \xrightarrow{\delta^1} \operatorname{Ext}^1_{\Gamma(m+2)}(E^1_{m+1}) \xrightarrow[\cong]{\delta^0} U^2_{m+1}. \tag{4.3}$$

The corresponding elements will be denoted by $\widehat{\theta}_{i,j}$ and $\widehat{\theta}_{p/k}$.

**Remark 4.4.** The choice of $\widehat{\theta}_{i,j}$ is not unique: the definition of $\widehat{\theta}_{i,j}$ has ambiguity up to elements of $\ker \delta^1$. In particular, the comodule $B_{m+1}$ is involved in $\ker \delta^1$ and we may tack any element of $B_{m+1}$ to $\widehat{\theta}_{i,j}$.

Recall the recursive formula (3.10) for the $\widehat{\ell}_i$, which are independent of $m$:

$$\widehat{\ell}_1 = \widehat{\lambda}_1, \quad \widehat{\ell}_2 = \widehat{\lambda}_2 + \ell_1\widehat{\lambda}_1^p, \quad \widehat{\ell}_3 = \widehat{\lambda}_3 + \ell_1\widehat{\lambda}_2^p + \ell_2\widehat{\lambda}_1^{p^2}. \tag{4.5}$$

On the other hand, the expression of $\widehat{v}_i$ in terms of $\widehat{\lambda}_i$ depends on $m$. For small values of $i$, we have

**Lemma 4.6.** *In $D^0_{m+1}$ for $m > 0$, we have*

$$\widehat{v}_1 = p\widehat{\lambda}_1,$$

$$\widehat{v}_2 = p\widehat{\lambda}_2 + (1 - p^{p-1})v_1\widehat{\lambda}_1^p - v_1^{p^{m+1}}\widehat{\lambda}_1,$$

$$\widehat{v}_3 \equiv p\widehat{\lambda}_3 - p^{p^2-1}v_2\widehat{\lambda}_1^{p^2} + \zeta \mod (v_1), \text{ where } \zeta = v_2\widehat{\lambda}_1^{p^2} - \begin{cases} 0 & (m = 1), \\ v_2^{p^{m+1}}\widehat{\lambda}_1 & (m \geq 2). \end{cases}$$

**Proof.** By (3.9) we have

$$p\widehat{\ell}_1 = \widehat{v}_1,$$

$$p\widehat{\ell}_2 = \widehat{v}_2 + \ell_1\widehat{v}_1^p + \widehat{\ell}_1\widehat{v}_1^{p^{m+1}},$$

$$p\widehat{\ell}_3 = \widehat{v}_3 + \ell_1\widehat{v}_2^p + \ell_2\widehat{v}_1^{p^2} + \begin{cases} v_1^{p^{m+2}}\widehat{\ell}_2 & (m = 1), \\ v_1^{p^{m+2}}\widehat{\ell}_2 + v_2^{p^{m+}}\widehat{\ell}_1 & (m \geq 2). \end{cases}$$

The result follows from (4.5) and the relations between $\ell_i$ and $v_i$. □

Define the element $\xi$ in $D^0_{m+1}$ by

$$\xi = v_2\widehat{v}_2^p - \begin{cases} 0 & (m = 1), \\ v_1^p v_2^{p^{m+1}}\widehat{\lambda}_1 & (m \geq 2). \end{cases}$$

**Lemma 4.7.** *For $m \geq 1$, we have $v_1^p \zeta \equiv \xi$ mod $(p^2, v_1^{p^{m+1}})$ in $E_{m+1}^1$.*

**Proof.** Note that $\widehat{v}_2 \equiv v_1 \widehat{\lambda}_1^p \bmod (p, v_1^{p^{m+1}})$. For $m \geq 2$

$$v_1^p \zeta = v_2 (v_1 \widehat{\lambda}_1^p)^p - v_1^p v_2^{p^{m+1}} \widehat{\lambda}_1 \equiv v_2 \widehat{v}_2^p - v_1^p v_2^{p^{m+1}} \widehat{\lambda}_1 = \xi$$

mod $(p^2, v_1^{p^{m+1}})$. The case $m = 1$ is similarly proved. □

**Proposition 4.8.** *Define $\widehat{\theta}_{p,j}$ for $j \geq 0$ by*

$$\widehat{\theta}_{p,j} = \widehat{v}_2^j \left( \frac{\widehat{v}_3^p}{p! \cdot p v_1} - \frac{\xi^p}{p! \cdot p v_1^{1+p^2}} \right). \tag{4.9}$$

*Then it is in $\operatorname{Ext}^0_{\Gamma(m+2)}(E_{m+1}^1/(v_1^\infty))$ and satisfies $\delta^0 \delta^1 (\widehat{\theta}_{p,j}) = \widehat{u}_{p,j}$.*

**Proof.** By Lemma 4.7 we see that

$$\frac{\widehat{v}_2^j \widehat{v}_3^p}{p! \cdot p v_1} \equiv \frac{\widehat{v}_2^j (p \widehat{\lambda}_3 - p^{p^2-1} v_2 \widehat{\lambda}_1^{p^2} + \zeta)^p}{p! \cdot p v_1} \equiv \frac{\widehat{v}_2^j (v_1^p \zeta)^p}{p! \cdot p v_1^{1+p^2}} \equiv \frac{\widehat{v}_2^j \xi^p}{p! \cdot p v_1^{1+p^2}}$$

mod $E_{m+1}^1/(v_1^\infty)$. Direct calculations show that $\widehat{\theta}_{p,j}$ is invariant over $\Gamma(m+2)$. Since $v_1^{-p^2-1} \xi^p / p^2$ is in $\ker \delta^1$, the second statement follows. □

**Proposition 4.10.** *Define $\widehat{\theta}_{i,j}$ for $0 < i \leq p$ and $j \geq 0$ by (4.9) and the downward induction on $i$:*

$$\widehat{\theta}_{i,j} = v_2^{-1} \widehat{r}_{p^2}(\widehat{\theta}_{i+1,j}) \qquad \text{for } 0 < i < p.$$

*Then they are in $\operatorname{Ext}^0_{\Gamma(m+2)}(E_{m+1}^1/(v_1^\infty))$ and satisfy $\delta^0 \delta^1 (\widehat{\theta}_{i,j}) = \widehat{u}_{i,j}$.*

**Proof.** The first statement is obvious since $\operatorname{Ext}^0_{\Gamma(m+2)}(E_{m+1}^1/(v_1^\infty))$ is a subcomodule of $E_{m+1}^1/(v_1^\infty)$. Since the second term of (4.9) is in $\ker \delta^1$ and each Quillen operation commutes with the connecting homomorphism, the second statement follows. □

The following lemma on Quillen operations is useful.

**Lemma 4.11.** *The $k$-fold iteration of $\widehat{r}_{p^j}$ is congruent to $k! \widehat{r}_{kp^j}$ modulo $p^j$.*

**Proof.** Since $r_s r_t = \binom{s+t}{s} r_{s+t}$, the $k$-fold iteration of $\widehat{r}_{p^j}$ is equal to

$$\frac{(kp^j)!}{(p^j!)^k} \widehat{r}_{kp^j},$$

where the coefficient is congruent to $k!$ modulo $p^j$. □

Then we have

**Proposition 4.12.** *Quillen operations on $\widehat{\theta}_{1,j}$ for $0 \le j \le p^2 - p$ are given by*

$$\widehat{r}_{p^2}(\widehat{\theta}_{1,j}) = 0 \quad and \quad \widehat{r}_p(\widehat{\theta}_{1,j}) = jv_2\widehat{\beta}_{j+p-1/p}$$

*up to unit scalar multiplication.*

**Proof.** By Lemma 4.11 $\widehat{r}_{p^2}(\widehat{\theta}_{1,j})$ is a unit multiple of $v_2^{-p+1}\widehat{r}_{p^3}(\widehat{\theta}_{p,j})$, and we can check $\widehat{r}_{p^3}(\widehat{\theta}_{p,j}) = 0$. Similarly, $\widehat{r}_p(\widehat{\theta}_{1,j})$ is a unit multiple of $v_2^{-p+1}\widehat{r}_{p^3-p^2+p}(\widehat{\theta}_{p,j})$, which can be computed by direct calculation. □

**Proposition 4.13.** *We have*

$$\psi(\widehat{\theta}_{i,j}) \equiv \sum_{0\le k<i} \widehat{t}_1^{kp^2} \otimes \frac{v_2^k\widehat{\theta}_{i-k,j}}{k!} \qquad \bmod (v_2^i).$$

**Proof.** Roughly speaking, this follows from $k!\,\widehat{r}_{kp^2}(\widehat{\theta}_{i,j}) = v_2^k\widehat{\theta}_{i-k,j}$ since $\widehat{r}_{p^2}(\widehat{\theta}_{i+1,j}) = v_2\widehat{\theta}_{i,j}$. More precisely, it is enough to consider $\psi(v_2^{p-i}\widehat{\theta}_{i,j}) \bmod (v_2^p)$ using the equality $v_2^{p-i}\widehat{\theta}_{i,j} = (p-i)!\,\widehat{r}_{(p-i)p^2}(\widehat{\theta}_{p,j})$. □

**Proposition 4.14.** *Define $\widehat{\theta}_{p/k}$ $(0 < k \le p)$ by*

$$\widehat{\theta}_{p/k} = \frac{\widehat{v}_3^p}{pv_1^k} - \frac{v_2^p\widehat{v}_2^{p^2}}{pv_1^{p^2+k}} + \frac{v_2^{p^{m+2}}\widehat{v}_2}{pv_1^{k+1}}.$$

*Then it is in $\mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1}/(v_1^\infty))$ and satisfies $\delta^0\delta^1(\widehat{\theta}_{p/k}) = \widehat{u}_{p/k}$. Moreover, it is $G(m+1)$-invariant: we have $\widehat{r}_j(\widehat{\theta}_{p/k}) = 0$ for all $j \ge 1$.*

**Proof.** By Lemma 4.6, modulo $E^1_{m+1}/(v_1^\infty)$

$$\begin{aligned}\widehat{\theta}_{p/k} &\equiv \frac{v_1^p\widehat{\lambda}_2^{p^2} + v_2^p\widehat{\lambda}_1^{p^3}}{pv_1^k} - \frac{v_2^p\cdot v_1^{p^2}\widehat{\lambda}_1^{p^3}}{pv_1^{p^2+k}} + \frac{v_2^{p^{m+2}}\cdot v_1\widehat{\lambda}_1^p}{pv_1^{k+1}}\\ &\equiv \frac{v_1^p\widehat{\lambda}_2^{p^2}}{pv_1^k} + \frac{(p\widehat{\lambda}_1)^{p^{m+2}}\cdot\widehat{\lambda}_1^p}{pv_1^k} \equiv 0\end{aligned}$$

for $m = 1$, and

$$\widehat{\theta}_{p/k} \equiv \frac{v_1^p\widehat{\lambda}_2^{p^2} + v_2^p\widehat{\lambda}_1^{p^3} - v_2^{p^{m+2}}\widehat{\lambda}_1^p}{pv_1^k} - \frac{v_2^p\cdot v_1^{p^2}\widehat{\lambda}_1^{p^3}}{pv_1^{p^2+k}} + \frac{v_2^{p^{m+2}}\cdot v_1\widehat{\lambda}_1^p}{pv_1^{k+1}} \equiv 0$$

for $m \ge 2$. The second statement follows since all terms in $\widehat{\theta}_{p/k}$ except for the leading term are in $\ker\delta^1$. The last statement follows from direct calculations. □

## 5. The homotopy groups of $T(m)_{(2)}$

In this section we determine the homotopy groups of $T(m)_{(2)}$ below dimensions $p|\widehat{v}_3|$ by analyzing the Cartan-Eilenberg $E_2$-term of Table 1 for $j = 2$. By Lemma 2.8 and 2.9 we have

**Proposition 5.1.** *Below dimension $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$, the Cartan-Eilenberg $E_2$-term of Table 1 for $j = 2$ satisfies $\tilde{E}_2^{s',0} = 0$ for $s' \geq 2$, and $\tilde{E}_2^{1,0}$ is isomorphic to the $A(m+1)$-module generated by*

$$\left\{\widehat{\beta}'_{i/t} \mid i \geq 2, 0 < t \leq \min(i-1, p)\right\} \cup \left\{\widehat{\beta}_{p^2/t} \mid p < t \leq p^2\right\}.$$

*Note that $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$ is larger than $p|\widehat{v}_3|$ if $m > 0$.*

Thus our remaining task is to determine the structure of

$$\tilde{E}_2^{s',s''} \cong \mathrm{Ext}^{s'}_{G(m+1)}(\overline{T}_m^{(2)} \otimes U_{m+1}^{s''+1}) \quad \text{for } s'' \geq 1.$$

Since this is a certain suspension of $\tilde{E}_2^{s',1}$ (i.e., tensored object with some power of $\widehat{b}_{2,0}$ and $\widehat{h}_{2,0}$), it suffices to treat the case $\tilde{E}_2^{s',1}$. Below dimension $p|\widehat{v}_3|$, define the $v_2$-torsion free $A(m+1)$-submodule $U^0$ of $v_2^{-1}U_{m+1}^2$ by adjoining the elements

$$\{v_2^{-i}\widehat{u}_{i,j} \mid 0 < i \leq p, j \geq 0\} \cup \{v_2^{-p}\widehat{u}_{p/k} \mid 2 \leq k \leq p\}$$

to $U_{m+1}^2$. Note that $U^0$ is a comodule since the congruence in Proposition 4.13 is modulo $v_2^i$ and the ignored elements have non-negative $v_2$-exponent after applying $v_2^{-i}$. We also define the quotient comodule $U^1$ by the following short exact sequence:

$$0 \longrightarrow U_{m+1}^2 \longrightarrow U^0 \longrightarrow U^1 \longrightarrow 0 \tag{5.2}$$

The Quillen operations on $v_2^{-p}\widehat{u}_{p/k} \in U^0$ are trivial by Proposition 4.14. The behavior of Quillen operations on $v_2^{-i}\widehat{u}_{i,j} \in U^0$ follows from Proposition 4.10, and it is demonstrated in (5.3) for $p = 5$, where each diagonal arrow represents the action of $\widehat{r}_{p^2}$ up to unit scalar multiplication and the elements in the rightmost column are out of our range except for $j = 0$.

$$
\begin{tikzcd}[column sep=small]
\vdots & \vdots & \vdots & \vdots & \vdots \\
\widehat{u}_{1,j} & \widehat{u}_{2,j} & \widehat{u}_{3,j} & \widehat{u}_{4,j} & \widehat{u}_{5,j} \\
v_2^{-1}\widehat{u}_{1,j} & v_2^{-1}\widehat{u}_{2,j} \arrow[ul] & v_2^{-1}\widehat{u}_{3,j} \arrow[ul] & v_2^{-1}\widehat{u}_{4,j} \arrow[ul] & v_2^{-1}\widehat{u}_{5,j} \arrow[ul] \\
 & v_2^{-2}\widehat{u}_{2,j} \arrow[ul] & v_2^{-2}\widehat{u}_{3,j} \arrow[ul] & v_2^{-2}\widehat{u}_{4,j} \arrow[ul] & v_2^{-2}\widehat{u}_{5,j} \arrow[ul] \\
 & & v_2^{-3}\widehat{u}_{3,j} \arrow[ul] & v_2^{-3}\widehat{u}_{4,j} \arrow[ul] & v_2^{-3}\widehat{u}_{5,j} \arrow[ul] \\
 & & & v_2^{-4}\widehat{u}_{4,j} \arrow[ul] & v_2^{-4}\widehat{u}_{5,j} \arrow[ul] \\
 & & & & v_2^{-5}\widehat{u}_{5,j} \arrow[ul]
\end{tikzcd}
\tag{5.3}
$$

**Proposition 5.4.** *$U^0$ is 2-free, and we have an isomorphism of $A(m+1)$-modules*

$$\operatorname{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^0) \cong A(m+1) \otimes \left\{v_2^{-1}\widehat{u}_{1,j}, v_2^{-p}\widehat{u}_{p/k} \mid j \geq 0, 2 \leq k \leq p\right\}.$$

**Proof.** By Lemma 1.12, $\operatorname{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^0)$ is additively isomorphic to

$$L_2(U^0) = \bigcap_{\ell \geq p^2} \ker \widehat{r}_\ell.$$

In (5.3) the only possible elements with trivial action of $\widehat{r}_{p^2}$ are $v_2^{-1}\widehat{u}_{1,j}$. Note that

$$\widehat{r}_\ell(v_2^{-1}\widehat{u}_{1,j}) = \delta^0\delta^1(v_2^{-1}\widehat{r}_\ell(\widehat{\theta}_{1,j}))$$

and $v_2^{-1}\widehat{r}_\ell(\widehat{\theta}_{1,j}) = 0$ for $\ell \neq 1, p^2$ because

$$\psi\left(\frac{\widehat{v}_2^j\widehat{v}_3}{pv_1}\right) = \frac{\widehat{v}_2^j(\widehat{v}_3 + v_2\widehat{t}_1^{p^2} - v_2^{p^{m+1}}\widehat{t}_1)}{pv_1}.$$

Indeed, we have $\widehat{r}_\ell(v_2^{-1}\widehat{u}_{1,j}) = 0$ even for $\ell = 1$ or $p^2$ because

$$v_2^{-1}\widehat{r}_1(\widehat{\theta}_{1,j}) = v_2^{p^{m+1}-1}\widehat{\beta}_j \quad \text{and} \quad v_2^{-1}\widehat{r}_{p^2}(\widehat{\theta}_{1,j}) = \widehat{\beta}_j$$

are in $\ker \delta^1$. Thus all Quillen operations on $v_2^{-1}\widehat{u}_{1,j}$ are trivial. Note that it is also shown that there is a bijection between $\operatorname{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^0)$ and $\operatorname{Ext}^0_{G(m+1)}(U^0)$.

The diagram (5.3) also suggests the equality of Poincaré series

$$g(U^0) = \frac{g(\operatorname{Ext}^0(U^0))}{1 - x^{p^2}} \qquad \text{where} \quad x = t^{|\widehat{v}_1|}$$

and we have

$$\begin{aligned}
g(\overline{T}_m^{(2)} \otimes U^0) = g(U^0) \cdot \frac{1-x^{p^2}}{1-x} &= \frac{g(\mathrm{Ext}^0(U^0))}{1-x} \\
&= g(\mathrm{Ext}^0(U^0)) \cdot g(G(m+1)/I) \\
&= g(\mathrm{Ext}^0(\overline{T}_m^{(2)} \otimes U^0)) \cdot g(G(m+1)/I)
\end{aligned}$$

which means that $U^0$ is 2-free. □

**Proposition 5.5.** *$U^1$ is 2-free, and we have an isomorphism of $A(m+1)$-modules*

$$\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^1) \cong A(m+1)/I_3 \otimes \{\widehat{u}_{i,j}/v_2 \mid i \geq 1, j \geq 0\}.$$

**Proof.** The analogous diagram to (5.3) for $p = 5$ is as follows:

$$\begin{array}{ccccccccc}
\widehat{u}_{1,j}/v_2 & & \widehat{u}_{2,j}/v_2 & & \widehat{u}_{3,j}/v_2 & & \widehat{u}_{4,j}/v_2 & & \widehat{u}_{5,j}/v_2 \\
 & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow & \\
 & & \widehat{u}_{2,j}/v_2^2 & & \widehat{u}_{3,j}/v_2^2 & & \widehat{u}_{4,j}/v_2^2 & & \widehat{u}_{5,j}/v_2^2 \\
 & & & \nwarrow & & \nwarrow & & \nwarrow & \\
 & & & & \widehat{u}_{3,j}/v_2^3 & & \widehat{u}_{4,j}/v_2^3 & & \widehat{u}_{5,j}/v_2^3 \\
 & & & & & \nwarrow & & \nwarrow & \\
 & & & & & & \widehat{u}_{4,j}/v_2^4 & & \widehat{u}_{5,j}/v_2^4 \\
 & & & & & & & \nwarrow & \\
 & & & & & & & & \widehat{u}_{5,j}/v_2^5
\end{array}$$

In this case $\mathrm{Ext}^0$ is generated by the elements in the top row. The 2-freeness of $U^1$ is similarly shown to $U^0$. □

**Proposition 5.6.** *Below dimension $p|\widehat{v}_3|$, the Cartan-Eilenberg $E_2$-term of Table 1 for $j = 2$ satisfies*

$$\widetilde{E}_2^{s',*+1} \cong E(\widehat{h}_{2,0}) \otimes P(\widehat{b}_{2,0}) \otimes \mathrm{Ext}^{s'}_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^2_{m+1})$$

*and*

$$\begin{aligned}
\widetilde{E}_2^{s',1} &= \mathrm{Ext}^{s'}_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^2_{m+1}) \\
&\cong \begin{cases}
A(m+1)/I_2 \otimes \{\widehat{u}_{1,i}, \widehat{u}_{p/k} \mid i \geq 0, 2 \leq k \leq p\} & \text{for } s' = 0, \\
A(m+2)/I_3 \otimes \{\widehat{\gamma}_\ell \mid \ell \geq 2\} & \text{for } s' = 1, \\
0 & \text{for } s' \geq 2
\end{cases}
\end{aligned}$$

*where $\widehat{\gamma}_\ell = \delta^2\left(\widehat{u}_{\ell,0}/v_2\right)$ and $\delta^2$ is the connecting homomorphism associated to (5.2). The operators behave as if they had bidegree $\widehat{h}_{2,0} \in \widetilde{E}_2^{0,1}$ and $\widehat{b}_{2,0} \in \widetilde{E}_2^{0,2}$.*

**Proof.** By Proposition 5.4 and 5.5, we have the 4-term exact sequence[5]

$$0 \longrightarrow \tilde{E}_2^{0,1} \longrightarrow \mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^0) \longrightarrow \mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^1) \longrightarrow \tilde{E}_2^{1,1} \longrightarrow 0$$

and $\tilde{E}_2^{s',1} = 0$ for $s' \geq 2$. Since the image of the middle map is

$$A(m+1)/I_2 \otimes \{\hat{u}_{1,i}/v_2 \mid j \geq 0\} \cong A(m+2)/I_3 \otimes \{\hat{u}_{1,0}/v_2\}$$

we obtain the result. □

By Proposition 5.1 and 5.6, Table 1 is reduced to the following one:

TABLE 2. The Cartan-Eilenberg $E_2$-term of (2.1) for $j = 2$.

| | | | | |
|---|---|---|---|---|
| | $\vdots$ | $\vdots$ | $\vdots$ | |
| $s'' = 2$ | $\mathrm{Ext}^0(\overline{T}_m^{(2)} \otimes U^3_{m+1})$ | $\mathrm{Ext}^1(\overline{T}_m^{(2)} \otimes U^3_{m+1})$ | 0 | $\cdots$ |
| $s'' = 1$ | $\mathrm{Ext}^0(\overline{T}_m^{(2)} \otimes U^2_{m+1})$ | $\mathrm{Ext}^1(\overline{T}_m^{(2)} \otimes U^2_{m+1})$ | 0 | $\cdots$ |
| $s'' = 0$ | $\ker \rho_*$ | described in Proposition 5.1 | 0 | $\cdots$ |
| | $s' = 0$ | $s' = 1$ | $s' = 2$ | |

**Proposition 5.7.** *Below dimension $p|\hat{v}_3|$, the Cartan-Eilenberg spectral sequence of Table 1 for $j = 2$ collapses, and we have the short exact sequence*

$$0 \longrightarrow \tilde{E}_\infty^{1,s''} \longrightarrow \mathrm{Ext}^{s''+2}_{\Gamma(m+1)}(T_m^{(2)}) \longrightarrow \tilde{E}_\infty^{0,s''+1} \longrightarrow 0$$

*which splits for $s'' \geq 1$, but not for $s'' = 0$.*

**Proof.** The spectral sequence collapses since we have only two columns in Table 2. The middle groups is isomorphic to $\mathrm{Ext}^{s''+1}_{\Gamma(m+1)}(T_m^{(2)} \otimes E^1_{m+1})$, and the short exact sequences follow by inspection of Table 2. For $s'' \geq 1$, it splits because $\tilde{E}_2^{1,s''}$ is $v_2$-torsion while $\tilde{E}_2^{0,s''+1}$ is $v_2$-torsion free by Proposition 5.6. For $s'' = 0$, for example, an element

$$\hat{u}_{1,0} \in \mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)} \otimes U^2_{m+1}) \cong \tilde{E}_2^{0,1}$$

is killed by $v_1$, however, its lift

$$\delta^0\delta^1(\hat{\theta}_{1,0}) = \delta^0\delta^1\left(\frac{\hat{v}_3}{pv_1} - \frac{v_2\hat{v}_2^p}{pv_1^{1+p}}\right) \in \mathrm{Ext}^2_{\Gamma(m+1)}(T_m^{(2)})$$

is not killed by $v_1$. Thus, it does not split. □

[5]The case $m = 0$ was described in [Rav04, Lemma 7.3.5].

**Theorem 5.8.** *Below dimension $p|\widehat{v}_3|$, the Adams-Novikov spectral sequence for $T(m)_{(2)}$ collapses.*

**Proof.** We have computed the Adams-Novikov $E_2^{n,*} = \mathrm{Ext}^n_{\Gamma(m+1)}(T_m^{(2)})$ for $n \geq 2$ and the shortest possible differential is $d_{2p-1} : E_2^{2,*} \to E_2^{2p+1,*}$. The first element in the target is $\widehat{h}_{2,0}\widehat{b}_{2,0}^{p-1}\widehat{u}_{1,0} \in E_2^{2p+1,*}$, and its total degree

$$2(p^{m+4} + p^{m+2} - p^2 - p) - 3$$

is larger than $p|\widehat{v}_3|$. □

## 6. The homotopy groups of $T(m)_{(1)}$

In this section we determine the homotopy groups of $T(m)_{(1)}$ below dimensions $p|\widehat{v}_3|$. To determine the Cartan-Eilenberg $E_2$-term of Table 1 for $j = 1$, we use the algebraic small descent spectral sequence of Theorem 1.17: For a $G(m+1)$-comodule $M$ and non-negative integer $i$, there is a spectral sequence converging to $\mathrm{Ext}_{G(m+1)}(\overline{T}_m^{(i)} \otimes_{A(m+1)} M)$ with

$$E_1^{*,t} \cong E(\widehat{h}_{1,j}) \otimes P(\widehat{b}_{1,j}) \otimes \mathrm{Ext}^t_{G(m+1)}(\overline{T}_m^{(i+1)} \otimes_{A(m+1)} M)$$

with $\widehat{h}_{1,j} \in E_1^{1,0}$, $\widehat{b}_{1,j} \in E_1^{2,0}$, and $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$. In particular, $d_1$ is induced by the action on $M$ of $r_{p^j}$ for $s$ even and $r_{(p-1)p^j}$ for $s$ odd. Note that $r_{(p-1)p^j}$ is congruent to the $(p-1)$-fold iteration of $r_{p^j}$ up to unit scalar multiplication.

The case $M = U_{m+1}^2$ is easy.

**Proposition 6.1.** *Below dimension $p|\widehat{v}_3|$, the algebraic small descent spectral sequence for $U_{m+1}^2$ collapses from the $E_2$-term, and*

$$\mathrm{Ext}^{*+k}_{G(m+1)}(\overline{T}_m^{(1)} \otimes U_{m+1}^2) \cong E(\widehat{h}_{1,1}) \otimes P(\widehat{b}_{1,1}) \otimes \mathrm{Ext}^k_{G(m+1)}(\overline{T}_m^{(2)} \otimes U_{m+1}^2).$$

**Proof.** Since the action of $\widehat{r}_p$ on $U_{m+1}^2$ is trivial by Corollary 4.1, the $E_1$-term coincides with the $E_2$-term. The differentials $d_2 : E_2^{s,1} \to E_2^{s+2,0}$ are also trivial since the source is $v_2$-torsion while the target is $v_2$-torsion free. By Proposition 5.6 the small descent spectral sequence has only two rows, and so $d_r = 0$ for $r \geq 3$. □

Hereafter we will denote $\widehat{u}_{1,i}$ by $\widehat{u}_i$ for short. Since

$$\widetilde{E}_2^{s',s''} \cong \mathrm{Ext}^{s'}_{G(m+1)}(\overline{T}_m^{(1)} \otimes U_{m+1}^{s''+1}) \quad \text{for } s'' \geq 1,$$

the following is a translation of Proposition 6.1.

**Corollary 6.2.** *Below dimension $p|\widehat{v}_3|$, the Cartan-Eilenberg $E_2$-term of Table 1*

$$\widetilde{E}_2^{*+s',*+1} \cong \mathrm{Ext}^{*+s'}_{G(m+1)}(\overline{T}_m^{(1)} \otimes U_{m+1}^{*+2})$$

*is isomorphic to*

$$E(\widehat{h}_{2,0},\widehat{h}_{1,1})\otimes P(\widehat{b}_{2,0},\widehat{b}_{1,1})\otimes\left\{\begin{array}{c} A(m+1)/I_2\otimes\left\{\widehat{u}_i,\widehat{u}_{p/k}\mid i\geq 0, 2\leq k\leq p\right\}\\ \oplus \\ A(m+2)/I_3\otimes\left\{\widehat{\gamma}_\ell\mid \ell\geq 2\right\}\end{array}\right.$$

*where the bidegree of elements are $\widehat{u}\in\tilde{E}_2^{0,1}$ and $\widehat{\gamma}\in\tilde{E}_2^{1,1}$ and the operators behave as if they had the bidegree $\widehat{h}_{2,0}\in\tilde{E}_2^{0,1}$, $\widehat{b}_{2,0}\in\tilde{E}_2^{0,2}$, $\widehat{h}_{1,1}\in\tilde{E}_2^{1,0}$ and $\widehat{b}_{1,1}\in\tilde{E}_2^{2,0}$.*

The algebraic small descent spectral sequence for $M=B_{m+1}$ was treated in [NR09], which we summarize here. Below dimension $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$ it collapses from $E_2$-term since $B_{m+1}$ is 2-free by Lemma 2.9, so we need to compute only $d_1$. On the elements of $\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(2)}\otimes B_{m+1})$ (2.10), we have

$$\widehat{r}_p(\widehat{\beta}'_{i/e_1})=\widehat{\beta}_{i-1/e_1-1},\quad \widehat{r}_p(\widehat{\beta}_{pi/e_1})=0\quad\text{and}\quad \widehat{r}_{p^2-p}(\widehat{\beta}'_{i/p})=\widehat{\beta}_{i-p+1/1}$$

up to unit scalar multiplication (cf. [NR09, Proposition B.2]). It may be helpful to demonstrate the behavior of $d_1$ for $p=3$. The following diagrams describes $d_1$ related to the first set of (2.10):

$$\begin{tikzcd}
\widehat{\beta}'_{3/1} & \widehat{\beta}'_{3/2} \arrow[dl, "\widehat{r}_3"'] & \widehat{\beta}'_{3/3} \arrow[dl, "\widehat{r}_3"'] \arrow[ddll, bend left, "\widehat{r}_6"] & & & & \widehat{\beta}_{5/3} \arrow[dl, "\widehat{r}_3"] \arrow[ddll, bend right, "\widehat{r}_6"'] \\
\widehat{\beta}'_{2/1} & \widehat{\beta}'_{2/2} \arrow[dl, "\widehat{r}_3"'] & & & & \widehat{\beta}_{4/2} \arrow[dl, "\widehat{r}_3"] & \widehat{\beta}_{4/3} \arrow[dl, "\widehat{r}_3"] \\
\widehat{\beta}'_{1/1} & & & & \widehat{\beta}_{3/1} & \widehat{\beta}_{3/2} & \widehat{\beta}_{3/3}
\end{tikzcd} \tag{6.3}$$

Corresponding to the diagonal containing $\widehat{\beta}'_{1/1}$, the subgroup of $E_1$ generated by

$$E(\widehat{h}_{1,1})\otimes P(\widehat{b}_{1,1})\otimes\{\widehat{\beta}'_{1/1},\dots,\widehat{\beta}'_{p/p}\}$$

reduces to simply $\{\widehat{\beta}'_{1/1}\}$ on passage to $E_2$. The similar argument is true for the diagonal containing $\widehat{\beta}_{p/1}$. On the other hand, corresponding to the diagonal containing $\widehat{\beta}'_{i/1}$ $(2\leq i\leq p)$ is the subgroup generated by

$$E(\widehat{h}_{1,1})\otimes P(\widehat{b}_{1,1})\otimes\{\widehat{\beta}'_{i/1},\dots,\widehat{\beta}'_{p/p-i+1}\}$$

which is reduced to $P(\widehat{b}_{1,1})\otimes\{\widehat{\beta}'_{i/1},\widehat{h}_{1,1}\widehat{\beta}'_{p/p-i+1}\}$. The similar argument is true for the diagonal containing $\widehat{\beta}_{p/i}$ $(2\leq i\leq p)$; the subgroup generated by

$$E(\widehat{h}_{1,1})\otimes P(\widehat{b}_{1,1})\otimes\{\widehat{\beta}_{p/i},\dots,\widehat{\beta}_{2p-i/p}\}$$

reduces to $P(\widehat{b}_{1,1})\otimes\{\widehat{\beta}_{p/i}, \widehat{h}_{1,1}\widehat{\beta}_{2p-i/p}\}$. In particular, the subgroups corresponding to $\widehat{\beta}'_{p/1}$ and $\widehat{\beta}_{p/p}$ survive to $E_2$ entirely.

**Remark 6.4.** In the diagram (6.3) we can read off the existence of certain Massey products. For example, if we have a relation $\widehat{r}_p(b) = a$, then we have the Massey product $\langle \widehat{h}_{1,1}, \widehat{h}_{1,1}, a\rangle$, as we will explain in Appendix A. In general, if we have a sequence

$$a_i \xrightarrow{\widehat{r}_p} a_{i-1} \xrightarrow{\widehat{r}_p} \cdots \xrightarrow{\widehat{r}_p} a_1 \qquad (0 < i < p) \tag{6.5}$$

then we would have the Massey product $\langle \widehat{h}_{1,1}, \dots, \widehat{h}_{1,1}, a_1\rangle$ with $i$-factors of $\widehat{h}_{1,1}$ whose representative has the leading term $\widehat{t}_1^p \otimes a_i$. In this paper we denote this Massey product by $\mu_i(a_1)$, although it is denoted by $\underline{pi}a_1$ in [Rav04, Definition 7.4.12].

Note that the entire configuration is $\widehat{v}_2^p$-periodic. The diagram containing $\widehat{\beta}_{p^2/1}$ corresponding to the right one of (6.3) is combined with the diagram for the second set of (2.10):

$$\begin{tikzcd}
 & & \widehat{\beta}_{11/3} \arrow[dl, "\widehat{r}_3"] \arrow[ddll, bend right, "\widehat{r}_6"'] & \cdots & \cdots & \widehat{\beta}_{11/9} \arrow[dl, "\widehat{r}_3"] \arrow[ddll, bend right, "\widehat{r}_6"'] \\
 & \widehat{\beta}_{10/2} \arrow[dl, "\widehat{r}_3"] & \cdots & \cdots & \widehat{\beta}_{10/8} \arrow[dl, "\widehat{r}_3"] & \widehat{\beta}_{10/9} \arrow[dl, "\widehat{r}_3"] \\
\widehat{\beta}_{9/1} & \cdots & \cdots & \widehat{\beta}_{9/7} & \widehat{\beta}_{9/8} & \widehat{\beta}_{9/9}
\end{tikzcd} \tag{6.6}$$

Then, the summand corresponding to $\widehat{\beta}_{p^2/k}$ $(1 \le k \le p^2 - p + 1)$ reduces to $\{\widehat{\beta}_{p^2/k}\}$, and the summand corresponding to $\widehat{\beta}_{p^2/p^2-\ell}$ $(0 \le \ell \le p-2)$ reduces to $P(\widehat{b}_{1,1}) \otimes \{\widehat{\beta}_{p^2/p^2-\ell}, \widehat{h}_{1,1}\widehat{\beta}_{p^2+\ell/p^2}\}$.

By these observations we have the following result:

**Proposition 6.7** ([NR09, Proposition 7.3]). *Below dimensions $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$, the Cartan-Eilenberg $E_2$-term of Table 1*

$$\tilde{E}_2^{*+1,0} = \mathrm{Ext}^*_{G(m+1)}(\overline{T}_m^{(1)} \otimes B_{m+1})$$

*has the following $A(m+1)/I_2$-basis:*

$$\begin{gathered}
P(\widehat{v}_2^p) \otimes \{\widehat{\beta}'_1, \widehat{\beta}_{p/1}\} \oplus \{\widehat{\beta}_{p^2/k} \mid 1 \le k \le p^2 - p + 1\} \\
\oplus \\
P(\widehat{b}_{1,1}) \otimes \left( \begin{gathered} P(\widehat{v}_2^p) \otimes \{\widehat{\beta}'_{i/1}, \widehat{h}_{1,1}\widehat{\beta}'_{p/p-i+1}, \widehat{\beta}_{p/i}, \widehat{h}_{1,1}\widehat{\beta}_{2p-i/p} \mid 2 \le i \le p\} \\ \oplus \\ \{\widehat{\beta}_{p^2/p^2-\ell}, \widehat{h}_{1,1}\widehat{\beta}_{p^2+\ell/p^2} \mid 0 \le \ell \le p-2\} \end{gathered} \right)
\end{gathered}$$

*subject to the caveat that $\widehat{v}_2\widehat{\beta}_{k/e} = \widehat{\beta}_{k+1/e}$. The bigrading of elements are (omitting unnecessary subscripts) $\widehat{\beta} \in \tilde{E}_2^{1,0}$ and the operators $\widehat{h}_{1,1}$ and $\widehat{b}_{1,1}$ behave as if they had the bidegrees given in Corollary 6.2.*

Note that the range of dimensions (i.e., $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$) exceeds $p|\widehat{v}_3|$ for $m > 0$.

Now we have determined the Cartan-Eilenberg $E_2$-term for $j = 1$. In the followings we will see that the spectral sequence has a rich pattern of differentials, which is essentially independent of $m$.

For the differential

$$\tilde{d}_2 : \tilde{E}_2^{s',1} = \mathrm{Ext}^{s'}_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^2_{m+1}) \longrightarrow \tilde{E}_2^{s'+2,0} = \mathrm{Ext}^{s'+1}_{G(m+1)}(\overline{T}_m^{(1)} \otimes B_{m+1})$$

we may ignore the $v_2$-torsion part of the source (i.e.,$\gamma$-elements) since the target is $v_2$-torsion free. For the other part, we have the following result[6].

**Lemma 6.8.** *The Cartan-Eilenberg spectral sequence of Table 1 for $j = 1$ has the following differentials:*

(i) $\tilde{d}_2(\widehat{u}_i) = iv_2\widehat{h}_{1,1}\widehat{\beta}_{i+p-1/p}$ *for* $i \not\equiv 0$ *mod* $p$.

(ii) $\tilde{d}_2(\widehat{h}_{1,1}\widehat{u}_i) = \binom{i}{p-1}v_2\widehat{b}_{1,1}\widehat{\beta}_{i+1/2}$ *for* $i \equiv -1$ *mod* $p$.

*All differentials commute with multiplication by $\widehat{b}_{1,1}$.*

**Proof.** We are considering the Cartan-Eilenberg spectral sequence for $T_m^{(1)} \otimes E^1_{m+1}$, and its $\mathrm{Ext}^{s'}$ for $s' > 0$ is a quotient of (isomorphic to for $s' > 1$) $\mathrm{Ext}^{s'-1}$ for $T_m^{(1)} \otimes E^1_{m+1}/(v_1^\infty)$, so we can work in the cobar complex over $G(m+1)$ for the latter comodule.

The differential (i) follows from $\widehat{r}_p(\widehat{u}_i) = iv_2\widehat{\beta}_{i+p-1/p}$ given by Proposition 4.12. We also have $\widehat{r}_{p^2-p}(\widehat{u}_i) = \binom{i}{p-1}v_2\widehat{\beta}_{i+1/2}$ and the differential (ii) by Lemma 4.11. ∎

Now the diagram (6.3)) for $p = 3$ is reviewed as follows. In each case the graph now has $2p + 1$ instead of $2p$ components, three of which are maximal:

$$\begin{tikzcd}
\widehat{\beta}'_{3/1} & \widehat{\beta}'_{3/2} \arrow[dl, "\widehat{r}_3"'] & \widehat{\beta}'_{3/3} \arrow[dl, "\widehat{r}_3"'] \arrow[ddll, bend left, "\widehat{r}_6"] \\
\widehat{\beta}'_{2/1} & \widehat{\beta}'_{2/2} \arrow[dl, "\widehat{r}_3"'] & \\
\widehat{\beta}'_{1/1} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \widehat{\beta}_{5/3} \arrow[ddll, bend right, "\widehat{r}_6"'] \arrow[dl, "\widehat{r}_3"', "\tilde{d}_2"] & v_2^{-1}\widehat{u}_2 \arrow[dl, "\tilde{d}_2"] \arrow[ddll, bend right] \\
 & \widehat{\beta}_{4/2} \arrow[dl, "\widehat{r}_3"'] & \widehat{\beta}_{4/3} \arrow[dl, "\widehat{r}_3"] & v_2^{-1}\widehat{u}_1 \arrow[dl, "\tilde{d}_2"] \\
\widehat{\beta}_{3/1} & \widehat{\beta}_{3/2} & \widehat{\beta}_{3/3} & v_2^{-1}\widehat{u}_0
\end{tikzcd}
\tag{6.9}$$

In fact, each $d_1$ in the small descent spectral sequence behaves as it were the Cartan-Eilenberg $\tilde{d}_2$. Note that the bigrading of elements in the small descent

[6] The result for $m = 0$ was described in [Rav04, Lemma 7.3.12].

spectral sequence are $\widehat{\beta} \in E_r^{0,2}$, $\widehat{u} \in E_r^{0,2}$ and $\widehat{\gamma} \in E_r^{0,3}$, and each operator has the same bigrading as that for Cartan-Eilenberg spectral sequence. In general, the small descent $d_r$ correspond to the Cartan-Eilenberg $\tilde{d}_{r+1}$ for $r \geq 1$. See Table 3.

TABLE 3. Bigradings of elements. Some subscripts have been omitted.

Cartan-Eilenberg spectral sequence for $j = 1$

| | | | | |
|---|---|---|---|---|
| $s'' = 3$ | $\widehat{b}_{2,0}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{u}$ | $\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}$ |
| $s'' = 2$ | $\widehat{h}_{2,0}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}$ | $\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}$ |
| $s'' = 1$ | $\widehat{u}$ | $\widehat{h}_{1,1}\widehat{u}$ | $\widehat{b}_{1,1}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}$ |
| $s'' = 0$ | $*$ | $\widehat{\beta}$ | $\widehat{h}_{1,1}\widehat{\beta}$ | $\widehat{b}_{1,1}\widehat{\beta}$ |
| | $s' = 0$ | $s' = 1$ | $s' = 2$ | $s' = 3$ |

small descent spectral sequence for $j = 1$

| | | | | |
|---|---|---|---|---|
| $s'' = 4$ | $\widehat{b}_{2,0}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{u}$ | $\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}$ |
| $s'' = 3$ | $\widehat{h}_{2,0}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}$ | $\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}$ |
| $s'' = 2$ | $\widehat{u}$ | $\widehat{h}_{1,1}\widehat{u}$ | $\widehat{b}_{1,1}\widehat{u}$ | $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}$ |
| | $\widehat{\beta}$ | $\widehat{h}_{1,1}\widehat{\beta}$ | $\widehat{b}_{1,1}\widehat{\beta}$ | $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}$ |
| $s'' = 1$ | $*$ | | | |
| | $s' = 0$ | $s' = 1$ | $s' = 2$ | $s' = 3$ |

**Remark 6.10.** In (6.9) the "virtual" element $v_2^{-1}\widehat{u}_i$ lives in $\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^0)$ but not in $\mathrm{Ext}^0_{G(m+1)}(\overline{T}_m^{(1)} \otimes U^2_{m+1})$. This means that $\widehat{h}_{1,1}\widehat{b}_{1,1}^k\widehat{\beta}_{i+p-1/p}$ is not actually trivial but $v_2$-torsion, and that it is chromatically renamed $\widehat{v}_2^i\widehat{b}_{1,1}^k\widehat{\gamma}_1$. This is a feature of the cases $m \geq 0$ and it does not happen for $m = 0$. For example, in the chromatic spectral sequence we have

$$d_e(v_2^{-1}\widehat{u}_1) = d_e\left(\frac{v_2^{-1}\widehat{v}_2\widehat{v}_3}{pv_1} - \frac{\widehat{v}_2^{p+1}}{pv_1^{p+1}}\right) = \frac{\widehat{v}_2\widehat{v}_3}{pv_1v_2} = \widehat{v}_2\widehat{\gamma}_1$$

$$\text{and}\qquad d_i(v_2^{-1}\widehat{u}_1) = -\frac{\widehat{v}_2^p\widehat{t}_1^p}{pv_1^p} - \frac{v_2^{p^{m+1}-1}\widehat{v}_2\widehat{t}_1}{pv_1} \equiv -\widehat{h}_{1,1}\widehat{\beta}_{p/p}.$$

The second term in $d_i$ is the product of $\widehat{t}_1$ with an invariant element $x$. It is ignored because we are working in $T(m)_{(1)}$; it is the coboundary of $\widehat{t}_1 \otimes x$.

It is also observed that $\widehat{b}_{1,1}^{k+1}\widehat{\beta}_{ip/2}$ is renamed $\widehat{v}_2^{ip-1}\widehat{h}_{1,1}\widehat{b}_{1,1}^{k}\widehat{\gamma}_1$. For example, we have

$$d_e(v_2^{-1}\widehat{h}_{1,1}\widehat{u}_{p-1}) = d_e\left(\widehat{t}_1^p\left(\frac{v_2^{-1}\widehat{v}_2^{p-1}\widehat{v}_3}{pv_1} - \frac{\widehat{v}_2^{2p-1}}{pv_1^{p+1}}\right)\right) = \frac{\widehat{v}_2^{p-1}\widehat{v}_3\widehat{t}_1^p}{pv_1v_2} = \widehat{v}_2^{p-1}\widehat{h}_{1,1}\widehat{\gamma}_1$$

and $\quad d_i(v_2^{-1}\widehat{h}_{1,1}\widehat{u}_{p-1}) = -\widehat{t}_1^p \otimes \dfrac{\widehat{v}_2^p\widehat{t}_1^{p^2-p}}{pv_1^2} + \cdots = \widehat{b}_{1,1}\widehat{\beta}_{p/2}.$

The following result concerns higher Cartan-Eilenberg differentials, and we will prove it in the next section.

**Theorem 6.11.** *The Cartan-Eilenberg spectral sequence of Table 1 for $j = 1$ has the following differentials and no others in our range of dimensions:*

(i) $\tilde{d}_3(\widehat{h}_{2,0}\widehat{u}_i) = v_2\widehat{b}_{1,1}\widehat{\beta}'_{i+1}$ *for* $i \not\equiv 0$ *mod* $p$.
(ii) $\tilde{d}_3(\widehat{h}_{2,0}^{\varepsilon}\widehat{b}_{2,0}^{k}\widehat{u}_i) = v_2\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{h}_{2,0}^{\varepsilon}\widehat{b}_{2,0}^{k-1}\widehat{u}_{i-1}$ *for* $i \not\equiv 0$ *mod* $p$, $k \geq 1$ *and* $\varepsilon = 0$ *or* $1$.
(iii) $\tilde{d}_{2k+3}(\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{b}_{2,0}^{k}\widehat{u}_i) = v_2^{k+1}\widehat{h}_{1,1}\widehat{b}_{1,1}^{k+1}\widehat{\beta}'_{i+1/k+1}$ *for* $i \equiv -1$ *mod* $p$ *and* $0 \leq k < p-1$.
(iv) $\tilde{d}_{2k+2}(\widehat{h}_{1,1}\widehat{b}_{2,0}^{k}\widehat{u}_i) = v_2^{k+1}\widehat{b}_{1,1}^{k+1}\widehat{\beta}_{i+1/k+2}$ *for* $i \equiv -1$ *mod* $p$ *and* $1 \leq k < p-1$ *(the case* $k = 0$ *is Lemma 6.8(ii)).*
(v) $\tilde{d}_{2p-1}(\widehat{h}_{1,1}\widehat{b}_{2,0}^{p-1}\widehat{u}_i) = v_2^{p-1}\widehat{b}_{1,1}^{p}\widehat{u}_{i-p+1}$ *for* $i \equiv -1$ *mod* $p$.

*All differentials commute with multiplication by* $\widehat{b}_{1,1}$.

Since each source of the stated differentials lies in $\tilde{E}_r^{0,*}$ or $\tilde{E}_r^{1,*}$, it cannot be the target of another differential. Moreover, each differential has maximal length for the bidegree of its source. Thus, the source should be a permanent cycle if a differential is trivial.

**Remark 6.12.** We can define a decreasing filtration on $B_{m+1}$ and $U_{m+1}$ by

$$||\widehat{\beta}'_{i/j}|| = i - j - 1, \quad ||\widehat{u}_i|| = i + [i/p], \quad \text{and} \quad ||p|| = ||v_1|| = ||v_2|| = 1.$$

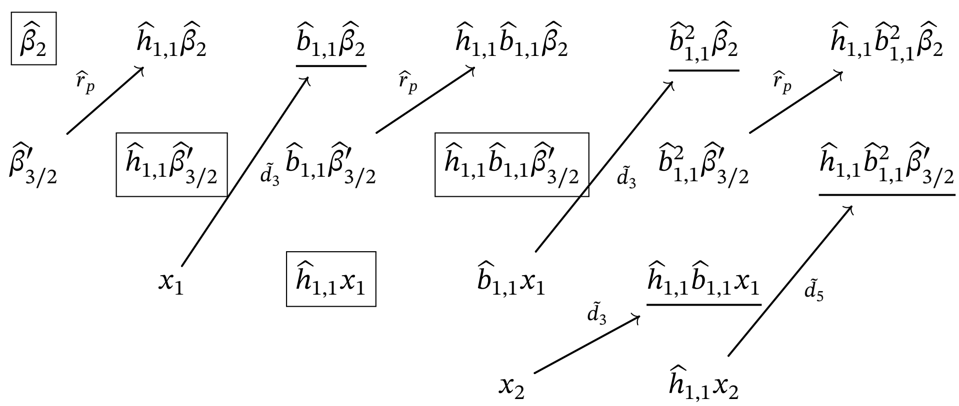
Then the source and target of each differential listed in Theorem 6.11 have the same filtration. A similar filtration for $m = 0$ is discussed in [Rav04, Lemma 7.4.6]. In (6.9) all elements along the same diagonal (e.g., $\widehat{\beta}_2$, $\widehat{\beta}'_{3/2}$, $\widehat{\beta}_{3/3}$ and $v_2^{-1}\widehat{u}_1$ in filtration 0) have the same filtration.

**Remark 6.13.** Again, we obtained the differentials of the form $d_r(x) = v_2^t y$, each of which doesn't kill $y$ but makes $y$ into a $v_2^t$-torsion element, as we have already seen in Remark 6.10. For example, the differential in (i) means that $\widehat{b}_{1,1}\widehat{\beta}'_{i+1}$ is killed by $v_2$; in the chromatic cobar complex we have

$$d(v_2^{-1}\widehat{h}_{2,0}\widehat{u}_i) = -\widehat{b}_{1,1}\widehat{\beta}'_{i+1} \pm \widehat{v}_2^{i}\widehat{h}_{2,0}\widehat{\gamma}_1,$$

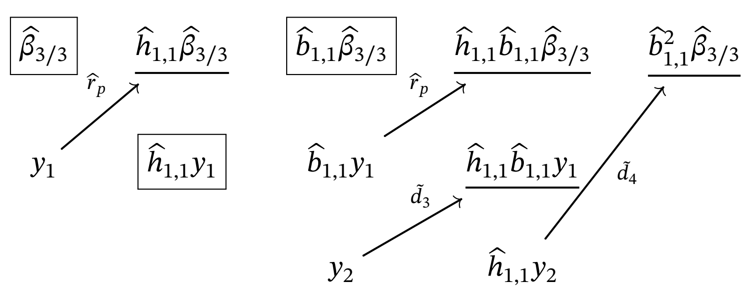
so $\pm\widehat{v}_2^i\widehat{h}_{2,0}\widehat{\gamma}_1$ is the new name for $\widehat{b}_{1,1}\widehat{\beta}'_{i+1}$. Similarly, $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{h}_{2,0}^{\varepsilon}\widehat{b}_{2,0}^{k-1}\widehat{u}_{i-1}$ is renamed $\widehat{v}_2^i\widehat{h}_{2,0}^{\varepsilon}\widehat{b}_{2,0}^k\widehat{\gamma}_1$ by (ii), and $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}'_p$ is renamed $\widehat{v}_2^{p-1}\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{\gamma}_1$ by (iii).

There are some patterns of differentails associated with each component of (6.9), which we now demonstrate for $p = 3$. For example, for $\widehat{\beta}_2$ we have the following diagram:




where $x_k = v_2^{-k}\widehat{h}_{2,0}\widehat{b}_{2,0}^{k-1}\widehat{u}_k$, and the boxed elements are permanent in the Cartan-Eilenberg spectral sequence. The underlined elements indeed survive, however, each of these changes into $v_2$-torsion element (cf. Remark 6.10 and 6.13). It is also observed that $\widehat{h}_{1,1}\widehat{\beta}'_{3/2}$, $\widehat{b}_{1,1}\widehat{\beta}_2$ and $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}'_{3/2}$ correspond to the Massey products $\mu_2(\widehat{\beta}_2)$, $\mu_1(\mu_2(\widehat{\beta}_2))$ and $\mu_2(\mu_1(\mu_2(\widehat{\beta}_2)))$ respectively (see Remark 6.4).

Similarly, for $\widehat{\beta}_{3/3}$ we have the following diagram:



where $y_k = v_2^{-k}\widehat{b}_{2,0}^{k-1}\widehat{u}_k$ and $\widehat{h}_{1,1}\widehat{\beta}_{3/3}$ is renamed $\widehat{v}_2\widehat{\gamma}_1$, and for $\widehat{\beta}_{3/2}$ we also have the following diagram:

$$\begin{tikzcd}
\boxed{\widehat{\beta}_{3/2}} & \widehat{h}_{1,1}\widehat{\beta}_{3/2} & \underline{\widehat{b}_{1,1}\widehat{\beta}_{3/2}} \\
\widehat{\beta}_{4/3} \arrow[ur, "\widehat{r}_3"] & \underline{\widehat{h}_{1,1}\widehat{\beta}_{4/3}} & \\
z \arrow[ur, "d_2"] & \widehat{h}_{1,1}z \arrow[uur, "d_2"'] &
\end{tikzcd}$$

where $z = v_2^{-1}\widehat{u}_2$, and we have $\widehat{h}_{1,1}\widehat{\beta}_{4/3} = \mu_2(\widehat{\beta}_{3/2})$.

Finally, we have the following result:

**Theorem 6.14.** *Below dimension $p|\widehat{v}_3|$, the Cartan-Eilenberg $\tilde{E}_\infty$-term of Table 1 for $j = 1$ is the direct sum of the followings:*

(i) *the $A(m+1)/I_2 \otimes P(\widehat{v}_2^p)$-module generated by*

$$\begin{gathered}
\left\{\widehat{\beta}'_1, \widehat{\beta}'_2, \dots, \widehat{\beta}'_p; \widehat{\beta}_{p/1}, \widehat{\beta}_{p/2}; \widehat{h}_{1,1}\widehat{\beta}'_p\right\} \\
\oplus \\
P(\widehat{b}_{1,1}) \otimes \left\{\widehat{h}_{1,1}\widehat{\beta}'_{p/p-i+1}, \widehat{\beta}_{p/j} \mid 2 \le i \le p-1, 3 \le j \le p\right\} \\
\oplus \\
E(\widehat{h}_{2,0}) \otimes P(\widehat{b}_{2,0}) \otimes \left(\begin{gathered} P(\widehat{b}_{1,1}) \otimes \{\widehat{u}_0\} \\ \oplus \\ \left\{\widehat{h}_{1,1}\widehat{u}_i \mid 0 \le i \le p-2\right\} \end{gathered}\right);
\end{gathered}$$

(ii) *the $A(m+1)/I_3 \otimes P(\widehat{v}_2^p)$-module generated by*

$$E(\widehat{h}_{2,0}) \otimes P(\widehat{b}_{1,1}, \widehat{b}_{2,0}) \otimes \left(\begin{gathered} E(\widehat{h}_{1,1}) \otimes \{\widehat{v}_2^{p-1}\widehat{\gamma}_1\} \\ \oplus \\ \{\widehat{v}_2^i\widehat{\gamma}_1 \mid 2 \le i \le p-2\} \\ \oplus \\ \{\widehat{v}_2\widehat{\gamma}_1\} \end{gathered}\right) \Bigg/ \left(\begin{gathered} \widehat{v}_2^{p-1}\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{\gamma}_1, \\ \widehat{v}_2^{p-1}\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{\gamma}_1, \\ \widehat{v}_2^i\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{\gamma}_1 \end{gathered}\right)$$

*where the second summand is only for $p \ge 5$;*

(iii) *the $A(m+1)/I_2$-module generated by*

$$\begin{gathered}
\left\{\widehat{\beta}_{p^2/k} \mid 1 \le k \le p^2 - p + 1\right\} \\
\oplus \\
P(\widehat{b}_{1,1}) \otimes \left(\begin{gathered} \left\{\widehat{\beta}_{p^2/p^2-\ell}, \widehat{h}_{1,1}\widehat{\beta}_{p^2+\ell/p^2} \mid 0 \le \ell \le p-2\right\} \\ \oplus \\ E(\widehat{h}_{1,1}, \widehat{h}_{2,0}) \otimes P(\widehat{b}_{2,0}) \otimes \left\{\widehat{u}_{p/k} \mid 2 \le k \le p\right\} \end{gathered}\right); \textit{and}
\end{gathered}$$

| $t-s$ | Element |
|---|---|
| 46 | $\widehat{\beta}_1$ |
| 98 | $\widehat{\beta}_2$ |
| 142 | $\widehat{\beta}_{3/3}$ |
| 146 | $\widehat{\beta}_{3/2}$ |
| 150 | $\widehat{\beta}_3$<br>$\widehat{\beta}'_3$ |
| 154 | $\widehat{u}_0$ |
| 189 | $\widehat{v}_2\widehat{\gamma}_1$ |
| 193 | $\widehat{h}_{1,1}\widehat{\beta}'_{3/2}$ |
| 197 | $\widehat{h}_{1,1}\widehat{\beta}'_3$ |
| 201 | $\widehat{h}_{1,1}\widehat{u}_0$ |
| 202 | $\widehat{\beta}_4$ |
| 205 | $\widehat{h}_{2,0}\widehat{u}_0$ |
| 240 | $\widehat{v}_2\widehat{h}_{2,0}\widehat{\gamma}_1$ |
| 241 | $\widehat{v}_2^2\widehat{\gamma}_1$ |
| 252 | $\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}_0$ |
| 253 | $\widehat{h}_{1,1}\widehat{u}_1$ |
| 254 | $\widehat{\beta}_5$ |
| 284 | $\widehat{b}_{1,1}\widehat{\beta}_{3/3}$ |
| 288 | $\widehat{v}_2^2\widehat{h}_{1,1}\widehat{\gamma}_1$ |
| 292 | $\widehat{v}_2^2\widehat{h}_{2,0}\widehat{\gamma}_1$ |

| $t-s$ | Element |
|---|---|
| 296 | $\widehat{b}_{1,1}\widehat{u}_0$ |
| 297 | $\widehat{\gamma}_2$ |
| 298 | $\widehat{\beta}_{6/3}$ |
| 302 | $\widehat{\beta}_{6/2}$ |
| 304 | $\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}_1$ |
| 306 | $\widehat{\beta}'_6$<br>$\widehat{\beta}_6$ |
| 308 | $\widehat{b}_{2,0}\widehat{u}_0$ |
| 310 | $\widehat{u}_3$ |
| 331 | $\widehat{v}_2\widehat{b}_{1,1}\widehat{\gamma}_1$ |
| 335 | $\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{\beta}'_{3/2}$ |
| 339 | $\widehat{v}_2^2\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{\gamma}_1$ |
| 343 | $\widehat{v}_2\widehat{b}_{2,0}\widehat{\gamma}_1$ |
| 344 | $\widehat{h}_{1,1}\widehat{\gamma}_2$ |
| 345 | $\widehat{v}_2^4\widehat{\gamma}_1$ |
| 347 | $\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{u}_0$ |
| 348 | $\widehat{h}_{2,0}\widehat{\gamma}_2$ |
| 349 | $\widehat{h}_{1,1}\widehat{\beta}'_{6/2}$<br>$\widehat{v}_2\widehat{\gamma}_2$ |
| 353 | $\widehat{h}_{1,1}\widehat{\beta}'_6$ |

| $t-s$ | Element |
|---|---|
| 355 | $\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{u}_0$ |
| 357 | $\widehat{h}_{1,1}\widehat{u}_3$ |
| 358 | $\widehat{\beta}_7$ |
| 359 | $\widehat{h}_{2,0}\widehat{b}_{2,0}\widehat{u}_0$ |
| 361 | $\widehat{h}_{2,0}\widehat{u}_3$ |
| 382 | $\widehat{v}_2\widehat{b}_{1,1}\widehat{h}_{2,0}\widehat{\gamma}_1$ |
| 383 | $\widehat{v}_2^2\widehat{b}_{1,1}\widehat{\gamma}_1$ |
| 394 | $\widehat{v}_2\widehat{h}_{2,0}\widehat{b}_{2,0}\widehat{\gamma}_1$ |
| 395 | $\widehat{v}_2^2\widehat{b}_{2,0}\widehat{\gamma}_1$<br>$\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{\gamma}_2$ |
| 396 | $\widehat{v}_2^4\widehat{h}_{2,0}\widehat{\gamma}_1$<br>$\widehat{v}_2\widehat{h}_{1,1}\widehat{\gamma}_2$ |
| 397 | $\widehat{v}_2^5\widehat{\gamma}_1$ |
| 400 | $\widehat{v}_2\widehat{h}_{2,0}\widehat{\gamma}_2$ |
| 401 | $\widehat{v}_2^2\widehat{\gamma}_2$ |
| 406 | $\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{b}_{2,0}\widehat{u}_0$ |
| 407 | $\widehat{h}_{1,1}\widehat{b}_{2,0}\widehat{u}_1$ |
| 408 | $\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{u}_3$ |
| 409 | $\widehat{h}_{1,1}\widehat{u}_4$ |
| 410 | $\widehat{\beta}_8$ |

FIGURE 1. The elements of $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*(T(1)_{(1)}))$ for $p=3$, and $t-s \le 426$.

(iv) *the $A(m+2)/I_3$-module generated by*

$$E(\widehat{h}_{1,1},\widehat{h}_{2,0}) \otimes P(\widehat{b}_{1,1},\widehat{b}_{2,0}) \otimes \left\{\widehat{\gamma}_\ell \mid \ell \ge 2\right\}.$$

**Remark 6.15.** Theorem 6.11 (iii) and (iv) mean that some elements in the second summand of Theorem 6.14 (i) have higher $v_2$-torsion. They should be renamed chromatically so as to be realized explicitly that they are $v_2$-torsion.

Now we have computed $\mathrm{Ext}^n_{\Gamma(m+1)}(T_m^{(1)})$ for $n \ge 2$. There is no Adams-Novikov differential in this range because the first element in filtration $\ge 2p+1$ is $\widehat{v}_2\widehat{b}_{1,1}^{p-1}\widehat{\gamma}_1$, which is not killed by $d_{2p-1}$. Thus, the Adams-Novikov spectral sequence for $T(m)_{(1)}$ collapses and Theorem 6.14 gives us the stable homotopy groups of $T(m)_{(1)}$. The elements for $(p,m)=(3,1)$ are listed in Figure 1 and depicted in Figure 2.

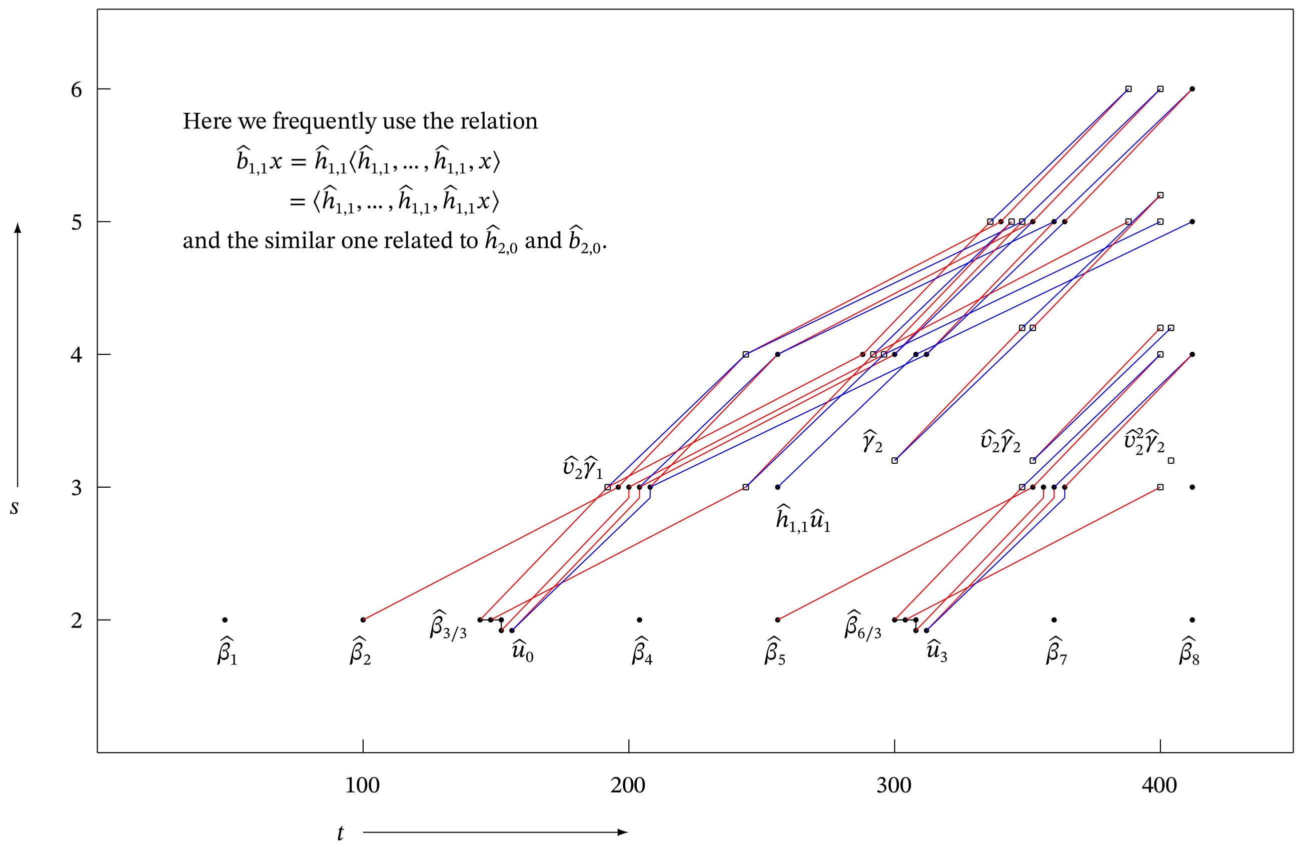



FIGURE 2. $\mathrm{Ext}_{BP_*(BP)}(BP_*(T(1)_{(1)}))$ for $p = 3$ in dimensions up to 426 dimension.

- Solid dots indicate $v_2$-torsion free elements, and squares indicate elements killed by $v_2$.
- Short vertical and horizontal lines indicate multiplication by $p$ and $v_1$.
- Red lines (resp. blue lines) indicate multiplication by $h_{2,1}$ (resp. $h_{3,0}$) and the Massey product operation $\langle h_{2,1}, h_{2,1}, -\rangle$ (resp. $\langle h_{3,0}, h_{3,0}, -\rangle$). The compositition of the two is the multiplication by $b_{2,1}$ (resp. $b_{3,0}$).

## 7. The proof of Theorem 6.11

In this section we give a detailed proof[7] of Theorem 6.11 for $m > 0$. As is stated in the proof of Lemma 6.8, our spectral sequence is a quotient of the Cartan-Eilenberg spectral sequence and it is enough to prove each differential by computing in $C_{\Gamma(m+1)}(T_m^{(1)} \otimes N^2)$.

**Lemma 7.1.** *For $m > 0$, we have a cocycle $\widehat{b}'_{2,0} = p^{-1}(v_1^p \widehat{b}_{1,1} + d(\widehat{t}_2^p))$ in the cobar complex over $\Gamma(m+1)$, which projects to $\widehat{b}_{2,0}$ in that over $\Gamma(m+2)$.*

**Proof.** Recall that we are using the symbols $\widehat{b}_{1,j}$ and $\widehat{b}_{2,0}$ for their cobar representatives, namely

$$\widehat{b}_{1,j} = p^{-1}d\left(\widehat{t}_1^{p^{j+1}}\right) = -\sum_{0<\ell<p^{j+1}} p^{-1}\binom{p^{j+1}}{\ell}\widehat{t}_1^{\ell} \otimes \widehat{t}_1^{p^{j+1}-\ell}$$

and
$$\begin{aligned}\widehat{b}_{2,0} &\equiv p^{-1}\left(\widehat{t}_2^p \otimes 1 + 1 \otimes \widehat{t}_2^p - (\widehat{t}_2 \otimes 1 + 1 \otimes \widehat{t}_2)^p\right) \\ &\equiv -\sum_{0<\ell<p} p^{-1}\binom{p}{\ell}\widehat{t}_2^{\ell} \otimes \widehat{t}_2^{p-\ell} \qquad \bmod (\widehat{t}_1).\end{aligned}$$

Then the result follows from $d(\widehat{t}_2^p) = (\widehat{t}_2^p \otimes 1 + 1 \otimes \widehat{t}_2^p - (\widehat{t}_2 \otimes 1 + v_1\widehat{b}_{1,0} + 1 \otimes \widehat{t}_2)^p)$. □

By Lemma 1.4 and Lemma 7.1, it follows that the product of any permanent cycle with $\widehat{b}_{2,0}$ is again a permanent cycle. This implies that each element in

$$\begin{gathered} A(m+1)/I_2 \otimes E(\widehat{h}_{1,1}, \widehat{h}_{2,0}) \otimes P(\widehat{b}_{1,1}, \widehat{b}_{2,0}) \otimes \{\widehat{u}_{p/k} \mid 2 \le k \le p\} \\ \bigoplus \\ A(m+2)/I_3 \otimes E(\widehat{h}_{1,1}, \widehat{h}_{2,0}) \otimes P(\widehat{b}_{1,1}, \widehat{b}_{2,0}) \otimes \{\widehat{\gamma}_2, \widehat{\gamma}_3, \dots\} \end{gathered}$$

is a permanent cycle, unlike the case $m = 0$.

**Lemma 7.2.** *Let $\overline{\widehat{t}_3}$ be the conjugation of $\widehat{t}_3$. Then we have*

$$\Delta(\overline{\widehat{t}_3}) = \overline{\widehat{t}_3} \otimes 1 + 1 \otimes \overline{\widehat{t}_3} - v_1\widehat{b}_{2,0} - v_2\widehat{b}_{1,1} + \begin{cases} \widehat{t}_1^{p^2} \otimes \widehat{t}_1 & \text{for } m = 1 \\ 0 & \text{for } m \ge 2. \end{cases}$$

*The difference between $\overline{\widehat{t}_3}$ and $-\widehat{t}_3$ has trivial image in $\Gamma(m+2)$.*

**Proof.** By definition, $\overline{\widehat{t}_3} = -\widehat{t}_3 + \widehat{t}_1^{1+p^2}$ for $m = 1$ and $\overline{\widehat{t}_3} = -\widehat{t}_3$ for $m \ge 2$. Since

$$\Delta(\widehat{t}_3) = \widehat{t}_3 \otimes 1 + 1 \otimes \widehat{t}_3 + v_1\widehat{b}_{2,0} + v_2\widehat{b}_{1,1} + \begin{cases} \widehat{t}_1 \otimes \widehat{t}_1^{p^2} & \text{for } m = 1 \\ 0 & \text{for } m \ge 2 \end{cases}$$

we have the result. □

[7] The case $m = 0$ was treated in [Rav04, §7.4].

**Proof of Theorem 6.11 (i).** We may use $\dfrac{\widehat{v}_2^i\widehat{v}_3}{pv_1}$ instead of $\widehat{u}_i$ because these have the same $\delta^1\delta^0$-image (4.3) into $U^2_{m+1}$. For $i>0$, we have

$$
\begin{aligned}
d\left(\widehat{t}_2\otimes 1\otimes \frac{\widehat{v}_2^i\widehat{v}_3}{pv_1}\right) &= \widehat{t}_2\otimes\left(v_2\widehat{t}_1^{p^2}-v_2^{p^{m+1}}\widehat{t}_1\right)\otimes 1\otimes\frac{\widehat{v}_2^i}{pv_1},\\
d\left(\widehat{t}_2\otimes\widehat{t}_1\otimes\frac{v_2^{p^{m+1}}\widehat{v}_2^i}{pv_1}\right) &= \widehat{t}_2\otimes\widehat{t}_1\otimes 1\otimes\frac{v_2^{p^{m+1}}\widehat{v}_2^i}{pv_1},\\
d\left(\widehat{t}_2\widehat{t}_1^{p^2}\otimes 1\otimes\frac{v_2\widehat{v}_2^i}{pv_1}\right) &= -\left(\widehat{t}_2\otimes\widehat{t}_1^{p^2}+\widehat{t}_1^{p^2}\otimes\widehat{t}_2\right)\otimes 1\otimes\frac{v_2\widehat{v}_2^i}{pv_1},\\
d\left(\widehat{t}_1^{p^2}\otimes 1\otimes\frac{v_2\widehat{v}_2^{i+1}}{(i+1)p^2v_1}\right) &= \widehat{t}_1^{p^2}\otimes\widehat{t}_2\otimes 1\otimes\frac{v_2\widehat{v}_2^i}{pv_1}+\widehat{b}_{1,1}\otimes 1\otimes\frac{v_2\widehat{v}_2^{i+1}}{(i+1)pv_1}.
\end{aligned}
$$

The sum of the preimages on the left represents $\widehat{h}_{2,0}\widehat{u}_i$; summing on the right gives the result. $\square$

**Proof of Theorem 6.11 (ii).** We give the proof for $k=1$ and $\varepsilon=1$. The general case follows by replacing $\widehat{b}_{2,0}$ by $\widehat{b}'_{2,0}$ (Lemma 7.1) and tensoring all equations on the left with the cocycle $(\widehat{b}'_{2,0})^{k-1}$.

We have $\eta_R(\widehat{v}_2)\equiv\widehat{v}_2+z \bmod I^{p^{m+1}}$, where $I=(p,v_1,\dots)$ and $z=v_1\widehat{t}_1^p+p\widehat{t}_2$. By this and Lemma 7.2 we have

$$
\begin{aligned}
d(\widehat{b}_{2,0}\otimes 1\otimes\widehat{u}_i) &= \widehat{b}_{2,0}\otimes d(1\otimes\widehat{u}_i)\\
&= -\widehat{b}_{2,0}\otimes v_2\sum_{0<k<p}\binom{i+p}{k}z^k\otimes 1\otimes\frac{\widehat{v}_2^{i+p-k}}{\binom{i+p}{p}pv_1^{p+1}}\\
&= -\widehat{b}_{2,0}\otimes v_2\left((i+p)\widehat{t}_1^p\otimes 1\otimes\frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^p}+\cdots\right),
\end{aligned}
$$

$$
\begin{aligned}
d&\left(-\widehat{t}_3\otimes v_2\left(-(i+p)\widehat{t}_1^p\otimes 1\otimes\frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^{p+1}}+\cdots\right)\right)\\
&= -(v_1\widehat{b}_{2,0}+v_2\widehat{b}_{1,1})\otimes v_2\left(-(i+p)\widehat{t}_1^p\otimes 1\otimes\frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p}pv_1^{p+1}}+\cdots\right)\\
&\quad -\widehat{t}_3\otimes -v_2(i+p)\widehat{t}_1^p\otimes\binom{i+p-1}{p}\widehat{t}_1^{p^2}\otimes 1\otimes\frac{\widehat{v}_2^{i-1}}{\binom{i+p}{p}pv_1}
\end{aligned}
$$

$$
\begin{aligned}
&= -\widehat{b}_{2,0} \otimes v_2 \left( -(i+p)\widehat{t}_1^p \otimes 1 \otimes \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p} p v_1^p} + \cdots \right) \\
&\qquad - v_2 \widehat{b}_{1,1} \otimes v_2 \left( -(i+p)\widehat{t}_1^p \otimes 1 \otimes \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p}{p} p v_1^{p+1}} + \cdots \right) \\
&\qquad + i v_2 \widehat{t}_3 \otimes \widehat{t}_1^p \otimes \widehat{t}_1^{p^2} \otimes 1 \otimes \frac{\widehat{v}_2^{i-1}}{p v_1},
\end{aligned}
$$

and

$$
\begin{aligned}
&d\left( -i\widehat{t}_3 \otimes \widehat{t}_1^p \otimes 1 \otimes \frac{\widehat{v}_2^{i-1}\widehat{v}_3}{p v_1} \right) \\
&\qquad = -i v_2 \widehat{b}_{1,1} \otimes \widehat{t}_1^p \otimes 1 \otimes \frac{\widehat{v}_2^{i-1}\widehat{v}_3}{p v_1} - i\widehat{t}_3 \otimes \widehat{t}_1^p \otimes v_2 \widehat{t}_1^{p^2} \otimes 1 \otimes \frac{\widehat{v}_2^{i-1}}{p v_1}.
\end{aligned}
$$

The sum of the preimages on the left represents $\widehat{b}_{2,0}\widehat{u}_i$, and the terms on the right add up to

$$
\begin{aligned}
&\widehat{b}_{1,1} \otimes \widehat{t}_1^p \otimes 1 \otimes \left( -\frac{i v_2 \widehat{v}_2^{i-1}\widehat{v}_3}{p v_1} + \frac{(i+p) v_2^2 \widehat{v}_2^{i+p-1}}{\binom{i+p}{p} p v_1^{p+1}} \right) + \cdots \\
&\qquad = -i v_2 \widehat{b}_{1,1} \otimes \widehat{t}_1^p \otimes 1 \otimes \left( \frac{\widehat{v}_2^{i-1}\widehat{v}_3}{p v_1} - \frac{\widehat{v}_2^{i+p-1}}{\binom{i+p-1}{p} p v_1^{p+1}} \right) + \cdots
\end{aligned}
$$

The inspection of $\tilde{E}_2$-terms described in Corollary 6.2 shows that the element represents $-i v_2 \widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}_{i-1}$ as claimed. □

To derive (iii), (iv) and (v) from (i) and (ii), we use Massey product arguments. Oberve Figure 3 for $p = 5$, in which each diagonal is similar to (6.5) and the arrows labeled $\tilde{d}_r$ are related to Cartan-Eilenberg differentials given in Lemma 6.8 and (ii); for example, the differential $\tilde{d}_3(\widehat{b}_{2,0}\widehat{u}_4) = v_2\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}_3$ is denoted

$$\widehat{b}_{2,0}\widehat{u}_4 \xrightarrow{\tilde{d}_3} v_2\widehat{b}_{1,1}\widehat{u}_3.$$

**Proof of Theorem 6.11 (iii).** For $k = 0$ this is a direct consequence of (i) via multiplication by $\widehat{h}_{1,1}$. We will illustrate with the case $i = p-1$ and $k \le 2$, and the other cases are similarly shown. For $k = 1$, we have the sequence analogous to that of Remark 6.4:

$$\widehat{b}_{2,0}\widehat{u}_{p-1} \xrightarrow{\tilde{d}_3} v_2\widehat{b}_{1,1}\widehat{u}_{p-2} \xrightarrow{\tilde{d}_2} v_2^2\widehat{b}_{1,1}\widehat{\beta}_{2p-3/p} \xrightarrow{\widehat{r}_p} \cdots \xrightarrow{\widehat{r}_p} v_2^2\widehat{b}_{1,1}\widehat{\beta}_{p/3}.$$

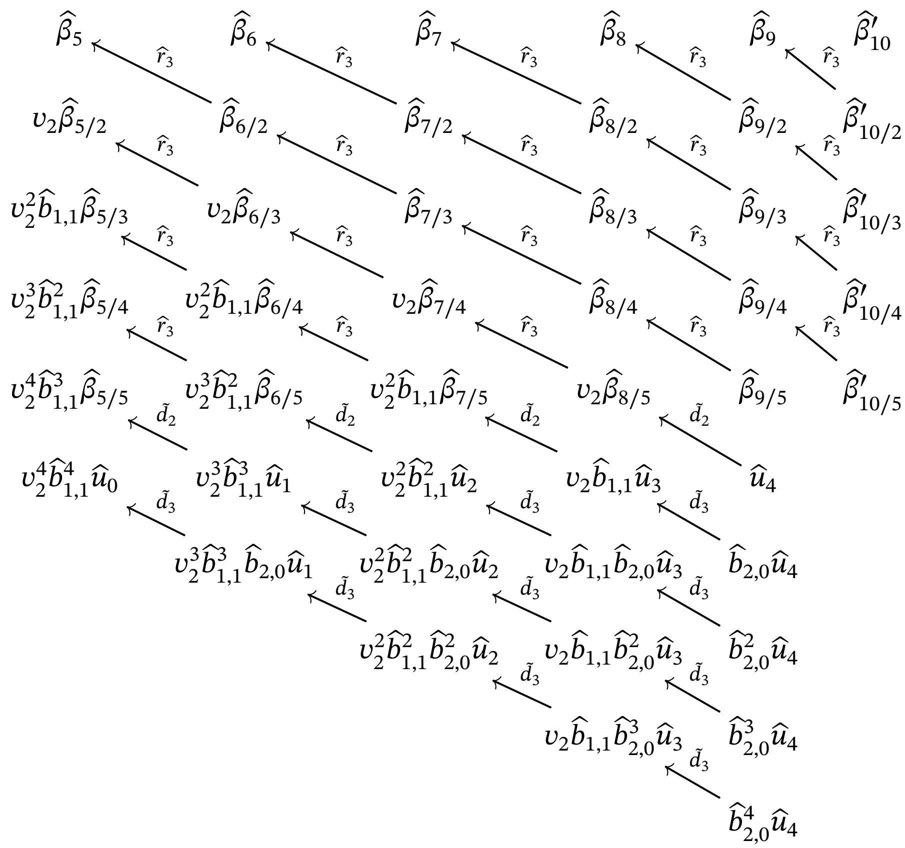


FIGURE 3. Differentials for the case $p = 5$.

This allows us to identify $v_2\widehat{h}_{1,1}\widehat{b}_{1,1}\widehat{u}_{p-2}$, up to unit scalar multiplication, with the Massey product $\mu_{p-1}(v_2^2\widehat{b}_{1,1}\widehat{\beta}_{p/3})$. It then follows that the differential on $\widehat{h}_{2,0}\widehat{h}_{1,1}(\widehat{b}_{2,0}\widehat{u}_{p-1})$ is the value of $\widehat{h}_{2,0}\widehat{h}_{1,1}\mu_{p-1}(v_2^2\widehat{b}_{1,1}\widehat{\beta}_{p/3})$. Now $\widehat{h}_{2,0}\widehat{h}_{1,1}$ (resp. $\widehat{b}_{1,1}$) is the image of $\widehat{\beta}_2$ (resp. $\widehat{\beta}_{p/p}$) under a suitable reduction map, so we have

$$\begin{aligned}
\tilde{d}_5(\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{b}_{2,0}\widehat{u}_{p-1}) &= \widehat{h}_{2,0}\widehat{h}_{1,1}\mu_{p-1}(v_2^2\widehat{b}_{1,1}\widehat{\beta}_{p/3}) = v_2^2\widehat{b}_{1,1}\widehat{\beta}_2\mu_{p-1}(\widehat{\beta}_{p/3}) \\
&= v_2^2\widehat{b}_{1,1}\mu_{p-1}(\widehat{\beta}_2)\widehat{\beta}_{p/3} \qquad \text{by Lemma A.8} \\
&= v_2^2\widehat{b}_{1,1}\mu_{p-1}(\widehat{\beta}_2)v_1^{p-3}\widehat{\beta}_{p/p} = v_2^2\widehat{b}_{1,1}^2 v_1^{p-3}\mu_{p-1}(\widehat{\beta}_2) \\
&= v_2^2\widehat{b}_{1,1}^2\mu_2(\widehat{\beta}_{p-1}) \qquad \text{by Example A.9} \\
&= v_2^2\widehat{b}_{1,1}^2\widehat{h}_{1,1}\widehat{\beta}'_{p/2}
\end{aligned}$$

as claimed. For $k = 2$, we have the sequence[8]

$$\widehat{b}_{2,0}^2\widehat{u}_{p-1} \xrightarrow{\tilde{d}_3} v_2\widehat{b}_{1,1}\widehat{b}_{2,0}\widehat{u}_{p-2} \xrightarrow{\tilde{d}_3} v_2^2\widehat{b}_{1,1}^2\widehat{u}_{p-3} \xrightarrow{\tilde{d}_2} v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{2p-4/p} \xrightarrow{\widehat{r}_p} \cdots \xrightarrow{\widehat{r}_p} v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{p/4}$$

By the similar argument to the case $k = 1$, we have

$$\begin{aligned}
\tilde{d}_7(\widehat{h}_{1,1}\widehat{h}_{2,0}\widehat{b}_{2,0}^2\widehat{u}_{p-1}) &= \widehat{h}_{2,0}\widehat{h}_{1,1}\mu_{p-1}(v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_{p/4}) = v_2^3\widehat{b}_{1,1}^2\widehat{\beta}_2\mu_{p-1}(\widehat{\beta}_{p/4})\\
&= v_2^3\widehat{b}_{1,1}^2\mu'_{p-1}(\widehat{\beta}_2)\widehat{\beta}_{p/4} \qquad \text{by Lemma A.8}\\
&= v_2^3\widehat{b}_{1,1}^2\mu'_{p-1}(\widehat{\beta}_2)v_1^{p-4}\widehat{\beta}_{p/p} = v_2^3\widehat{b}_{1,1}^3v_1^{p-4}\mu'_{p-1}(\widehat{\beta}_2)\\
&= v_2^3\widehat{b}_{1,1}^3\mu'_3(\widehat{\beta}_{p-2}) \qquad \text{by Example A.9}\\
&= v_2^3\widehat{b}_{1,1}^3\widehat{h}_{1,1}\widehat{\beta}'_{p/3}
\end{aligned}$$

as claimed. □

**Proof of Theorem 6.11 (iv) and (v).** We have the sequence

$$\widehat{b}_{2,0}^k\widehat{u}_{p-1} \xrightarrow{\tilde{d}_3} \cdots \xrightarrow{\tilde{d}_3} v_2^k\widehat{b}_{1,1}^k\widehat{u}_{p-1-k} \xrightarrow{\tilde{d}_2} v_2^{k+1}\widehat{b}_{1,1}^k\widehat{\beta}_{2p-2-k/p} \xrightarrow{\widehat{r}_p} \cdots \xrightarrow{\widehat{r}_p} v_2^{k+1}\widehat{b}_{1,1}^k\widehat{\beta}_{p/k+2}$$

for $1 \le k < p-1$, and

$$\widehat{b}_{2,0}^{p-1}\widehat{u}_{p-1} \xrightarrow{\tilde{d}_3} \cdots \xrightarrow{\tilde{d}_3} v_2^{p-1}\widehat{b}_{1,1}^{p-1}\widehat{u}_0$$

for $k = p-1$. Thus we have

$$\tilde{d}_r(\widehat{b}_{2,0}^k\widehat{u}_{p-1}) = \begin{cases}\mu_{p-1}(v_2^{k+1}\widehat{b}_{1,1}^k\widehat{\beta}_{p/k+2}) & \text{for } k < p-1\\ \mu_{p-1}(v_2^{p-1}\widehat{b}_{1,1}^{p-1}\widehat{u}_0) & \text{for } k = p-1\end{cases}$$

up to unit scalar multiplication. Since $\widehat{h}_{1,1}\mu_{p-1}(x) = \widehat{b}_{1,1}x$ we have

$$\tilde{d}_r(\widehat{h}_{1,1}\widehat{b}_{2,0}^k\widehat{u}_{p-1}) = \begin{cases}v_2^{k+1}\widehat{b}_{1,1}^{k+1}\widehat{\beta}_{p/k+2} & \text{for } k < p-1\\ v_2^{p-1}\widehat{b}_{1,1}^{p}\widehat{u}_0 & \text{for } k = p-1\end{cases}$$

as claimed. □

## Appendix A. Massey products

Here we recall the definition and properties of Massey products very briefly (cf. [Rav04, A1.4]) and prove some results used in this paper. Let $C$ be a differential graded algebra, which makes $H^*(C)$ a graded algebra. For $x \in C$ or $x \in H^*(C)$, let $\overline{x} = (-1)^{1+\deg(x)}x$, where $\deg(x)$ denotes the total degree: the sum of its internal and cohomogical degrees of $x$. Then we have $d(\overline{x}) = -\overline{d(x)}$, $\overline{(xy)} = -\overline{x}\,\overline{y}$, and $d(xy) = d(x)y - \overline{x}d(y)$.

Let $\alpha_k \in H^*(C)$ $(k = 1, 2, \dots)$ be a finite collection of elements and with representative cocycles $a_{k-1,k} \in C$. When $\overline{\alpha_1}\alpha_2 = 0$ and $\overline{\alpha_2}\alpha_3 = 0$, there are cochains $a_{0,2}$ and $a_{1,3}$ such that $d(a_{0,2}) = \overline{a_{0,1}}a_{1,2}$ and $d(a_{1,3}) = \overline{a_{1,2}}a_{2,3}$, and we have a cocycle $b_{0,3} = \overline{a_{0,2}}a_{2,3} + \overline{a_{0,1}}a_{1,3}$. The corresponding class in $H^*(C)$

[8]Note that we may assume that $p \ge 5$ since $0 \le k < p-1$.

represents the Massey product $\langle\alpha_1,\alpha_2,\alpha_3\rangle$, which is the coset comprising all cohomology classes represented by such $b_{0,3}$ for all possible choices of $a_{i,j}$. Two choices of $a_{0,2}$ or $a_{1,3}$ differ by a cocycle. The **indeterminacy** of $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ is the set

$$\alpha_1 H^{|\alpha_2\alpha_3|}(C)+H^{|\alpha_1\alpha_2|}(C)\alpha_3.$$

If the triple product contains zero, then one such choice yields a $b_{0,3}$ which is the coboundary of a cochain $a_{0,3}$.

If we have two 3-fold Massey products $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ and $\langle\alpha_2,\alpha_3,\alpha_4\rangle$ containing zero, then the $a_{i-1,i}$ and $a_{i-2,i}$ can be chosen so that there are cochains $a_{0,3}$ and $a_{1,4}$ with $d(a_{0,3})=b_{0,3}$ and $d(a_{1,4})=b_{1,4}$, and the 4-fold Massey product $\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle$ represented by the cocycle $b_{0,4}=\overline{a_{0,3}}a_{3,4}+\overline{a_{0,2}}a_{2,4}+\overline{a_{0,1}}a_{1,4}$. More generally, if we have cocycles $b_{j,k}$ and cochains $a_{j,k}$ satisfying

$$b_{j,k}=\sum_{j<\ell<k}\overline{a_{j,\ell}}a_{\ell,k}\quad\text{for } i\le j<k\le i+n \tag{A.1}$$

and $d(a_{j,k})=b_{j,k}$ for $0<k-j<n$, then we have the $n$-fold Massey products $\langle\alpha_{i+1},\dots,\alpha_{i+n}\rangle$ represented by $b_{i,i+n}$. The cochains $a_{j,k}$ chosen above are called the **defining system** for the Massey product.

If two products $\langle\alpha_1,\dots,\alpha_{n-1}\rangle$ and $\langle\alpha_2,\dots,\alpha_n\rangle$ are strictly defined (meaning all the lower order products in sight have trivial indeterminacy), then we have

$$\alpha_1\langle\alpha_2,\dots,\alpha_n\rangle=\langle\overline{\alpha}_1,\dots,\overline{\alpha_{n-1}}\rangle\alpha_n.$$

In fact, we can relax the hypothesis of strict definition in the following way.

**Lemma A.2.** *Suppose that $\langle\alpha_1,\dots,\alpha_{n-1}\rangle$ and $\langle\alpha_2,\dots,\alpha_n\rangle$ are defined and have representatives $x$ and $y$ respectively with the common defining system $a_{i,j}$ $(0<i<j<n)$. Then, the cocycle $\overline{x}a_{n-1,n}$ is cohomologous to $a_{0,1}y$.*

**Proof.** If both $x$ and $y$ contain zero, then we would have cochains $a_{1,n}$ and $a_{0,n-1}$ satisfying $d(a_{0,n-1})=x$ and $d(a_{1,n})=y$. Hence we could define the cocycle $b_{0,n}$ (A.1). In that case we would have

$$\begin{aligned} d(b_{0,n}) &= d(\overline{a_{0,1}}a_{1,n})+d(\overline{a_{0,n-1}}a_{n-1,n})+d(\tilde{b}_{0,n})\\ &= -a_{0,1}y+\overline{x}a_{n-1,n}+d(\tilde{b}_{0,n})=0 \end{aligned}$$

where

$$\tilde{b}_{0,n}=\sum_{1<i<n-1}\overline{a_{0,i}}a_{i,n}.$$

Even if $x$ and $y$ do not contain zero, so we don't have cochains $a_{1,n}$ and $a_{0,n-1}$, we can still define $\tilde{b}_{0,n}$. A routine calculation gives the desired value of $d(\tilde{b}_{0,n})$. □

We also have Massey products in the spectral sequence associated with a filtered differential graded algebra or a filtered differential graded module over a filtered differential graded algebra. Though our Cartan-Eilenberg spectral sequence is not associated with such a filtration, we can get around this as follows.

Let $T_m^* = \bigoplus_{i\geq 0} T_m^i$ be a bigraded comodule algebra with $i$ being the second grading and the algebra structure given by the pairings $T_m^i \otimes T_m^j \to T_m^{i+j}$.

Recall that for a Hopf algebroid $(A, \Gamma)$ and a comodule algebra $M$ the cup product in the cobar complex $C = C_\Gamma(M)$ is given by

$$\begin{aligned}(\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m_1) \cup (\gamma_{s+1} \otimes \cdots \otimes \gamma_{s+t} \otimes m_2)\\ = \gamma_1 \otimes \cdots \otimes \gamma_s \otimes m_1^{(1)}\gamma_{s+1} \otimes \cdots \otimes m_1^{(t)}\gamma_{s+t} \otimes m_1^{(t+1)} m_2\end{aligned}$$

where $\gamma_i \in \Gamma(m+1)$ and $m_j \in M$, and $m_1^{(1)} \otimes \cdots \otimes m_1^{(t+1)}$ is the iterated coproduct on $m_1$. The coboundary operator is a derivation with respect to this product and $C$ is a filtered differential graded algebra; we have

$$d(x \cup y) = d(x) \cup y + (-1)^{\deg(x)} x \cup d(y).$$

Now we have consider the two quadrigraded Cartan-Eilenberg spectral sequences:

$$\mathrm{Ext}_{G(m+1)}(\mathrm{Ext}_{\Gamma(m+2)}(T_m^*)) \implies \mathrm{Ext}_{\Gamma(m+1)}(T_m^*), \tag{A.3}$$

which is associated with a filtration on $C = C_{\Gamma(m+1)}(T_m^*)$, and

$$\mathrm{Ext}_{G(m+1)}(\mathrm{Ext}_{\Gamma(m+2)}(T_m^* \otimes E_{m+1}^1)) \implies \mathrm{Ext}_{\Gamma(m+1)}(T_m^* \otimes E_{m+1}^1), \tag{A.4}$$

which is associated with a filtration on $C' = C_{\Gamma(m+1)}(T_m^* \otimes E_{m+1}^1)$. We may regard the Cartan-Eilenberg spectral sequence of (2.1) as a quotient of the degree $p^i - 1$ component of (A.4).

Since $C'$ is a left differential module over $C$, (A.4) is a module over (A.3). Then we can make a similar product $\langle \alpha_1, \dots, \alpha_j\rangle$ with $\alpha_i \in H^*(C)$ $(1 \leq i < j)$ and $\alpha_j \in H^*(C')$ under certain conditions. In particular, we will be interested in Massey products of the form

$$\mu_k(y) = \langle \widehat{h}_{1,1}, \dots, \widehat{h}_{1,1}, y\rangle \qquad \text{and} \qquad \mu_k'(x) = \langle x, \widehat{h}_{1,1}, \dots, \widehat{h}_{1,1}\rangle \tag{A.5}$$

with $k$ factors $\widehat{h}_{1,1}$. For $1 < k < p$, $\mu_k(y)$ is defined only if $0 \in \mu_{k-1}(y)$. If $\mu_k(\mu_{p-k}(y))$ is defined for some $k$, then it contains $\widehat{b}_{1,1}y$.

**Remark A.6.** $\widehat{h}_{1,1} \in \mathrm{Ext}^1_{\Gamma(m+1)}(T_m^{p-1})$ is represented in the cobar complex by

$$x = -d(\widehat{t}_1^p) = \left(\widehat{t}_1 \otimes 1 + 1 \otimes \widehat{t}_1\right)^p - 1 \otimes \widehat{t}_1^p \equiv \widehat{t}_1^p \otimes 1 \qquad \text{mod } (p),$$

which means that $\widehat{h}_{1,1}$ becomes trivial when we pass to $\mathrm{Ext}^1_{\Gamma(m+1)}(T_m^p)$. Similarly, we have

$$x \cup x = d\left(x \cup \widehat{t}_1^p\right) = d\left(\sum_{i>0} \binom{p}{i} \widehat{t}_1^i \otimes \widehat{t}_1^{2p-i}\right).$$

Thus $\widehat{h}_{1,1} \cup \widehat{h}_{1,1} \in \mathrm{Ext}^2_{\Gamma(m+1)}(T_m^{2p-2})$ maps trivially to $\mathrm{Ext}^2_{\Gamma(m+1)}(T_m^{2p-1})$.

**Lemma A.7.** *Let $x_1 = x$ as above and define $x_i$ inductively on $i$ by*

$$x_i = \left(x_{i-1} \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_{i-1}\right)/i \qquad (1 < i < p).$$

*Then $x_i$ is in $C_{\Gamma(m+1)}(T_m^{(i-1)(p-1)})$ and it satisfies*

$$x_i \equiv (-1)^{i+1}\widehat{t}_1^{ip} \otimes 1/i! \mod (p) \qquad \textit{and} \qquad d(x_i) = \sum_{0<j<i} x_j \cup x_{i-j}.$$

**Proof.** We will prove these statements by induction. For the first statement, let us assume that $x_i \in C_{\Gamma(m+1)}(T_m^{(i-1)(p-1)})$. This means that it has the form $c\widehat{t}_1^{i+p-1} \otimes \widehat{t}_1^{(i-1)(p-1)}$ modulo $C_{\Gamma(m+1)}(T_m^{(i-1)(p-1)-1})$ for some scalar $c$, and so we have

$$\begin{aligned} x_{i+1} &= (x_i \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_i)/(i+1) \\ &\equiv c(\widehat{t}_1^{i+p-1} \otimes \widehat{t}_1^{(i-1)(p-1)+p} - \widehat{t}_1^{i+p-1} \otimes \widehat{t}_1^{(i-1)(p-1)+p})/(i+1) \equiv 0 \end{aligned}$$

modulo $C_{\Gamma(m+1)}(T_m^{i(p-1)})$. For the congruence, we see that

$$\begin{aligned} (i+1)!x_{i+1} &= i!(x_i \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_i) \equiv (-1)^{i+1}\left((\widehat{t}_1^{ip} \otimes 1) \cup \widehat{t}_1^p - \widehat{t}_1^p \cup (\widehat{t}_1^{ip} \otimes 1)\right) \\ &= (-1)^{i+1}\left(\widehat{t}_1^{ip} \otimes \widehat{t}_1^p - \widehat{t}_1^{(i+1)p} \otimes 1 - \widehat{t}_1^{ip} \otimes \widehat{t}_1^p\right) = (-1)^{i+2}\widehat{t}_1^{(i+1)p} \otimes 1. \end{aligned}$$

For the derivation formula, we see that

$$\begin{aligned} &(i+1)d(x_{i+1}) - x_i \cup x_1 - x_1 \cup x_i \\ &\quad = d(x_i) \cup \widehat{t}_1^p - \widehat{t}_1^p \cup d(x_i) \\ &\quad = \left(\sum_{0<j<i} x_j \cup x_{i-j}\right) \cup \widehat{t}_1^p - \widehat{t}_1^p \cup \left(\sum_{0<j<i} x_j \cup x_{i-j}\right) \\ &\quad = \sum_{0<j<i} \left(x_j \cup (x_{i-j} \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_{i-j}) + (x_j \cup \widehat{t}_1^p - \widehat{t}_1^p \cup x_j) \cup x_{i-j}\right) \\ &\quad = \sum_{0<j<i} \left((i+1-j)x_j \cup x_{i+1-j} + (j+1)x_{j+1} \cup x_{i-j}\right) \\ &\quad = (i+1) \sum_{1<j<i} x_j \cup x_{i+1-j}. \end{aligned}$$

□

The following result follows easily from Lemma A.7.

**Lemma A.8.** *Suppose that $\alpha, \beta \in \operatorname{Ext}_{\Gamma(m+1)}(T_m^h \otimes E_{m+1}^2)$ are represented by cocycles $a_1$ and $b_1$, and that there are cochains*

$$a_i, b_i \in C_{\Gamma(m+1)}(T_m^{h+(i-1)(p-1)} \otimes E_{m+1}^2) \quad \textit{for } 1 < i \le k$$

*satisfying*

$$d(a_i) = \sum_{0<j<i} a_{i-j} \cup x_j \quad \textit{and} \quad d(b_i) = \sum_{0<j<i} x_j \cup b_{i-j},$$

*where $x_j$ are as in Lemma A.7. Then the Massey products*

$$\mu'_k(\alpha), \mu_k(\beta) \in \mathrm{Ext}_{\Gamma(m+1)}(T_m^{h+k(p-1)} \otimes E_{m+1}^2)$$

*are defined and are represented by the cocycles*

$$\sum_{0<i<k+1} a_{k+1-i} \cup x_i \quad \text{and} \quad \sum_{0<i<k+1} x_i \cup b_{k+1-i}.$$

*Moreover, we have $\alpha\mu_k(\beta) = \mu'_k(\overline{\alpha})\beta$ using these representatives.*

Here are two examples of such products.

**Example A.9.** For $0 < k < p$ and $\ell > 0$, the Massey product $\mu_k(\widehat{\beta}'_{p\ell-k+1})$ is defined and it is represented by

$$\sum_{0<i<k+1} x_i \cup (-1)^{k-i}\frac{(p\ell-k)!}{(p\ell-i)!}\widehat{\beta}'_{p\ell+1-i/k+1-i}.$$

We have an equality $v_1\mu_k(\widehat{\beta}'_{p\ell+1-k}) = \mu_{k-1}(\widehat{\beta}'_{p\ell+2-k})/(k-1-p\ell)$ for $k > 1$.

**Example A.10.** For $0 < k < p$ and $\ell > 0$, the Massey product $\mu_k(\widehat{\beta}_{p\ell/p+2-k})$ is defined and it is represented by

$$x_1 \cup v_2^{-1}\widehat{u}_{p\ell+k-1-p} + \sum_{1<i<k+1} x_i \cup (-1)^{i+1}\frac{(p\ell+k)!}{\ell(p\ell+k-i)!}\widehat{\beta}_{p\ell+k-i/p+2-i}.$$

## References


[Haz77] Hazewinkel, Michiel. Constructing formal groups. I. The local one dimensional case. *J. Pure Appl. Algebra* **9** (1976/77), no. 2, 131–149. MR0463182, Zbl 0363.14011, doi: 10.1016/0022-4049(77)90061-5. 233

[Nak08] Nakai, Hirofumi. An algebraic generalization of image $J$. *Homology Homotopy Appl.* **10** (2008), no. 3, 321–333. MR2475627, Zbl 1170.55004, doi: 10.4310/HHA.2008.v10.n3.a14. 235, 239, 240

[NR09] Nakai, Hirofumi; Ravenel, Douglas C. On $\beta$-elements in the Adams-Novikov spectral sequence. *J. Topol.* 2 (2009), no. 2, 295–320. MR2529298, Zbl 1229.55015, doi: 10.1112/jtopol/jtp012. 237, 238, 239, 240, 251, 252

[Rav] Ravenel, Douglas C. The first differential in the Adams-Novikov spectral sequence for the spectrum $T(m)$. Available at `https://people.math.rochester.edu/faculty/doug/mypapers/first.pdf` 236

[Rav86] Ravenel, Douglas C. Complex cobordism and stable homotopy groups of spheres. Pure and Applied Mathematics, 121. *Academic Press, Inc., Orlando, FL,* 1986. xx+413 pp. ISBN: 0125834306. MR0860042, Zbl 0608.55001, doi: 10.1090/chel/347. Errata available at `https://people.math.rochester.edu/faculty/doug/mybooks/1errata.pdf`. 231

[Rav02] Ravenel, Douglas C. The method of infinite descent in stable homotopy theory. I. *Recent progress in homotopy theory (Baltimore, MD, 2000),* 251–284, Contemp. Math., 293, *Amer. Math. Soc., Providence, RI,* 2002. MR1890739, Zbl 1006.55010, doi: 10.1090/conm/293/04951. 231, 232

[Rav04] Ravenel, Douglas C. Complex cobordism and stable homotopy groups of spheres. 2nd ed. *Providence, RI: AMS Chelsea Publishing.* xix, 395 p. (2004). ISBN: 082182967X. MR0860042, Zbl 1073.55001, doi: 10.1090/chel/347.H. 231, 232, 236, 238, 249, 252, 253, 255, 260, 264

[Tod67] TODA, HIROSI. An important relation in homotopy groups of spheres. *Proc. Japan Acad.* 43 (1967), 839–842. MR0230310, Zbl 0169.25603, doi: 10.3792/pja/1195521423. 236

[Tod68] TODA, HIROSI. Extended $p$-th powers of complexes and applications to homotopy theory. *Proc. Japan Acad.* 44 (1968), 198–203. MR0230311, Zbl 0181.26405, doi: 10.3792/pja/1195521243. 236

(Hirofumi Nakai) TOKYO CITY UNIVERSITY, 1-28-1 TAMAZUTSUMI, TOKYO 158-8557, JAPAN
hnakai@tcu.ac.jp

(Douglas C. Ravenel) UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14627, USA
doug.ravenel@rochester.edu

This paper is available via http://nyjm.albany.edu/j/2024/30-8.html.