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Real-variable theory of matrix-weighted weak Triebel-Lizorkin spaces

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ABSTRACT. We introduce matrix-weighted weak Triebel-Lizorkin spaces and establish the equivalence between the corresponding weak discrete sequence spaces. In the scalar unweighted case, we first prove the boundedness of almost diagonal operators on the weak discrete Triebel-Lizorkin space and then extend this result to the matrix-weighted setting. Furthermore, we provide a characterization of these spaces in terms of molecules. Additionally, we demonstrate the equivalence between the continuous function spaces defined via a sequence of reducing operators and those defined directly by matrix weights. These results ultimately establish a complete connection between matrix-weighted weak Triebel-Lizorkin spaces and their discrete or sequence space analogues. Within this framework, we develop several characterizations of matrix-weighted weak Triebel-Lizorkin spaces: First, using the doubling property of matrix weights and the Fefferman-Stein inequality, we obtain the characterization of matrix-weighted weak Triebel-Lizorkin spaces in terms of the Peetre maximal function. Second, combining the Peetre maximal function with the Fefferman-Stein inequality, we derive the Lusin area function characterization of matrix-weighted weak Triebel-Lizorkin spaces. Third, we utilize reducing operators and the Fefferman-Stein inequality to provide the Littlewood-Paley g_λ^*-function characterization of matrix-weighted weak Triebel-Lizorkin spaces. Finally, as an application, the boundedness of the classical Calderón-Zygmund operator on these spaces is obtained.

CONTENTS

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1. Introduction

Lizorkin [22, 23] and Triebel [32] have independently studied what is now known as Triebel-Lizorkin space since the 1970s. Besov spaces first appeared in the 1960s, introduced by the Soviet mathematician Oleg Vladimirovich Besov. Many classical function spaces such as Lebesgue spaces, Hardy spaces, Sobolev spaces, Lipschitz spaces, etc., are special cases of (homogeneous) Besov spaces or (homogeneous) Triebel-Lizorkin spaces (see [33] for details). More theories and applications of these two types of spaces can be found in [11, 13]. In [17], Danqing He considered the characterization of the square function of weak Hardy spaces. In [16], Grafakos and Danqing He discussed various characterizations of maximal functions for these spaces and presented an interpolation theorem for $H^{p,\infty}$ from the initial strong H^{p_0} and H^{p_1} estimates ($p_0 < p < p_1$), as well as they introduced the weak Triebel- Lizorkin spaces. Obviously, the usual Triebel-Lizorkin spaces are subsets of weak Triebel-Lizorkin spaces. In [36], Xianjie Yan, Dachun Yang, Wen Yuan, and Ciqiang Zhuo introduced variable weak Hardy spaces and obtained the characterizations of atoms, molecules, Lusin area functions, Littlewood-Paley g-functions, or g^*_λ-functions of variable weak Hardy spaces. Wenchang Li and Jingshi Xu [20] obtained the equivalent quasi-norms of the Peetre maximal functions for weak Triebel-Lizorkin spaces, as well as atomic decompositions. After that, they established vector-valued estimates for variable exponent weak Lebesgue spaces in [21], and then introduced weak Triebel-Lizorkin spaces with variable integrability, summability, and smoothness. They provided equivalent quasi-norms for these spaces using Peetre maximal functions and obtained the boundedness of the φ-transform and their atomic and molecular decompositions on these spaces.

On the other hand, the theory of scalar A_p weights originated from Muckenhoupt [24] and Hunt, Muckenhoupt, and Wheeden [19]. It has now been extended to matrix weights. Matrix weights were developed in the 1990s, and scalar methods cannot be directly applied in matrix-weighted spaces. In 1997, in order to address some meaningful problems related to multivariate stationary stochastic processes and Toeplitz operators (see [31]), Treil and Volberg [30] introduced Muckenhoupt A_2 matrix weights and extended the Hunt–Muckenhoupt–Wheeden theorem to the vector-valued case. Subsequently, Nazarov and Treil [25] introduced Muckenhoupt A_p matrix weights, extended the theory from $p = 2$ to $1 < p < \infty$, and obtained the boundedness of the Hilbert transform on the matrix-weighted Lebesgue space $L^p(W)$. Volberg [34] provided an alternative proof using methods from classical Littlewood-Paley theory. In 2016, Cruz-Uribe et al. [7] applied the theory of A_p matrix weights on

Euclidean spaces to study degenerate Sobolev spaces. For more research on matrix-weighted function spaces and their applications, see [6, 8, 9, 10].

In recent years, Frazier and Roudenko [15] introduced the homogeneous Triebel-Lizorkin spaces with matrix weights, denoted by $\dot{F}_p^{\alpha,q}(W)$, where $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, and $q \in (0,\infty]$, through the discrete Littlewood-Paley g-function. Frazier and Roudenko [15] proved that for any given $p \in (1,\infty)$, $L^p(W) = \dot{F}_p^{0,2}(W)$; and for any $k \in \mathbb{N}$, $F_p^{k,2}(W)$ coincides with the matrix-weighted Sobolev space $L_k^p(W)$. Frazier and Roudenko [15] also demonstrated that a vector-valued function $\vec{f}$ belongs to $\dot{F}_p^{\alpha,q}(W)$ if and only if its φ-transform coefficients belong to the sequence space $\dot{f}_p^{\alpha,q}(W)$. As an application of the above results, Frazier and Roudenko [15] obtained the boundedness of Calderón-Zygmund operators on $\dot{F}_p^{\alpha,q}(W)$. Qi Wang, Dachun Yang, and Yangyang Zhang et al. [35] provided several real-variable characterizations of $\dot{F}_p^{\alpha,q}(W)$. As an application, they proved the boundedness of Fourier multipliers on this space under the generalized Hörmander condition.

In addition, Dachun Yang et al. [37, 38] introduced Besov-type spaces $\dot{B}_{p,q}^{s,\tau}$ and Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}$ with a new Morrey parameter $\tau \in [0,\infty)$ and developed the real-variable theory for these spaces. They also demonstrated that when $\tau = 0$, these spaces not only include the well-known Besov and Triebel-Lizorkin spaces $\dot{B}_{p,q}^{s}$ and $\dot{F}_{p,q}^{s}$, but also encompass other function spaces such as Morrey spaces and Q spaces. In recent years, for any $A \in \{B,F\}$, $s \in \mathbb{R}$, $\tau \in [0,\infty)$, $p \in (0,\infty)$, $q \in (0,\infty]$, and Muckenhoupt A_p matrix weight W, Bu Fan et al. [1, 2, 3] introduced the matrix-weighted Besov-Triebel-Lizorkin-type spaces $\dot{A}_{p,q}^{s,\tau}(W)$ on $\mathbb{R}^n$ and developed their real-variable theory, including φ-transform characterizations, molecular and wavelet characterizations, as well as the boundedness of pseudo-differential operators, trace operators, and Calderón-Zygmund operators. In particular, $\dot{A}_{p,q}^{s,0}(W)$ coincides with the matrix-weighted Besov-Triebel-Lizorkin space $\dot{A}_{p,q}^{s}(W)$. Recently, for any $A \in \{B,F\}$, $s \in \mathbb{R}$, $\tau \in [0,\infty)$, $p \in (0,\infty)$, $q \in (0,\infty]$, and any matrix $A_{p,\infty}$ weight W, Bu Fan et al. [4] studied the matrix-weighted Besov-Triebel-Lizorkin-type spaces $A_{p,q}^{s,\tau}(W)$ on $\mathbb{R}^n$. Subsequently, inspired by the invariance of integrability indices in Triebel-Lizorkin spaces, Dachun Yang et al. [5] introduced the generalized matrix-weighted Besov-Triebel-Lizorkin-type spaces $\dot{A}_{p,q}^{s,\nu}(W)$ on $\mathbb{R}^n$ with broad generality, where $A \in \{B,F\}$, $s \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, ν is a growth function, and W is a matrix $A_{p,\infty}$ weight. They developed the real-variable theory for these spaces. Building on [5], they established the boundedness of pseudo-differential operators, trace operators, and Calderón-Zygmund operators on the space $\dot{A}_{p,q}^{s,\nu}(W)$ in [39]. Moreover, the space $\dot{A}_{p,q}^{s,\nu}(W)$ includes the matrix-weighted Besov-Triebel-Lizorkin-type spaces $\dot{A}_{p,q}^{s,\tau}(W)$, and in particular, the matrix-weighted Besov-Triebel-Lizorkin spaces $\dot{A}_{p,q}^{s}(W)$. However, the weak Triebel-Lizorkin spaces with matrix weights have not yet been studied in the literature. Based on this gap, this paper will introduce the weak Triebel-Lizorkin spaces with matrix weights and provide real-variable characterizations of these spaces.

The organization of this paper is as follows:

In Section 2 we begin by providing definitions of fundamental concepts and establishing notational conventions. In Section 3 we recall properties of classical scalar A_p weights and the A_p matrix weight class. In Section 4 we introduce the definition of homogeneous matrix-weighted weak Triebel-Lizorkin spaces $\dot{F}^{\alpha,q}_{p,\infty}(W)$ and $\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})$, along with their discrete norms $\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}$ and $\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(W)}$. Here, $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, W is an A_p matrix weight, and $\{A_Q\}$ is its associated sequence of reducing operators. This section presents several key lemmas essential for proving the main results of the paper and establishes the norm equivalence $\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(W)}$. In Section 5 we prove the boundedness of scalar, unweighted almost diagonal operators on the scalar, unweighted weak discrete Triebel-Lizorkin space $\dot{f}^{\alpha,q}_{p,\infty}$. Subsequently, this result is extended to establish the boundedness of almost diagonal operators on the weighted space $\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})$. Section 6 is molecular and atomic characterizations of these spaces. In Section 7 we prove the equivalence of the spaces $\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})$ and $\dot{F}^{\alpha,q}_{p,\infty}(W)$. In Section 8 we characterize the space $\dot{F}^{\alpha,q}_{p,\infty}(W)$ using the Peetre maximal function, the Lusin area function, and the Littlewood-Paley g^*_λ-function. Finally, in Section 9 we establish the boundedness of classical convolution-type Calderón-Zygmund operators on the matrix-weighted weak Triebel-Lizorkin spaces.

2. Preliminaries

To state the following results, we first introduce some notation. Let $f \lesssim g$ mean $f \le Cg$ for some positive constant C. Let $f \sim g$ denote $f \lesssim g$ and $g \lesssim f$. Let $\mathbb{N} := \{1, 2, ...\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}^n_+ := (\mathbb{Z}_+)^n$. For any measurable set $E \subset \mathbb{R}^n$, let $|E|$ be its measure. Define $\fint_E f(x)\,\mathrm{d}x := \frac{1}{|E|}\int_E f(x)\,\mathrm{d}x$. Define $\tilde{g}(x) := g(-x)$. Let $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$, $B := B(x_B, r_B)$, and $aB := B(x_B, ar_B)$. For $p \in [1,\infty]$, let p' denote its conjugate exponent, i.e., $1/p + 1/p' = 1$. For a measurable set $E \subset \mathbb{R}^n$, let $\mathbf{1}_E$ be its characteristic function. For any $j \in \mathbb{Z}$, define $\varphi_j(x) := 2^{jn}\varphi(2^j x)$. Define the dyadic cubes $Q_{jk} := \prod_{i=1}^n [2^{-j}k_i, 2^{-j}(k_i+1))$, $\mathcal{D} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, and $\mathcal{D}_j := \{Q \in \mathcal{D} : \ell(Q) = 2^{-j}\}$, where $\ell(Q)$ denotes the side length of the dyadic cube Q. For $Q = Q_{jk}$, define

$$\varphi_Q(x) := 2^{jn/2}\varphi(2^j x - k) = |Q|^{-1/2}\varphi((x - x_Q)/\ell(Q)),$$

where $x_Q = 2^{-j}k$ denotes the lower-left corner of the dyadic cube Q. Similarly, define $\psi_Q(x)$. For a vector-valued function $\vec{f}$, define

$$\langle \vec{f}, g\rangle := (\langle f_1, g\rangle, \dots, \langle f_m, g\rangle)^T.$$

$\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions on $\mathbb{R}^n$. $\mathcal{S}'(\mathbb{R}^n)$ is its topological dual space.

$$\mathcal{S}_\infty(\mathbb{R}^n) := \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x) x^\gamma \mathrm{d}x = 0,\ \gamma \in \mathbb{Z}_+^n\right\},$$

it is a subspace of $\mathcal{S}(\mathbb{R}^n)$. $\mathcal{S}'_\infty(\mathbb{R}^n)$ denotes the topological dual space of $\mathcal{S}_\infty(\mathbb{R}^n)$. $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$. Then $\mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. For any $m \in \mathbb{N}$,

$$(\mathcal{S}'_\infty(\mathbb{R}^n))^m := \{\vec{f} := (f_1, \dots, f_m)^T : i \in \{1, \dots, m\},\ f_i \in \mathcal{S}'_\infty(\mathbb{R}^n)\}.$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\hat{\varphi}$ denotes its Fourier transform, and for any $\xi \in \mathbb{R}^n$, it is defined as follows:

$$\hat{\varphi}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi} \mathrm{d}x.$$

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, $\langle \hat{f}, \varphi\rangle := \langle f, \hat{\varphi}\rangle$. For any $f \in \mathcal{S}(\mathbb{R}^n)$, $\check{f}$ denotes its inverse Fourier transform, and it is defined as follows:

$$\check{f}(x) := (2\pi)^{n/2} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi x} \mathrm{d}\xi.$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, denote $\operatorname{supp}\varphi := \overline{\{x \in \mathbb{R}^n : \varphi(x) \neq 0\}}$. Let f and g be measurable functions on $\mathbb{R}^n$. The convolution of f and g is defined by $(f * g)(x) = \int_{\mathbb{R}^n} f(x-t) g(t)\,\mathrm{d}t$. Define $\varphi_j * \vec{f} := (\varphi_j * f_1, \dots, \varphi_j * f_m)^T$.

The Lebesgue space $L^p(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that $\|f\|_{L^p(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{L^p(\mathbb{R}^n)} := \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, & p \in (0, \infty), \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|, & p = \infty. \end{cases}$$

Let $0 < p \le \infty$, denote the weak Lebesgue space by $L^{p,\infty}(\mathbb{R}^n)$, which is the set of Lebesgue measurable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} := \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}} < \infty.$$

For brevity, we shall henceforth abbreviate the weak-L^p space $L^{p,\infty}(\mathbb{R}^n)$ as $L^{p,\infty}$.

Remark 2.1. *For $0 < A < \infty$, by definition it follows that, $\||f|^A\|_{L^{p,\infty}} = \|f\|^A_{L^{pA,\infty}}$.*

Let $p \in (0, \infty]$, $q \in (0, \infty]$, the space $L^{p,\infty}(\ell^q)$ is defined as the set of all complex-valued measurable function sequences $\{f_j\}_{j \in \mathbb{Z}}$ on $\mathbb{R}^n$ satisfy

$$\left\|\{f_j\}\right\|_{L^{p,\infty}(\ell^q)} := \left\|\|\{f_j\}\|_{\ell^q}\right\|_{L^{p,\infty}} = \left\|\left\{\sum_{j \in \mathbb{Z}} \left|f_j\right|^q\right\}^{\frac{1}{q}}\right\|_{L^{p,\infty}} < \infty.$$

3. Muckenhoupt matrix weights

Let $L^1_{\text{loc}}(\mathbb{R}^n)$ be the set of all locally integrable functions on $\mathbb{R}^n$ (functions that are integrable on every compact subset). Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined as

$$\mathcal{M}f(x) := \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y,\ \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q centered at x with sides parallel to the coordinate axes.

Next, we first review the concept of the classical $A_p(\mathbb{R}^n)$-weights.

Definition 3.1. *Let $1 < p < \infty$ and ω be a positive measurable function on $\mathbb{R}^n$. If*

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q \omega(x)\,\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,\mathrm{d}x\Big)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q, then ω is called an A_p weight function, denoted by $\omega \in A_p$, or we say that ω satisfies the A_p condition. If

$$\mathcal{M}\omega(x) \le C\omega(x)$$

for almost every x, then ω is called an A_1 weight function, denoted by $\omega \in A_1$. Here, $\mathcal{M}$ is the Hardy-Littlewood maximal operator and $A_\infty(\mathbb{R}^n)$ denotes the set $\bigcup\limits_{1\le p<\infty} A_p(\mathbb{R}^n)$.

Let $m \in \mathbb{N}$, and let A be an $m \times m$ complex-valued matrix. A is said to be positive definite if for any $\vec{z} \in \mathbb{C}^m \setminus\{0\}$, $(A\vec{z}, \vec{z}) > 0$. A is said to be non-negative definite if for any $\vec{z} \in \mathbb{C}^m\setminus\{0\}$, $(A\vec{z}, \vec{z}) \ge 0$. $M(\mathbb{C}^m)$ is denotes the set of all $m\times m$ non-negative definite complex-valued matrix throughout this paper.

Regarding the matrix A as a bounded linear operator on $\mathbb{C}^m$, we denote its operator norm by $\|A\|$,

$$\|A\| := \sup_{\vec{z}\in\mathbb{C}^m\setminus\{0\}} \frac{|A\vec{z}|}{|\vec{z}|},$$

where $\vec{z} := (z_1, \dots, z_m)^T \in \mathbb{C}^m$, and $|\vec{z}| := \Big(\sum\limits_{j=1}^m |z_j|^2\Big)^{1/2}$. For any non-negative definite matrices A, $B \in M(\mathbb{C}^m)$, we have $\|AB\| = \|BA\|$. The norm here is the same as the one defined above (see Lemma 2.3 in [3]).

Definition 3.2. *Let $m \in \mathbb{N}$, and let A be a positive definite $m \times m$ complex-valued matrix, satisfying that there exists an invertible $m \times m$ complex-valued matrix P and a diagonal matrix* $\operatorname{diag}(\lambda_1, \dots, \lambda_m)$*, where $\{\lambda_1, \dots, \lambda_m\} \subset \mathbb{R}_+$, such that $A = P \operatorname{diag}(\lambda_1, \dots, \lambda_m)P^{-1}$. Then for any $\alpha \in \mathbb{R}$, A^α is defined as*

$$A^\alpha := P \operatorname{diag}(\lambda_1^\alpha, \dots, \lambda_m^\alpha)P^{-1}.$$

Remark 3.3. *From the knowledge of algebra, it is known that the definition of A^α is independent of the diagonalization decomposition form. See Proposition 1.1 in* [18].

Next, we review the concept of matrix weights.

Definition 3.4. *Let $m \in \mathbb{N}$. A matrix-valued function $W : \mathbb{R}^n \to M(\mathbb{C}^m)$ is called a matrix weight if each entry of W is a locally integrable function and $W(x)$ is invertible for almost every $x \in \mathbb{R}^n$.*

The concept of A_p matrix weights is derived from the p.1226, (1.1) in [14] and Definition 3.2 in [27].

Definition 3.5. *Let $m \in \mathbb{N}$, $p \in (0, \infty)$. A matrix weight W is called an A_p-matrix weight, denoted by $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, if*
(i) when $p \in (0, 1]$, there exists a positive constant C such that for any cube $Q \subset \mathbb{R}^n$,

$$\operatorname*{ess\,sup}_{y \in Q} \fint_Q \left\| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) \right\|^p \mathrm{d}x \le C;$$

(ii) when $p \in (1, \infty)$, there exists a positive constant C such that for any cube $Q \subset \mathbb{R}^n$,

$$\fint_Q \left(\fint_Q \left\| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) \right\|^{p'} \mathrm{d}y \right)^{\frac{p}{p'}} \mathrm{d}x \le C.$$

The expression $\|W^{1/p}(x)W^{-1/p}(y)\|$ in the above formula refers to the matrix (operator) norm. The definition below is derived from the p.1230 in [14].

Definition 3.6. *Let $p \in (0, \infty)$. We say that a matrix weight W is a doubling matrix weight of order p if for all $\vec{y} \in \mathbb{C}^m$, the scalar measures $\omega_{\vec{y}}(x) := |W^{1/p}(x)\vec{y}|^p$ are uniformly doubling: that is, there exists $C > 0$ such that for all cubes $Q \subseteq \mathbb{R}^n$ and all $\vec{y} \in \mathbb{C}^m$,*

$$\int_{2Q} \omega_{\vec{y}}(x) \mathrm{d}x \le C \int_Q \omega_{\vec{y}}(x) \mathrm{d}x, \tag{3.1}$$

where $2Q$ is the cube concentric with Q, having twice the side length of Q.
If $C = 2^\beta$ is the smallest constant for which (3.1) holds, then we say that β is the doubling exponent of W.

The following lemma is obtained from Lemma 2.1 in [14] when $0 < p \le 1$, and from Lemma 5.3 in [34] and p. 196 in [29] when $p > 1$.

Lemma 3.7. *Let $p \in (0, \infty)$. If $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, then W is a doubling matrix weight of order p.*

The following definition is Definition 2.14 in [35].

Definition 3.8. *Let $m \in \mathbb{N}$, $p \in (0,\infty)$, and W be a matrix weight from $\mathbb{R}^n$ to $M(\mathbb{C}^m)$. A sequence of positive definite $m \times m$ matrices $\{A_Q\}_{Q\in\mathcal{D}}$ is called a sequence of reducing operators of order p for W if there exist positive constants C_1 and C_2 such that for any $\vec{z} \in \mathbb{C}^m$ and $Q \in \mathcal{D}$,*

$$C_1|A_Q\vec{z}| \le \left(⨍_Q |W^{\frac{1}{p}}(x)\vec{z}|^p \mathrm{d}x\right)^{\frac{1}{p}} \le C_2|A_Q\vec{z}|.$$

The following definition is Definition 2.1 in [15].

Definition 3.9. *Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of positive definite matrices, $\beta \in (0,\infty)$, $p \in (0,\infty)$, and $r \in (0,\infty)$.*
(i) If there exists a positive constant C such that for any $Q, P \in \mathcal{D}$,

$$\|A_QA_P^{-1}\|^p \le C \max\left\{\left(\frac{\ell(P)}{\ell(Q)}\right)^n, \left(\frac{\ell(Q)}{\ell(P)}\right)^{\beta-n}\right\}\left(1+\frac{|x_Q-x_P|}{\max\{\ell(P),\ell(Q)\}}\right)^\beta,$$

then we say that the sequence $\{A_Q\}_{Q\in\mathcal{D}}$ is strongly doubling of order (β, p).
(ii) If there exists a positive constant C such that for any $k, \ell \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$,

$$\|A_{Q_{jk}}A_{Q_{j\ell}}^{-1}\| \le C(1+|k-\ell|)^r,$$

then we say that the sequence $\{A_Q\}_{Q\in\mathcal{D}}$ is weakly doubling of order r.

Remark 3.10. *By the definition above, if the sequence $\{A_Q\}_{Q\in\mathcal{D}}$ is strongly doubling of order $(\beta,\ p)$, then by the arbitrariness of P and Q, taking $\ell(P) = \ell(Q) = 2^{-j}$, we have $\|A_{Q_{jk}}A_{Q_{j\ell}}^{-1}\| \le C(1+|k-\ell|)^{\beta/p}$, that is, $\{A_Q\}_{Q\in\mathcal{D}}$ is also weakly doubling of order $r := \beta/p$.*

The following lemma explains the connection between doubling weights W and doubling sequences $\{A_Q\}_{Q\in\mathcal{D}}$.

Lemma 3.11. *(Lemma 2.2 in [15]) Let $p \in (0,\infty)$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. Then, $\{A_Q\}_{Q\in\mathcal{D}}$ is weakly doubling of order β/p, where β is the doubling exponent of W.*

Lemma 3.12. *(Lemmas 3.2 and 3.3 in [15]) Let $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. Then there exists $\delta > 0$ such that*
(i) $p \in (0,1]$,

$$\sup_{Q\in\mathcal{D}} \operatorname*{ess\,sup}_{x\in Q} \|A_QW^{-\frac{1}{p}}(x)\| < \infty.$$

(ii) $p \in (1,\infty)$,

$$\sup_{Q\in\mathcal{D}} ⨍_Q \|A_QW^{-\frac{1}{p}}(x)\|^\eta \mathrm{d}x < \infty,\ \eta < p' + \delta.$$

(iii) $p \in (0,\infty)$,

$$\sup_{Q\in\mathcal{D}} ⨍_Q \|W^{\frac{1}{p}}(x)A_Q^{-1}\|^\eta \mathrm{d}x < \infty,\ \eta < p + \delta.$$

(iv) $p \in (0, \infty)$,

$$\sup_{Q\in\mathcal{D}} \fint_Q \sup_{P\in\mathcal{D}: x\in P\subseteq Q} \|W^{\frac{1}{p}}(x)A_P^{-1}\|^{\eta} \mathrm{d}x < \infty,\ \eta < p+\delta.$$

4. Definition of weakly homogeneous matrix-weighted Triebel-Lizorkin spaces

To introduce the weakly homogeneous matrix-weighted Triebel-Lizorkin spaces, we need the following definition.

Definition 4.1. *Let the function $\varphi : \mathbb{R}^n \to \mathbb{C}$ satisfy the following conditions:*
(T1) $\varphi \in \mathcal{S}(\mathbb{R}^n)$;
(T2) $\operatorname{supp} \hat{\varphi} \subseteq \{\xi : \ 1/2 \le |\xi| \le 2\}$;
(T3) $|\hat{\varphi}(\xi)| \ge C > 0,\ 3/5 \le |\xi| \le 5/3$.
Then φ is called admissible, denoted by $\varphi \in \mathcal{A}$.

For simplicity, in the following content, let $\mathbf{\Phi}$ denote the collection of sequences $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ generated by all φ satisfying (T1), (T2), and (T3). Here, $\varphi_j(\cdot) := 2^{jn}\varphi(2^j\cdot)$.

Definition 4.2. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and let W be a matrix weight from $\mathbb{R}^n$ to $M(\mathbb{C}^m)$. Suppose $\{\varphi_j\}_{j\in\mathbb{Z}_+} \in \mathbf{\Phi}$. Then the weakly homogeneous matrix-weighted Triebel-Lizorkin space is defined as*

$$\dot{F}^{\alpha,q}_{p,\infty}(W) := \Big\{\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m : \ \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} < \infty\Big\},$$

where

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} := \left\|\Big\{\sum_{j\in\mathbb{Z}} |2^{j\alpha}W^{1/p}(\varphi_j * \vec{f})|^q\Big\}^{\frac{1}{q}}\right\|_{L^{p,\infty}}.$$

The usual modification is made when $q = \infty$.

Definition 4.3. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of $m \times m$ non-negative definite matrices. Suppose $\{\varphi_j\}_{j\in\mathbb{Z}_+} \in \mathbf{\Phi}$. Then the weak-$\{A_Q\}$ Triebel-Lizorkin space is defined as*

$$\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\}) := \Big\{\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m : \ \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} < \infty\Big\},$$

where

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} := \left\|\Big\{\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} \big(2^{j\alpha}|A_Q\varphi_j * \vec{f}|\mathbf{1}_Q\big)^q\Big\}^{\frac{1}{q}}\right\|_{L^{p,\infty}}.$$

The usual modification is made when $q = \infty$.

Definition 4.4. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, then the scalar, unweighted weak discrete Triebel-Lizorkin space $\dot{f}^{\alpha,q}_{p,\infty}$ is the set of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{D}}$ such that*

$$\|s\|_{\dot{f}^{\alpha,q}_{p,\infty}} = \left\|\Big\{\sum_{Q}(|Q|^{-\alpha/n-1/2}|s_Q|\mathbf{1}_Q)^q\Big\}^{1/q}\right\|_{L^{p,\infty}} < \infty.$$

Motivated by the definition of the scalar, unweighted weak discrete Triebel-Lizorkin space $\dot{f}^{\alpha,q}_{p,\infty}$, we provide the definitions of the matrix-weighted weak discrete Triebel-Lizorkin space and the weak discrete $\{A_Q\}$-Triebel-Lizorkin space, where A_Q is a reducing operator for W.

Definition 4.5. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and W be a matrix weight from $\mathbb{R}^n$ to $M(\mathbb{C}^m)$, and let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of $m\times m$ non-negative definite matrices. The matrix-weighted weak discrete Triebel-Lizorkin space $\dot{f}^{\alpha,q}_{p,\infty}(W)$ is the set of all sequences $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$ such that*

$$\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(W)} = \left\|\Big(\sum_{Q\in\mathcal{D}} (|Q|^{-\alpha/n-1/2}|W^{1/p}\vec{s}_Q|\mathbf{1}_Q)^q\Big)^{1/q}\right\|_{L^{p,\infty}} < \infty.$$

The weak discrete $\{A_Q\}$-Triebel-Lizorkin space $\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})$ is the set of all sequences $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$ such that

$$\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} = \left\|\Big(\sum_{Q\in\mathcal{D}} (|Q|^{-\alpha/n-1/2}|A_Q\vec{s}_Q|\mathbf{1}_Q)^q\Big)^{1/q}\right\|_{L^{p,\infty}} < \infty.$$

The following Lemmas 4.6 and 4.7 respectively correspond to (2.8) and (2.9) in [15].

Lemma 4.6. *Let $\{\varphi_j\}_{j\in\mathbb{Z}_+} \in \mathbf{\Phi}$. Suppose that $\{A_Q\}_{Q\in\mathcal{D}}$ is a sequence of positive definite matrices that is weakly doubling of order $r \in (0,\infty)$. Then, for any given $A \in (0,1]$, $R \in (0,\infty)$, there exists a positive constant C depending on A and R such that for any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,*

$$\sup_{x\in Q_{jk}} |A_{Q_{jk}}(\varphi_j * \vec{f})(x)|^A \le C \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(R-r)} \times 2^{jn}\int_{Q_{j\ell}} |A_{Q_{j\ell}}\varphi_j * \vec{f}(s)|^A \mathrm{d}s.$$

Lemma 4.7. *Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator, and let $\eta > n$. Then there exists a positive constant C such that for any $j \in \mathbb{Z}$ and any complex-valued measurable function h on $\mathbb{R}^n$,*

$$\sum_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-\eta}2^{jn}\int_{Q_{j\ell}} |h(s)|\mathrm{d}s\mathbf{1}_{Q_{jk}} \le C\mathcal{M}(h).$$

The following lemma is Corollary 3.8 of the reference [15].

Lemma 4.8. *Let $0 < p < \infty$, $0 < q \le \infty$, and $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$. Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. For any $j \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and $f \in L^1_{loc}(\mathbb{R}^n)$, define*

$$\gamma_j(x) := \sum_{Q\in\mathcal{D}_j} \|W^{\frac{1}{p}}(x)A_Q^{-1}\|\mathbf{1}_Q(x);$$

and

$$E_j(f) := \sum_{Q\in\mathcal{D}_j} \left(\fint_Q f(y)\mathrm{d}y \right) \mathbf{1}_Q.$$

Then there exists a positive constant C such that for any sequence of measurable functions $\{f_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}^n$,

$$\|\{\gamma_j E_j(f_j)\}_{j\in\mathbb{Z}}\|_{L^p(\ell^q)} \le C\|\{E_j(f_j)\}_{j\in\mathbb{Z}}\|_{L^p(\ell^q)}.$$

Based on the above results, we have the following Lemma 4.9.

Lemma 4.9. *Let $0 < p < \infty$, $0 < q \le \infty$, and $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$. Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. For any $j \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and $f \in L^1_{loc}(\mathbb{R}^n)$, define*

$$\gamma_j(x) := \sum_{Q\in\mathcal{D}_j} \|W^{\frac{1}{p}}(x)A_Q^{-1}\|\mathbf{1}_Q(x);$$

and

$$E_j(f) := \sum_{Q\in\mathcal{D}_j} \left(\fint_Q f(y)\mathrm{d}y \right) \mathbf{1}_Q.$$

Then there exists a positive constant C such that for any sequence of measurable functions $\{f_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}^n$,

$$\|\{\gamma_j E_j(f_j)\}_{j\in\mathbb{Z}}\|_{L^{p,\infty}(\ell^q)} \le C\|\{E_j(f_j)\}_{j\in\mathbb{Z}}\|_{L^{p,\infty}(\ell^q)}.$$

Proof. For $0 < p < \infty$, $0 < q \le \infty$. Fix q, and choose $0 < p_1 < p < p_2 < \infty$. Let $\vec{F} = \{E_j(f_j)\}$ and $|\vec{F}| = \|\{E_j(f_j)\}\|_{\ell^q}$, at height $\alpha > 0$, split $\vec{F}$, defining $\vec{F}_\alpha = \vec{F}\mathbf{1}_{|\vec{F}|>\alpha}$, $\vec{F}^\alpha = \vec{F}\mathbf{1}_{|\vec{F}|\le\alpha}$, and $\vec{F} = \vec{F}_\alpha + \vec{F}^\alpha$. It is easy to verify that

$$|\{|\vec{F}_\alpha| > \lambda\}| = \begin{cases} D_{\vec{F}}(\lambda), & \lambda > \alpha, \\ D_{\vec{F}}(\alpha), & \lambda \le \alpha, \end{cases}$$

and

$$|\{|\vec{F}^\alpha| > \lambda\}| = \begin{cases} 0, & \lambda > \alpha, \\ D_{\vec{F}}(\lambda) - D_{\vec{F}}(\alpha), & \lambda \le \alpha, \end{cases}$$

where $D_{\vec{F}}(\lambda) = |\{|\vec{F}| > \lambda\}|$. Therefore,

$$\begin{aligned} \||\vec{F}_\alpha|\|_{L^{p_1}}^{p_1} &= p_1 \int_0^\infty \lambda^{p_1-1} D_{\vec{F}_\alpha}(\lambda)\mathrm{d}\lambda \\ &= p_1\Big(\int_0^\alpha \lambda^{p_1-1} D_{\vec{F}_\alpha}(\lambda)\mathrm{d}\lambda + \int_\alpha^\infty \lambda^{p_1-1} D_{\vec{F}_\alpha}(\lambda)\mathrm{d}\lambda \Big) \\ &= p_1\Big(\int_0^\alpha \lambda^{p_1-1} D_{\vec{F}}(\alpha)\mathrm{d}\lambda + \int_\alpha^\infty \lambda^{p_1-1} D_{\vec{F}}(\lambda)\mathrm{d}\lambda \Big) \\ &\le p_1\Big(D_{\vec{F}}(\alpha)\frac{\alpha^{p_1}}{p_1} + \||\vec{F}|\|_{L^{p,\infty}}^p \int_\alpha^\infty \lambda^{p_1-1-p}\mathrm{d}\lambda \Big) \end{aligned}$$

$$\begin{aligned}&\le p_1\Big(|||\vec{F}|||^p_{L^{p,\infty}}\frac{\alpha^{p_1-p}}{p_1}-|||\vec{F}|||^p_{L^{p,\infty}}\frac{\alpha^{p_1-p}}{p_1-p}\Big)\\&=\frac{p}{p-p_1}\alpha^{p_1-p}|||\vec{F}|||^p_{L^{p,\infty}}.\end{aligned}\tag{4.1}$$

Similarly, the same computation applied to $\vec{F}^\alpha$ gives

$$\begin{aligned}|||\vec{F}^\alpha|||^{p_2}_{L^{p_2}}&=p_2\int_0^\infty\lambda^{p_2-1}D_{\vec{F}^\alpha}(\lambda)\mathrm{d}\lambda\\&=p_2\Big(\int_0^\alpha\lambda^{p_2-1}D_{\vec{F}^\alpha}(\lambda)\mathrm{d}\lambda+\int_\alpha^\infty\lambda^{p_2-1}D_{\vec{F}^\alpha}(\lambda)\mathrm{d}\lambda\Big)\\&=p_2\int_0^\alpha\lambda^{p_2-1}\big(D_{\vec{F}}(\lambda)-D_{\vec{F}}(\alpha)\big)\mathrm{d}\lambda\\&\le p_2\Big(|||\vec{F}|||^p_{L^{p,\infty}}\frac{\alpha^{p_2-p}}{p_2-p}-D_{\vec{F}}(\alpha)\frac{\alpha^{p_2}}{p_2}\Big)\\&=\frac{p_2}{p_2-p}\alpha^{p_2-p}|||\vec{F}|||^p_{L^{p,\infty}}-D_{\vec{F}}(\alpha)\alpha^{p_2}\\&\le\frac{p_2}{p_2-p}\alpha^{p_2-p}|||\vec{F}|||^p_{L^{p,\infty}}.\end{aligned}\tag{4.2}$$

Then by Lemma 4.8, (4.1), (4.2), we have

$$\begin{aligned}&\Big|\Big\{\big\|\{\gamma_jE_j(f_j)\}\big\|_{\ell^q}>\lambda\Big\}\Big|\\&\le\Big|\Big\{\Big\|\Big\{\gamma_jE_j(f_j)\mathbf{1}_{|\vec{F}|>\alpha}\Big\}\Big\|_{\ell^q}>\frac{\lambda}{2}\Big\}\Big|+\Big|\Big\{\Big\|\Big\{\gamma_jE_j(f_j)\mathbf{1}_{|\vec{F}|\le\alpha}\Big\}\Big\|_{\ell^q}>\frac{\lambda}{2}\Big\}\Big|\\&\le C_{p_1,q}\left(\frac{2}{\lambda}\right)^{p_1}\int_{\mathbb{R}^n}\Big\|\Big\{\gamma_jE_j(f_j)\mathbf{1}_{|\vec{F}|>\alpha}\Big\}\Big\|^{p_1}_{\ell^q}\mathrm{d}x\\&\qquad+C_{p_2,q}\left(\frac{2}{\lambda}\right)^{p_2}\int_{\mathbb{R}^n}\Big\|\Big\{\gamma_jE_j(f_j)\mathbf{1}_{|\vec{F}|\le\alpha}\Big\}\Big\|^{p_2}_{\ell^q}\mathrm{d}x\\&=C_{p_1,q}\left(\frac{2}{\lambda}\right)^{p_1}\Big\|\Big\|\Big\{\gamma_jE_j(f_j)\mathbf{1}_{|\vec{F}|>\alpha}\Big\}\Big\|_{\ell^q}\Big\|^{p_1}_{L^{p_1}}\\&\qquad+C_{p_2,q}\left(\frac{2}{\lambda}\right)^{p_2}\Big\|\Big\|\Big\{\gamma_jE_j(f_j)\mathbf{1}_{|\vec{F}|\le\alpha}\Big\}\Big\|_{\ell^q}\Big\|^{p_2}_{L^{p_2}}\\&\le C_{p_1,q}\left(\frac{2}{\lambda}\right)^{p_1}\Big\|\Big\|\Big\{E_j(f_j)\mathbf{1}_{|\vec{F}|>\alpha}\Big\}\Big\|_{\ell^q}\Big\|^{p_1}_{L^{p_1}}\\&\qquad+C_{p_2,q}\left(\frac{2}{\lambda}\right)^{p_2}\Big\|\Big\|\Big\{E_j(f_j)\mathbf{1}_{|\vec{F}|\le\alpha}\Big\}\Big\|_{\ell^q}\Big\|^{p_2}_{L^{p_2}}\\&\le C_{p_1,q}\left(\frac{2}{\lambda}\right)^{p_1}\frac{p}{p-p_1}\alpha^{p_1-p}|||\vec{F}|||^p_{L^{p,\infty}}\\&\qquad+C_{p_2,q}\left(\frac{2}{\lambda}\right)^{p_2}\frac{p_2}{p_2-p}\alpha^{p_2-p}|||\vec{F}|||^p_{L^{p,\infty}}\end{aligned}$$

$$\leq \left(\frac{p}{p-p_1}+\frac{p_2}{p_2-p}\right)2^{p_2}(C_{p_1,q})^{\frac{p_2-p}{p_2-p_1}}(C_{p_2,q})^{\frac{p-p_1}{p_2-p_1}}\lambda^{-p}|||\vec{F}|||^p_{L^{p,\infty}},$$

setting $\alpha=\lambda\gamma$, $\gamma=\left(\frac{C_{p_1,q}}{C_{p_2,q}}\right)^{\frac{1}{p_2-p_1}}$. Then, there exists a constant $C>0$, such that for any sequence of measurable functions $\{f_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}^n$,

$$\|\{\gamma_j E_j(f_j)\}\|_{L^{p,\infty}(\ell^q)}\leq C\|\{E_j(f_j)\}\|_{L^{p,\infty}(\ell^q)}. \qquad \square$$

The following lemma is the Fefferman-Stein vector-valued maximal inequality in weak Lebesgue spaces. Its proof can be found in Proposition 4 of reference [17].

Lemma 4.10. *Let $p\in(1,\infty)$, and $q\in(1,\infty]$. Then, there exists a positive constant C such that for any sequence of measurable functions $\{f_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}^n$,*

$$\left\|\left\{\sum_{j\in\mathbb{Z}}\left(\mathcal{M}(f_j)\right)^q\right\}^{\frac{1}{q}}\right\|_{L^{p,\infty}}\leq C\left\|\left\{\sum_{j\in\mathbb{Z}}|f_j|^q\right\}^{\frac{1}{q}}\right\|_{L^{p,\infty}}.$$

The following lemma is extremely useful. It states that for $\delta\in(0,1)$, if for each cube Q, $E_Q\subset Q$, and $|E_Q|\geq\delta|Q|$, then $\mathbf{1}_{E_Q}$ can replace $\mathbf{1}_Q$ in the space $\dot{f}^{\alpha,q}_{p,\infty}$.

Lemma 4.11. *Let $\alpha\in\mathbb{R}$, $p\in(0,\infty)$, $q\in(0,\infty]$, $\delta\in(0,1)$. Suppose for any dyadic cube $Q\in\mathcal{D}$, $E_Q\subset Q$ is a measurable set and $|E_Q|\geq\delta|Q|$. Then for any sequence $s=\{s_Q\}_{Q\in\mathcal{D}}$,*

$$\|s\|_{\dot{f}^{\alpha,q}_{p,\infty}}\sim\left\|\left(\sum_Q\left(|Q|^{-\alpha/n}|s_Q|\widetilde{\mathbf{1}}_{E_Q}\right)^q\right)^{1/q}\right\|_{L^{p,\infty}},$$

where $\widetilde{\mathbf{1}}_{E_Q}=|E_Q|^{-1/2}\mathbf{1}_{E_Q}$.

Proof. Since $\widetilde{\mathbf{1}}_{E_Q}\leq\delta^{-1/2}\mathbf{1}_Q$, it follows that

$$\left\|\left(\sum_Q\left(|Q|^{-\alpha/n}|s_Q|\widetilde{\mathbf{1}}_{E_Q}\right)^q\right)^{1/q}\right\|_{L^{p,\infty}}\leq C\|s\|_{\dot{f}^{\alpha,q}_{p,\infty}}.$$

Furthermore, for all $A>0$, $\mathbf{1}_Q\leq\delta^{-1/A}(\mathcal{M}(\mathbf{1}^A_{E_Q}))^{1/A}$, where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Choose $A\in(0,1)$ small enough so that $p/A>1$, $q/A>1$, by the Fefferman-Stein vector-valued maximal inequality in weak Lebesgue spaces, we have

$$\begin{aligned}\|s\|_{\dot{f}^{\alpha,q}_{p,\infty}}&\leq\delta^{-1/A}\left\|\left(\sum_Q\left(\mathcal{M}\left(|Q|^{-\alpha/n}|s_Q|\widetilde{\mathbf{1}}_{E_Q}\right)^A\right)^{q/A}\right)^{A/q}\right\|^{1/A}_{L^{p/A,\infty}}\\&\lesssim\delta^{-1/A}\left\|\left(\sum_Q\left(|Q|^{-\alpha/n}|s_Q|\widetilde{\mathbf{1}}_{E_Q}\right)^q\right)^{1/q}\right\|_{L^{p,\infty}}.\end{aligned}$$

Combining these results, we obtain

$$\|s\|_{\dot f^{\alpha,q}_{p,\infty}} \sim \Big\|\Big(\sum_{Q}\big(|Q|^{-\alpha/n}|s_Q|\widetilde{\mathbf{1}}_{E_Q}\big)^q\Big)^{1/q}\Big\|_{L^{p,\infty}}. \qquad \square$$

Theorem 4.12. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$. Suppose $\{A_Q\}_{Q\in\mathcal{D}}$ is a sequence of reducing operators of order p for W. Then there exists a constant C such that for any sequence $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$,*

$$\|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(W)}.$$

Proof. (i) Next, we begin by proving: $\|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(\{A_Q\})}$. For $j \in \mathbb{Z}$, define $\gamma_j(x)$ and $E_j(f)$ as in Lemma 4.9. For $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$, define

$$f_j = \sum_{Q\in\mathcal{D}_j} |Q|^{-\alpha/n-1/2}|A_Q\vec{s}_Q|\mathbf{1}_Q.$$

Note that on each $Q \in \mathcal{D}_j$, f_j is constant. Thus, $E_j(f_j) = f_j$. Let

$$g_j := \sum_{Q\in\mathcal{D}_j} |Q|^{-\alpha/n-1/2}|W^{1/p}\vec{s}_Q|\mathbf{1}_Q.$$

Then we have

$$g_j \le \sum_{Q\in\mathcal{D}_j} |Q|^{-\alpha/n-1/2}\|W^{-1/p}A_Q^{-1}\||A_Q\vec{s}_Q|\mathbf{1}_Q = \gamma_j f_j = \gamma_j E_j(f_j).$$

From this and Lemma 4.9, we deduce that

$$\begin{aligned}\|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(W)} = \|\{g_j\}\|_{L^{p,\infty}(\ell^q)} \le \|\{\gamma_j E_j(f_j)\}\|_{L^{p,\infty}(\ell^q)} \lesssim \|\{E_j(f_j)\}\|_{L^{p,\infty}(\ell^q)}\\ = \|\{f_j\}\|_{L^{p,\infty}(\ell^q)} = \|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(\{A_Q\})}.\end{aligned}$$

(ii) To prove that $\|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(W)}$, it remains to show

$$\|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(W)}.$$

Below, we consider two cases for p:

Case 1: $0 < p \le 1$. By (i) of Lemma 3.12,

$$|A_Q\vec{s}_Q|\mathbf{1}_Q \le \|A_Q W^{-1/p}\||W^{1/p}\vec{s}_Q|\mathbf{1}_Q \le C|W^{1/p}\vec{s}_Q|\mathbf{1}_Q \ a.e.,$$

which implies $\|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{s}\|_{\dot f^{\alpha,q}_{p,\infty}(W)}$.

Case 2: $p > 1$. Select $\eta = 1$ in (ii) of Lemma 3.12 to obtain

$$\sup_{Q\in\mathcal{D}} \fint_Q \|A_Q W^{-1/p}(x)\mathrm{d}x \le C.$$

For any $Q \in \mathcal{D}$, define the set

$$E_Q := \{x \in Q : \ \|A_Q W^{-1/p}(x)\| \le 2C\},$$

based on this, Chebyshev's inequality yields

$$|Q \setminus E_Q| \le \int_{Q\setminus E_Q} \frac{\|A_Q W^{-1/p}(x)\|}{2C}\mathrm{d}x \le \frac{|Q|}{2C}\fint_Q \|A_Q W^{-1/p}(x)\|\mathrm{d}x \le \frac{1}{2}|Q|.$$

Therefore, we have $|E_Q| \geq 1/2|Q|$, and for this we can apply Lemma 4.11 and the inequality $|A_Q\vec{s}_Q| \leq \|A_Q W^{-1/p}\||W^{1/p}\vec{s}_Q|$,

$$\begin{aligned}\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} &\sim \left\| \Big(\sum_{Q\in\mathcal{D}} \big(|Q|^{-\alpha/n-1/2}|A_Q\vec{s}_Q|\mathbf{1}_{E_Q}\big)^q \Big)^{1/q} \right\|_{L^{p,\infty}} \\ &\lesssim \left\| \Big(\sum_{Q\in\mathcal{D}} \big(|Q|^{-\alpha/n-1/2}|W^{1/p}\vec{s}_Q|\mathbf{1}_{E_Q}\big)^q \Big)^{1/q} \right\|_{L^{p,\infty}} \\ &\sim \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(W)}. \end{aligned}$$

□

5. Boundedness of almost diagonal matrices

First, recall the definition of scalar, unweighted almost diagonal operators (see equation (3.1) in [12]).

Definition 5.1. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$. A matrix $A = \{a_{QP}\}_{Q,P}$ is called almost diagonal if there exists $\epsilon > 0$ such that the matrix satisfies*

$$\sup_{Q,P} |a_{QP}|/\omega_{QP}(\epsilon) < \infty, \tag{5.1}$$

denoted as $A \in \mathbf{ad}^{\alpha,\mathbf{q}}_{\mathbf{p}}(\epsilon)$, where

$$\omega_{QP}(\epsilon) = \left(\frac{\ell(Q)}{\ell(P)}\right)^{\alpha} \left(1 + \frac{|x_Q - x_P|}{\max(\ell(P),\ \ell(Q))}\right)^{-n/\min(1,\ p,\ q)-\epsilon} \times \min\left(\left(\frac{\ell(Q)}{\ell(P)}\right)^{(n+\epsilon)/2},\ \left(\frac{\ell(P)}{\ell(Q)}\right)^{(n+\epsilon)/2+n/\min(1,\ p,\ q)-n}\right).$$

For the needs of subsequent proofs, we will also require a weighted version of almost diagonal matrices. This differs from the above definition of almost diagonality because it involves the doubling exponent β. The definition was first introduced in [28].

Definition 5.2. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and $\beta \geq n$. A matrix $B = \{b_{QP}\}_{Q,P\in\mathcal{D}}$ is called almost diagonal if there exists $C > 0$ such that for all $Q, P \in \mathcal{D}$,*

$$|b_{QP}| \leq C\omega_{QP},$$

denoted by $B \in \mathbf{ad}^{\alpha,\mathbf{q}}_{\mathbf{p}}(\beta)$. Here,

$$\omega_{QP} = \left(1 + \frac{|x_Q - x_P|}{\max(\ell(Q),\ell(P))}\right)^{-R} \min\left\{\left(\frac{\ell(P)}{\ell(Q)}\right)^{\alpha_1} \left(\frac{\ell(Q)}{\ell(P)}\right)^{\alpha_2}\right\} \tag{5.2}$$

for some

$$\begin{cases} \alpha_1 > -\alpha - n/2 + (\beta - n)/p + n/\min(1,p,q), \\ \alpha_2 > \alpha + n/2 + n/p, \\ R > n/\min(1,p,q) + \beta/p. \end{cases}$$

Remark 5.3. *The matrix $B = \{b_{QP}\}_{Q,P\in\mathcal{D}}$ acts on a sequence $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$ via componentwise matrix multiplication: $B\vec{s} = \vec{t} = \{\vec{t}_Q\}_{Q\in\mathcal{D}}$, where $\vec{t}_Q = \sum_{P\in\mathcal{D}} b_{QP}\vec{s}_P$.*

We now first consider the boundedness of scalar almost diagonal operators on the scalar, unweighted weak discrete Triebel-Lizorkin space $\dot{f}^{\alpha,q}_{p,\infty}$.

Lemma 5.4. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. Then on the scalar, unweighted weak discrete Triebel-Lizorkin space $\dot{f}^{\alpha,q}_{p,\infty}$, almost diagonal operators are bounded.*

Proof. Without loss of generality, we may assume $\alpha = 0$, since this case implies the general case. Let $J = \min(1, p, q) - \delta$, where $\delta > 0$ is sufficiently small. Suppose $B = \{b_{QP}\}_{Q,P\in\mathcal{D}}$ is an almost diagonal operator on $\dot{f}^{\alpha,q}_{p,\infty}$. We first decompose the matrix $B = \{b_{QP}\}_{Q,P\in\mathcal{D}}$ as follows:

$$(Bs)_Q = \sum_P b_{QP}s_P = \sum_{\ell(P)\leq\ell(Q)} b_{QP}s_P + \sum_{\ell(P)>\ell(Q)} b_{QP}s_P =: (B_0 s)_Q + (B_1 s)_Q.$$

If $\ell(Q) = 2^{-v}$, and $x \in Q$, then by $B \in \mathbf{ad}^{\alpha,\mathbf{q}}_{\mathbf{p}}(\epsilon)$ and Lemma A.2 in [12], we obtain

$$\begin{aligned}\sum_{\ell(P)>\ell(Q)} b_{QP}s_P &\lesssim \sum_{\mu<v}\sum_{\ell(P)=2^{-\mu}} 2^{(\mu-v)(n+\epsilon)/2}(1+2^{\mu}|x_Q - x_P|)^{-n/J-\epsilon}|s_p| \\ &\lesssim \sum_{\mu<v} 2^{(\mu-v)(n+\epsilon)/2}\Big\{\mathcal{M}\Big(\sum_{\ell(P)=2^{-\mu}}|s_P|^J\mathbf{1}_P\Big)(x)\Big\}^{1/J}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\{\sum_{v\in\mathbb{Z}}\sum_{\ell(Q)=2^{-v}}\left(|Q|^{-\frac{1}{2}}|(B_1 s)_Q|\mathbf{1}_Q\right)^q\right\}^{\frac{1}{q}} \\ &\lesssim \left\{\sum_{v\in\mathbb{Z}}\sum_{\ell(Q)=2^{-v}}\left(2^{\frac{vn}{2}}\sum_{\mu<v}2^{(\mu-v)\frac{n+\epsilon}{2}}\left(\mathcal{M}\Big(\sum_{\ell(P)=2^{-\mu}}|s_P|^J\mathbf{1}_P\Big)(x)\right)^{\frac{1}{J}}\right)^q\right\}^{\frac{1}{q}} \\ &= \left\{\sum_{\mu\in\mathbb{Z}}\sum_{v=\mu+1}^{\infty}\left(2^{\frac{vn}{2}}2^{(\mu-v)\frac{n+\epsilon}{2}}\left(\mathcal{M}\Big(\sum_{\ell(P)=2^{-\mu}}|s_P|^J\mathbf{1}_P\Big)(x)\right)^{\frac{1}{J}}\right)^q\right\}^{\frac{1}{q}}.\end{aligned}$$

Combining this with the Fefferman-Stein vector-valued maximal inequality in $L^{p,\infty}$, we immediately arrive at the following conclusion,

$$
\begin{aligned}
&\|B_1 s\|_{\dot{f}^{0,q}_{p,\infty}} \\
&= \left\| \left\{ \sum_{v\in\mathbb{Z}} \sum_{\ell(Q)=2^{-v}} \left(|Q|^{-\frac{1}{2}} |(B_1 s)_Q| \mathbf{1}_Q \right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&\lesssim \left\| \left\{ \sum_{\mu\in\mathbb{Z}} \sum_{v=\mu+1}^{\infty} \left(2^{\frac{vn}{2}} 2^{(\mu-v)\frac{n+\epsilon}{2}} \left(\mathcal{M}\Big(\sum_{\ell(P)=2^{-\mu}} |s_P|^J \mathbf{1}_P \Big)(x) \right)^{\frac{1}{J}} \right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&\lesssim \left\| \left\{ \sum_{\mu\in\mathbb{Z}} \sum_{\ell(P)=2^{-\mu}} \left(|P|^{-\frac{1}{2}} |s_P| \mathbf{1}_P \right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} = \|s\|_{\dot{f}^{0,q}_{p,\infty}}.
\end{aligned}
$$

We now estimate $B_0 s$. For $x \in Q$, since $B \in \mathbf{ad}^{\alpha,\mathbf{q}}_{\mathbf{p}}(\epsilon)$, we have

$$
\begin{aligned}
\sum_{\ell(P)\le\ell(Q)} b_{QP} s_P \lesssim \sum_{\mu\ge v} \sum_{\ell(P)=2^{-\mu}} & 2^{(v-\mu)((n+\epsilon)/2+n/J-n)} \\
& \times (1+2^v|x_Q-x_P|)^{-n/J-\epsilon} |s_P|,
\end{aligned}
$$

thus, by Lemma A.3 in [12], we obtain

$$
\begin{aligned}
&\left\{ \sum_{v\in\mathbb{Z}} \sum_{\ell(Q)=2^{-v}} \left(|Q|^{-\frac{1}{2}} |(B_0 s)_Q| \mathbf{1}_Q \right)^q \right\}^{\frac{1}{q}} \\
&\quad \lesssim \left\{ \sum_{v\in\mathbb{Z}} \left(\sum_{\ell(Q)=2^{-v}} |Q|^{-\frac{1}{2}} \sum_{\mu\ge v} \sum_{\ell(P)=2^{-\mu}} 2^{(v-\mu)(\frac{n+\epsilon}{2}+\frac{n}{J}-n)} \right.\right. \\
&\qquad\qquad \left.\left. \times (1+2^v|x_Q-x_P|)^{-\frac{n}{J}-\epsilon} |s_P| \mathbf{1}_Q \right)^q \right\}^{\frac{1}{q}} \\
&\quad = \left\{ \sum_{v\in\mathbb{Z}} \left(\sum_{\mu\ge v} \sum_{\ell(P)=2^{-\mu}} 2^{\frac{(v-\mu)n}{2}} |P|^{-\frac{1}{2}} 2^{(v-\mu)(\frac{n+\epsilon}{2}+\frac{n}{J}-n)} \right.\right. \\
&\qquad\qquad \left.\left. \times \sum_{\ell(Q)=2^{-v}} (1+2^v|x_Q-x_P|)^{-\frac{n}{J}-\epsilon} |s_P| \mathbf{1}_P \right)^q \right\}^{\frac{1}{q}}
\end{aligned}
$$

$$\lesssim \left\{ \sum_{v\in\mathbb{Z}} \left(\sum_{\mu\geq v} \sum_{\ell(P)=2^{-\mu}} 2^{\frac{(v-\mu)n}{2}} |P|^{-\frac{1}{2}} 2^{(v-\mu)(\frac{n+\varepsilon}{2}+\frac{n}{J}-n)} \right.\right.$$

$$\left.\left. \times \left(\mathcal{M}\Big(\sum_{\ell(P)=2^{-\mu}} |s_P|^J \mathbf{1}_P \Big) \mathbf{1}_P \right)^{\frac{1}{J}} \right)^q \right\}^{\frac{1}{q}}$$

$$\lesssim \left\{ \sum_{\mu\in\mathbb{Z}} \sum_{\ell(P)=2^{-\mu}} |P|^{-\frac{1}{2}} \left(\sum_{v=-\infty}^{\mu} 2^{(v-\mu)(\frac{\varepsilon}{2}+\frac{n}{J})} \right.\right.$$

$$\left.\left. \times \left(\mathcal{M}\Big(\sum_{\ell(P)=2^{-\mu}} |s_P|^J \mathbf{1}_P \Big) \mathbf{1}_P \right)^{\frac{1}{J}} \right)^q \right\}^{\frac{1}{q}}$$

$$\lesssim \left\{ \sum_{\mu\in\mathbb{Z}} \sum_{\ell(P)=2^{-\mu}} \left(|P|^{-\frac{1}{2}} \left(\mathcal{M}\Big(\sum_{\ell(P)=2^{-\mu}} |s_P|^J \mathbf{1}_P \Big) \mathbf{1}_P \right)^{\frac{1}{J}} \right)^q \right\}^{\frac{1}{q}}.$$

From this, and then applying the Fefferman-Stein vector-valued maximal inequality on the space $L^{p,\infty}$, we conclude that

$$\begin{aligned}
&\|B_0 s\|_{\dot f^{0,q}_{p,\infty}} \\
&= \left\| \left\{ \sum_{v\in\mathbb{Z}} \sum_{\ell(Q)=2^{-v}} \left(|Q|^{-\frac{1}{2}} |(B_0 s)_Q| \mathbf{1}_Q \right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&\lesssim \left\| \left\{ \sum_{\mu\in\mathbb{Z}} \sum_{\ell(P)=2^{-\mu}} \left(|P|^{-\frac{1}{2}} \left(\mathcal{M}\Big(\sum_{\ell(P)=2^{-\mu}} |s_P|^J \mathbf{1}_P \Big) \mathbf{1}_P \right)^{\frac{1}{J}} \right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&\lesssim \left\| \Big\{ \sum_{\mu\in\mathbb{Z}} \sum_{\ell(P)=2^{-\mu}} \left(|P|^{-\frac{1}{2}} |s_P| \mathbf{1}_P \right)^q \Big\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} = \|s\|_{\dot f^{0,q}_{p,\infty}}.
\end{aligned}$$

In summary, the almost diagonal operator B is bounded on the space $\dot f^{\alpha,q}_{p,\infty}$. □

Next, in the weighted case, consider the boundedness of almost diagonal matrices on the space $\dot f^{\alpha,q}_{p,\infty}(\{A_Q\})$.

Theorem 5.5. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$. Suppose the sequence of non-negative definite matrices $\{A_Q\}_{Q\in\mathcal{D}}$ is strongly doubling of order (β, p) for some $\beta > 0$. If $B \in \mathbf{ad}_{\mathbf{p}}^{\alpha,\mathbf{q}}(\beta)$, then B is bounded on the space $\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})$.*

Proof. For $\vec{s} \in \dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})$, define $\vec{t} = B\vec{s}$, $B = \{b_{QP}\}_{Q,P\in\mathcal{D}}$. Define a scalar sequence $t_A = \{t_{A,Q}\}_{Q\in\mathcal{D}}$ with $t_{A,Q} = |A_Q\vec{t_Q}|$, and similarly $s_A = \{s_{A,Q}\}_{Q\in\mathcal{D}}$ with $s_{A,Q} = |A_Q\vec{s_Q}|$. Then

$$\|\vec{t}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} = \|t_A\|_{\dot{f}^{\alpha,q}_{p,\infty}} \text{ and } \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} = \|s_A\|_{\dot{f}^{\alpha,q}_{p,\infty}},$$

where $\dot{f}^{\alpha,q}_{p,\infty}$ is the scalar, unweighted weak discrete Triebel-Lizorkin space defined earlier. Let $G = \{\gamma_{QP}\} = \left\{\omega_{QP}\|A_Q A_P^{-1}\|\right\}$, and from this, along with $\vec{t_Q} = \sum_{P\in\mathcal{D}} b_{QP}\vec{s_P}$, $B \in \mathbf{ad}_{\mathbf{p}}^{\alpha,\mathbf{q}}(\beta)$, we have

$$\begin{aligned} t_{A,Q} = |A_Q\vec{t_Q}| = |A_Q \sum_{P\in\mathcal{D}} b_{QP}\vec{s_P}| &\le \sum_{P\in\mathcal{D}} |b_{QP}||A_Q\vec{s_P}| \\ &\lesssim \sum_{P\in\mathcal{D}} \omega_{QP}\|A_Q A_P^{-1}\||A_P\vec{s_P}| = \sum_{P\in\mathcal{D}} \gamma_{QP} s_{A,P}. \end{aligned}$$

That is, $t_{A,Q} \lesssim (G(s_A))_Q$. Since $\{A_Q\}_{Q\in\mathcal{D}}$ is strongly doubling of order $(\beta,\ p)$, then γ_{QP} satisfies the scalar unweighted almost diagonal condition (5.1), which meaning G is a scalar unweighted almost diagonal operator. Therefore, by Lemma 5.4, it follows that G is bounded on $\dot{f}^{\alpha,q}_{p,\infty}$. Thus,

$$\|\vec{t}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} = \|t_A\|_{\dot{f}^{\alpha,q}_{p,\infty}} \lesssim \|G(s_A)\|_{\dot{f}^{\alpha,q}_{p,\infty}} \lesssim \|s_A\|_{\dot{f}^{\alpha,q}_{p,\infty}} = \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

Thus, B is bounded on the space $\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})$. □

6. Molecular characterization

Below, we recall the definitions of smooth molecules and atoms.

Definition 6.1. *Let $N \in \mathbb{Z}_+$. If*
(i) supp $a_Q \subseteq 3Q$,
(ii) $\int x^\gamma a_Q(x)\mathrm{d}x = 0,\ |\gamma| \le N$,
(iii) $|D^\gamma a_Q(x)| \le c_\gamma\, \ell(Q)^{-|\gamma|-n/2},\ |\gamma| \ge 0$.
then the function $a_Q \in \mathcal{D}(\mathbb{R}^n)$ is called a smooth N-atom.

Definition 6.2. *Let $0 < \delta \le 1$, $M > 0$, and N, $K \in \mathbb{Z}$, if there exist $\epsilon > 0$, $C > 0$, such that for all $Q \in \mathcal{D}$,*
(i) $\int x^\gamma m_Q(x)\mathrm{d}x = 0,\ |\gamma| \le N$,
(ii) $|m_Q(x)| \le C|Q|^{-\frac{1}{2}}\left(1 + \frac{|x-x_Q|}{\ell(Q)}\right)^{-\max(M,N+1+n+\epsilon)}$,
(iii) $|D^\gamma m_Q(x)| \le C|Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}}\left(1 + \frac{|x-x_Q|}{\ell(Q)}\right)^{-M},\ |\gamma| \le K$,
(iv) if $|\gamma| = K$,

$$|D^\gamma m_Q(x) - D^\gamma m_Q(y)| \le C|Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}-\frac{\delta}{n}}|x-y|^\delta$$

$$\times \sup_{|z|\le|x-y|}\left(1+\frac{|x-z-x_Q|}{\ell(Q)}\right)^{-M},$$

then $\{m_Q\}_{Q\in\mathcal{D}}$ is called a family of smooth $(N,\ K,\ M,\ \delta)$-molecules.

Obviously, a smooth atom is a molecule.

The following lemma is the discrete version of the Calderón reproducing formula (see Lemma 2.1 in [38]).

Lemma 6.3. *Let $\varphi\in\mathcal{A}$, and define $\hat{\psi}=\hat{\varphi}/(\sum_{j\in\mathbb{Z}}|\widehat{\varphi}_j|^2)$, then $\psi\in\mathcal{A}$, and for all $\xi\neq 0$, $\sum_{j\in\mathbb{Z}}\overline{\widehat{\varphi}_j}(\xi)\widehat{\psi}_j(\xi)=1$. Under this condition, assuming further that both $\operatorname{supp}\hat{\varphi}$, $\operatorname{supp}\hat{\psi}$ are compact and bounded away from the origin. Consequently, for any $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$,*

$$f=\sum_{j\in\mathbb{Z}}2^{-jn}\sum_{k\in\mathbb{Z}^n}(\widetilde{\varphi}_j*f)(2^{-j}k)\psi_j(x-2^{-j}k)=\sum_Q\langle f,\varphi_Q\rangle\psi_Q$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Theorem 6.4. *Let $\alpha\in\mathbb{R}$, $p\in(0,\infty)$, $q\in(0,\infty]$. Assume that the sequence of non-negative definite matrices $\{A_Q\}_{Q\in\mathcal{D}}$ is strongly doubling of order $(\beta,\ p)$. Let $0<\delta\le 1$, $M>0$, and $N,\ K\in\mathbb{Z}$, satisfy $N>-\alpha+(\beta-n)/p+n/\min(1,p,q)-n-1$, $K+\delta>\alpha+n/p$, $M>n/\min(1,p,q)+\beta/p$. Suppose $\{m_Q\}_{Q\in\mathcal{D}}$ is a family of smooth (N,K,M,δ)-molecules, $\vec{s}=\{\vec{s}_Q\}_{Q\in\mathcal{D}}\in\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})$, and $\vec{f}=\sum_{Q\in\mathcal{D}}\vec{s}_Q m_Q\in(\mathcal{S}'_\infty(\mathbb{R}^n))^m$, then $\vec{f}=\sum_{Q\in\mathcal{D}}\vec{s}_Q m_Q\in\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})$ and*

$$(\mathbf{a})\ \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}\lesssim\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

In particular, if $\vec{s}_Q=\langle\vec{f},\varphi_Q\rangle$, $\vec{f}\in(\mathcal{S}'_\infty(\mathbb{R}^n))^m$, $\varphi\in\mathcal{A}$, then

$$(\mathbf{b})\ \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}\lesssim\|\{\langle\vec{f},\varphi_Q\rangle\}_{Q\in\mathcal{D}}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

Proof. First, we prove the conclusion $(\mathbf{a})$. For $Q\in\mathcal{D}$, let

$$g_Q=|Q|^{1/2}|A_Q\varphi_j*\sum_{P\in\mathcal{D}}\vec{s}_P m_P|.$$

Then

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}=\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}}\left(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}g_Q\mathbf{1}_Q\right)^q\right\}^{\frac{1}{q}}\right\|_{L^{p,\infty}}.$$

Notice that for any $P,\ Q\in\mathcal{D},\ x\in Q$,

$$1+\frac{|x-x_P|}{\max\{\ell(P),\ell(Q)\}}\sim 1+\frac{|x_Q-x_P|}{\max\{\ell(P),\ell(Q)\}}.$$

Therefore, by Lemma 2.8 in [15] and (5.2), and for $x\in Q$, $|Q|^{1/2}|\varphi_j*m_P(x)|\lesssim\omega_{QP}$. Thus,

$$g_Q\mathbf{1}_Q\le\sum_{P\in\mathcal{D}}|Q|^{\frac{1}{2}}|\varphi_j*m_P||A_Q\vec{s}_P|\mathbf{1}_Q\lesssim\sum_{P\in\mathcal{D}}\omega_{QP}\|A_QA_P^{-1}\||A_P\vec{s}_P|\mathbf{1}_Q.$$

Define G and s_A as in Theorem 5.5, then by the above,

$$g_Q \mathbf{1}_Q \lesssim \sum_{P\in\mathcal{D}} \gamma_{QP} s_{A,P} \mathbf{1}_Q \lesssim (G(s_A))_Q \mathbf{1}_Q.$$

And by Lemma 5.4, we know that G is bounded on the scalar, unweighted space $\dot{f}^{\alpha,q}_{p,\infty}$, hence

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|G(s_A)\|_{\dot{f}^{\alpha,q}_{p,\infty}} \lesssim \|s_A\|_{\dot{f}^{\alpha,q}_{p,\infty}} = \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

We next prove the conclusion (**b**). For any $\varphi \in \mathcal{A}$, let $\hat{\psi} = \hat{\varphi}/(\sum_{j\in\mathbb{Z}} |\widehat{\varphi}_j|^2)$, by Lemma 6.3, for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, $f = \sum_{Q\in\mathcal{D}} \langle f, \varphi_Q\rangle \psi_Q$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Accordingly, for any vector-valued function $\vec{f} := (f_1, \dots, f_m)^T \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we naturally have $\vec{f} = \sum_{Q\in\mathcal{D}} \langle \vec{f}, \varphi_Q\rangle \psi_Q := \sum_{Q\in\mathcal{D}} \vec{s}_Q \psi_Q \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, and for any possible N, K, M, δ, $\{\psi_Q\}_{Q\in\mathcal{D}}$ is a family of smooth (N, K, M, δ)-molecules. Then by conclusion (**a**), we immediately have

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\{\langle \vec{f}, \varphi_Q\rangle\}_{Q\in\mathcal{D}}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}. \qquad \square$$

7. Equivalence of the spaces $\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})$ and $\dot{F}^{\alpha,q}_{p,\infty}(W)$

Lemma 7.1. *Let $\varphi \in \mathcal{A}$. Define $\widetilde{\varphi}(x) = \overline{\varphi(-x)}$. Suppose that the sequence of non-negative definite matrices $\{A_Q\}_{Q\in\mathcal{D}}$ is weakly doubling of any order $r > 0$. Then for $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, there exists a constant C such that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,*

$$\left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}} \left(2^{j\alpha} \sup_{x\in Q} |A_Q \varphi_j * \vec{f}(x)| \mathbf{1}_Q \right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \le C \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})},$$

and

$$\left\| \{\langle \vec{f}, \widetilde{\varphi}_Q\rangle\} \right\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} \le C \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

Proof. Choose a sufficiently small $A \in (0,1)$ such that $p/A > 1$ and $q/A > 1$. For any $R > 0$, by Lemma 4.6

$$\begin{aligned} \sum_{Q\in\mathcal{D}} (2^{j\alpha} \sup_Q |A_Q \varphi_j * \vec{f}| \mathbf{1}_Q)^q &= \sum_{k\in\mathbb{Z}^n} 2^{j\alpha q} \sup_{x\in Q_{jk}} |A_{Q_{jk}} \varphi_j * \vec{f}(x)|^q \mathbf{1}_{Q_{jk}}(x) \\ &\lesssim \Big| \sum_{k\in\mathbb{Z}^n} \sum_{\ell\in\mathbb{Z}^n} (1 + |k-\ell|)^{-A(R-r)} 2^{jn} \int_{Q_{j\ell}} |2^{j\alpha} A_{Q_{j\ell}} \varphi_j * \vec{f}(s)|^A \mathrm{d}s \mathbf{1}_{Q_{jk}} \Big|^{\frac{q}{A}}. \end{aligned}$$

Accordingly, also $A \in (0,1)$, and by the arbitrariness of $R > 0$ and $r > 0$, choose R sufficiently large and r sufficiently small such that $A(R-r) > n$, then applying Lemma 4.7 with $h = \sum_{Q\in\mathcal{D}} (2^{j\alpha} |A_Q \varphi_j * \vec{f}| \mathbf{1}_Q)^A$,

$$\sum_{Q\in\mathcal{D}} (2^{j\alpha} \sup_Q |A_Q \varphi_j * \vec{f}| \mathbf{1}_Q)^q \lesssim \left(\mathcal{M}\Big(\sum_{Q\in\mathcal{D}} \left(2^{j\alpha} |A_Q \varphi_j * \vec{f}| \mathbf{1}_Q \right)^A \Big) \right)^{q/A}.$$

Since $p/A > 1$ and $q/A > 1$, by Lemma 4.10,

$$\begin{aligned}
&\Big\|\Big\{\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}}\big(2^{j\alpha}\sup_{Q}|A_Q\varphi_j*\vec{f}|\mathbf{1}_Q\big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^{p,\infty}}\\
&\lesssim\Big\|\Big\{\sum_{j\in\mathbb{Z}}\Big(\mathcal{M}\big(\sum_{Q\in\mathcal{D}}(2^{j\alpha}|A_Q\varphi_j*\vec{f}|\mathbf{1}_Q)^A\big)\Big)^{\frac{q}{A}}\Big\}^{\frac{1}{q}}\Big\|_{L^{p,\infty}}\\
&=\Big\|\Big\{\sum_{j\in\mathbb{Z}}\Big(\mathcal{M}\big(\sum_{Q\in\mathcal{D}}(2^{j\alpha}|A_Q\varphi_j*\vec{f}|\mathbf{1}_Q)^A\big)\Big)^{\frac{q}{A}}\Big\}^{\frac{A}{q}}\Big\|_{L^{p/A,\infty}}^{\frac{1}{A}}\\
&\lesssim\Big\|\Big\{\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}}\big(2^{j\alpha}|A_Q\varphi_j*\vec{f}|\mathbf{1}_Q\big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^{p,\infty}}=\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}.
\end{aligned}$$

Furthermore, $|Q_{jk}|^{-1/2}\langle\vec{f},\widetilde{\varphi}_{Q_{jk}}\rangle=\varphi_j*\vec{f}(x_{Q_{jk}})$, therefore

$$\begin{aligned}
\Big\|\{\langle\vec{f},\widetilde{\varphi}_Q\rangle\}\Big\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}&=\Big\|\Big\{\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}}\big(|Q|^{-\alpha/n}|A_Q\varphi_j*\vec{f}(x_Q)|\mathbf{1}_Q\big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^{p,\infty}}\\
&\lesssim\Big\|\Big\{\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}}\big(2^{j\alpha}\sup_{x\in Q}|A_Q\varphi_j*\vec{f}(x)|\mathbf{1}_Q\big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^{p,\infty}}\\
&\lesssim\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}.
\end{aligned}$$

□

Theorem 7.2. *Let $m\in\mathbb{N}$, $\alpha\in\mathbb{R}$, $0<p<\infty$, $0<q\le\infty$, $W\in A_p(\mathbb{R}^n,\mathbb{C}^m)$, $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of p-order reducing operators for W, and $\{\varphi_j\}_{j\in\mathbb{Z}_+}\in\mathbf{\Phi}$. Then for any $\vec{f}\in(\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we have*

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}\sim\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})},$$

where the positive constant of equivalence is independent of $\vec{f}$.

Proof. First, we prove

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}\lesssim\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.$$

Let $\varphi\in\mathcal{A}$ be the test function defined in both $\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})$ and $\dot{F}^{\alpha,q}_{p,\infty}(W)$. Set $\widetilde{\varphi}(x)=\overline{\varphi(-x)}$. First, we prove

$$\|\{\langle\vec{f},\widetilde{\varphi}_Q\rangle\}_{Q\in\mathcal{D}}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}\lesssim\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.\tag{7.1}$$

i) When $0<p\le1$, by part (i) of Lemma 3.12, we have

$$|A_{Q_{j\ell}}\varphi_j*\vec{f}(x)|^A\le\|A_{Q_{j\ell}}W^{-1/p}\||W^{1/p}\varphi_j*\vec{f}(x)|^A\lesssim|W^{1/p}\varphi_j*\vec{f}(x)|^A\ \ a.e.$$

Furthermore, since $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, the sequence $\{A_Q\}$ is weakly doubling of order $r > 0$. Therefore, for any $A \in (0,1)$, $R > 0$, by Lemma 4.6, we have

$$\sup_{x \in Q_{jk}} |A_{Q_{jk}} \varphi_j * \vec{f}(x)|^A \lesssim \sum_{\ell \in \mathbb{Z}^n} (1 + |k - \ell|)^{-A(R-r)} 2^{jn} \int_{Q_{j\ell}} |W^{1/p} \varphi_j * \vec{f}(x)|^A \mathrm{d}x.$$

Thus,

$$\begin{aligned} \sum_{Q \in \mathcal{D}} (2^{j\alpha} \sup_Q |A_Q \varphi_j * \vec{f}| \mathbf{1}_Q)^q &= \sum_{k \in \mathbb{Z}^n} 2^{j\alpha q} \sup_{x \in Q_{jk}} |A_{Q_{jk}} \varphi_j * \vec{f}(x)|^{A \cdot \frac{q}{A}} \mathbf{1}_{Q_{jk}}(x) \\ &\lesssim \Big| \sum_{k \in \mathbb{Z}^n} \sum_{\ell \in \mathbb{Z}^n} (1 + |k - \ell|)^{-A(R-r)} 2^{jn} \int_{Q_{j\ell}} |2^{j\alpha} W^{\frac{1}{p}} \varphi_j * \vec{f}(x)|^A \mathrm{d}x \mathbf{1}_{Q_{jk}} \Big|^{\frac{q}{A}}. \end{aligned}$$

Accordingly, also $A \in (0,1)$, and by the arbitrariness of $R > 0$, $r > 0$, we choose R sufficiently large and r sufficiently small such that $\eta = A(R - r) > n$ holds. Then, setting $h = |2^{j\alpha} W^{1/p} \varphi_j * \vec{f}|^A$ in Lemma 4.7, we have

$$\sum_{Q \in \mathcal{D}} (2^{j\alpha} \sup_Q |A_Q \varphi_j * \vec{f}| \mathbf{1}_Q)^q \lesssim \left(\mathcal{M} \left(|2^{j\alpha} W^{1/p} \varphi_j * \vec{f}|^A \right) \right)^{\frac{q}{A}}.$$

Therefore,

$$\begin{aligned} &\left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}} \left(2^{j\alpha} \sup_Q |A_Q \varphi_j * \vec{f}| \mathbf{1}_Q \right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\ &\lesssim \left\| \left\{ \sum_{j \in \mathbb{Z}} \left(\mathcal{M} \left(|2^{j\alpha} W^{\frac{1}{p}} \varphi_j * \vec{f}|^A \right) \right)^{\frac{q}{A}} \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\ &= \left\| \left\{ \sum_{j \in \mathbb{Z}} \left(\mathcal{M} \left(|2^{j\alpha} W^{\frac{1}{p}} \varphi_j * \vec{f}|^A \right) \right)^{\frac{q}{A}} \right\}^{\frac{A}{q}} \right\|_{L^{p/A,\infty}}^{\frac{1}{A}} \\ &\lesssim \left\| \left\{ \sum_{j \in \mathbb{Z}} \left| 2^{j\alpha} W^{\frac{1}{p}} \varphi_j * \vec{f} \right|^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\ &= \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \end{aligned}$$

Here, the last inequality follows from $p/A > 1$, and $q/A > 1$, using Lemma 4.10. Since $|Q_{jk}|^{-1/2} \langle \vec{f}, \widetilde{\varphi}_{Q_{jk}} \rangle = \varphi_j * \vec{f}(x_{Q_{jk}})$, we have

$$\|\{\langle \vec{f}, \widetilde{\varphi}_Q \rangle\}_{Q \in \mathcal{D}}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} = \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}} \left(2^{j\alpha} |A_Q \varphi_j * \vec{f}(x_Q)| \mathbf{1}_Q \right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}}$$

$$\leq \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}} \left(2^{j\alpha} \sup_{Q} |A_Q\varphi_j * \vec{f}|\mathbf{1}_Q\right)^q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}}$$
$$\lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.$$

Since $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, the sequence $\{A_Q\}_{Q\in\mathcal{D}}$ is strongly doubling of order (β, p). Then, replacing φ with $\widetilde{\varphi} \in \mathcal{A}$ in Theorem 6.4, we have

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\{\langle \vec{f}, \widetilde{\varphi}_Q\rangle\}_{Q\in\mathcal{D}}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

Hence, for $p \in (0,1]$, we have $\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}$.

ii) When $1 < p < \infty$, by the proof of Theorem 3.5 in reference [15], we have

$$\sup_{x\in Q_{jk}} |A_{Q_{jk}}\varphi_j * \vec{f}(x)|^{At} \lesssim \sum_{\ell\in\mathbb{Z}^n} (1 + |k - \ell|)^{-A(R-r)} 2^{jn} \int_{Q_{j\ell}} |W^{1/p}\varphi_j * \vec{f}|^{At} \mathrm{d}x.$$

Replacing A with At and using the same method as in the case $0 < p \leq 1$, we can similarly prove

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.$$

Next, we prove $\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}$. Define

$$h_j(x) = 2^{j\alpha}|W^{1/p}(x)\varphi_j * \vec{f}(x)|,$$

and

$$k_j = \sum_{Q\in\mathcal{D}} |Q|^{-\alpha/n}\Big(\sup_{x\in Q} |A_Q\varphi_j * \vec{f}(x)|\Big)\mathbf{1}_Q, \quad j \in \mathbb{Z}.$$

Each k_j is constant on cubes $Q \in \mathcal{D}$. Then,

$$h_j \leq \sum_{Q\in\mathcal{D}} |Q|^{-\alpha/n}\|W^{1/p}A_Q^{-1}\||A_Q\varphi_j * \vec{f}|\mathbf{1}_Q \leq \gamma_j k_j.$$

Here, $\gamma_j(x) = \sum_{Q\in\mathcal{D}} \|W^{1/p}(x)A_Q^{-1}\|\mathbf{1}_Q$, defined identically to that in Lemma 4.9. From this and by Lemma 4.9,

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} = \|\{h_j\}\|_{L^{p,\infty}(\ell^q)} \leq \|\{\gamma_j k_j\}\|_{L^{p,\infty}(\ell^q)} = \|\{\gamma_j E_j(k_j)\}\|_{L^{p,\infty}(\ell^q)}$$
$$\lesssim \|\{E_j(k_j)\}\|_{L^{p,\infty}(\ell^q)} = \|\{k_j\}\|_{L^{p,\infty}(\ell^q)} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})},$$

where the last step follows from Lemma 7.1. □

Since the definitions of the spaces $\dot{F}^{\alpha,q}_{p,\infty}(A_Q)$ and $\dot{F}^{\alpha,q}_{p,\infty}(W)$ directly involve φ, they appear to depend on the choice of φ. However, the following lemma shows that this is not the case. It demonstrates that the definitions of both spaces $\dot{F}^{\alpha,q}_{p,\infty}(A_Q)$ and $\dot{F}^{\alpha,q}_{p,\infty}(W)$ are independent of the choice of $\varphi \in \mathcal{A}$.

Theorem 7.3. *Let $0 < p < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of p-order reducing operators for W. Then for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$ and $\varphi^{(1)}, \varphi^{(2)} \in \mathcal{A}$, we have*

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(1)})} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(2)})} \text{ and } \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W,\varphi^{(1)})} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W,\varphi^{(2)})}.$$

Proof. For $\varphi^{(1)}, \varphi^{(2)} \in \mathcal{A}$, denote the space defined by $\varphi^{(1)}$ as $\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\}, \varphi^{(1)})$ and the space defined by $\varphi^{(2)}$ as $\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\}, \varphi^{(2)})$. Define $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$ by $\vec{s}_Q = \langle \vec{f}, \widetilde{\varphi}^{(1)}_Q\rangle$, and define $\vec{t} = \{\vec{t}_Q\}_{Q\in\mathcal{D}}$ by $\vec{t}_Q = \langle \vec{f}, \varphi^{(2)}_Q\rangle$. Accordingly, choose $\widehat{\psi} = \widehat{\varphi}^{(1)}/\sum_{j\in\mathbb{Z}} |\widehat{\varphi^{(1)}_j}|^2$ and $\widehat{\tau} = \widehat{\varphi}^{(2)}/\sum_{j\in\mathbb{Z}} |\widehat{\varphi^{(2)}_j}|^2$. Then $\psi, \tau \in \mathcal{A}$, and for all $\xi \neq 0$, we have

$$\sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi^{(1)}_j}}(\xi)\widehat{\psi_j}(\xi) = 1 \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi^{(2)}_j}}(\xi)\widehat{\tau}_j(\xi) = 1.$$

From this, using Lemma 6.3, for $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we have $\vec{f} = \sum_Q \langle \vec{f}, \varphi^{(1)}_Q\rangle \psi_Q$, with convergence in $(\mathcal{S}'_\infty(\mathbb{R}^n))^m$.

Moreover, note that for all $\xi \neq 0$,

$$\sum_{j\in\mathbb{Z}} \overline{\widehat{\widetilde{\psi}}_j}(\xi)\widehat{\widetilde{\varphi}^{(1)}_j}(\xi) = \sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi^{(1)}_j}}(\xi)\widehat{\psi_j}(\xi) = 1.$$

From this, by replacing $\vec{f}$, $\varphi^{(1)}$, and ψ in $\vec{f} = \sum_Q \langle \vec{f}, \varphi^{(1)}_Q\rangle \psi_Q$ with $\varphi^{(2)}_Q$, $\widetilde{\psi}$, and $\widetilde{\varphi}^{(1)}$, respectively, we obtain

$$\varphi^{(2)}_Q = \sum_{P\in\mathcal{D}} \langle \varphi^{(2)}_Q, \widetilde{\psi}_P\rangle \widetilde{\varphi}^{(1)}_P.$$

Furthermore, since $\varphi^{(2)}_Q \in \mathcal{S}_0$, where

$$\mathcal{S}_0 := \{g \in \mathcal{S} : \text{for all multi-indices } \alpha, D^\alpha \hat{g}(0) = 0\},$$

then $\varphi^{(2)}_Q = \sum_{P\in\mathcal{D}} \langle \varphi^{(2)}_Q, \widetilde{\psi}_P\rangle \widetilde{\varphi}^{(1)}_P$ converges in $\mathcal{S}$.

Therefore, for $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\vec{t}_Q = \langle \vec{f}, \varphi^{(2)}_Q\rangle = \Big\langle \vec{f}, \sum_{P\in\mathcal{D}} \langle \varphi^{(2)}_Q, \widetilde{\psi}_P\rangle \widetilde{\varphi}^{(1)}_P \Big\rangle = \sum_{P\in\mathcal{D}} \langle \varphi^{(2)}_Q, \widetilde{\psi}_P\rangle \vec{s}_P.$$

Set $b_{QP} = \langle \varphi^{(2)}_Q, \widetilde{\psi}_P\rangle$ and the matrix $B = \{b_{QP}\}_{Q,P\in\mathcal{D}}$. Also, $\{\psi_P\}_{P\in\mathcal{D}}$ is a family of smooth (N, K, M, δ)-molecules for all possible N, K, M, δ. Accordingly, by Lemma 2.8 in reference [15], we have

$$b_{QP} = \langle \varphi^{(2)}_Q, \widetilde{\psi}_P\rangle = |Q|^{1/2}\varphi^{(2)}_j * \psi_P(x_Q) \leq C|Q|^{1/2}2^{(kn)/2}\omega_{QP} \leq C\omega_{QP},$$

that is, the matrix $B = \{b_{QP}\}_{Q,P\in\mathcal{D}}$ is almost diagonal, $B \in \mathbf{ad}^{\alpha,\mathbf{q}}_{\mathbf{p}}(\beta)$. Then by Theorem 5.5, B is bounded on the space $\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})$. From this and Lemma 7.1, it follows that

$$\|\vec{t}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} = \|B\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} \leq C\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} \leq C\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(1)})}.$$

Replacing φ with $\varphi^{(2)}$ in Theorem 6.4, we have

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(2)})} \leq C\|\vec{t}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

Combining the above, we obtain

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(2)})} \leq C\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(1)})}.$$

By interchanging $\varphi^{(2)}$ and $\varphi^{(1)}$, we obtain the equivalence of $\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(1)})}$ and $\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(2)})}$.

Moreover, by Theorem 7.2, we have

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W,\varphi^{(1)})} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(1)})} \text{ and } \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W,\varphi^{(2)})} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi^{(2)})}.$$

Therefore,

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W,\varphi^{(1)})} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W,\varphi^{(2)})}. \qquad \square$$

Corollary 7.4. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of p-order reducing operators for W. Set $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$, where $\vec{s}_Q := \langle \vec{f}, \varphi_Q\rangle$. Then for any $\varphi \in \mathcal{A}$ and $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we have*

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

Proof. Replacing $\widetilde{\varphi}$ with φ in Lemma 7.1 and using Theorem 7.3, we obtain

$$\|\{\langle \vec{f}, \varphi_Q\rangle\}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} \leq C\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\widetilde{\varphi})} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\},\varphi)}.$$

Combining this with Theorem 6.4, we have

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\{\langle \vec{f}, \varphi_Q\rangle\}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

Since $\vec{s}_Q = \langle \vec{f}, \varphi_Q\rangle$ and $\varphi \in \mathcal{A}$, it follows immediately that

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})}. \qquad \square$$

Combining the above results with Theorem 7.2 and Theorem 4.12, we obtain the following equivalent norms:

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(W)},$$

establishing complete equivalences between the matrix-weighted weak Triebel-Lizorkin spaces and their sequence space counterparts.

8. Characterizations via maximal functions: Peetre, Lusin, and g^*_λ

In this section, the matrix-weighted weak Triebel-Lizorkin space $\dot{F}^{\alpha,q}_{p,\infty}(W)$ is characterized using the Peetre maximal function, the Lusin area function, and the Littlewood-Paley g^*_λ-function.

Similar to the classical Peetre maximal function in [26], the concept of the matrix-weighted Peetre maximal function is introduced in [35]. Let $p \in (0, \infty)$,

$m \in \mathbb{N}$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, and $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$. For any given $j \in \mathbb{Z}$ and $a \in (0,\infty)$, $x \in \mathbb{R}^n$, define

$$(\varphi_j^* \vec{f})_a^{(W,p)}(x) := \sup_{y\in\mathbb{R}^n} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|}{(1+2^j|x-y|)^a}.$$

Theorem 8.1. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, $a \in (n/\min\{1,p,q\}+\beta/p, \infty)$, where β is the doubling exponent of W. Let $\{\varphi_j\}_{j\in\mathbb{Z}_+} \in$ $\mathbf{\Phi}$. Then $\vec{f} \in \dot{F}^{\alpha,q}_{p,\infty}(W)$ if and only if $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$ and $\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(W)} < \infty$, where*

$$\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(W)} := \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \Big((\varphi_j^*\vec{f})_a^{(W,p)} \Big)^q \right\}^{1/q} \right\|_{L^{p,\infty}}.$$

When $q = \infty$, the usual modification is made. Moreover, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(W)},$$

where the positive equivalence constants are independent of $\vec{f}$.

Proof. By the definition of $(\varphi_j^*\vec{f})_a^{(W,p)}(x)$, we have

$$|W^{\frac{1}{p}}(\varphi_j * \vec{f})(x)| \le (\varphi_j^*\vec{f})_a^{(W,p)}(x),$$

which immediately implies

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.$$

Therefore, to prove $\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}$, it suffices to show

$$\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \tag{8.1}$$

Let $\{A_Q\}_{Q\in\mathcal{D}}$ be the sequence of p-th order reducing operators for W, and for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, let

$$\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} := \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} \sup_{y\in\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1+2^j|\cdot - y|)^{aq}} \mathbf{1}_Q \right\}^{1/q} \right\|_{L^{p,\infty}}, \tag{8.2}$$

where $j \in \mathbb{Z}$. To prove (8.1), we first prove that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}. \tag{8.3}$$

By (3.10) in reference [35], for any given $A \in (0,1]$, any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, and $x \in Q_{jk}$,

$$\sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^A}{(1+2^j|x-y|)^{aA}}$$

$$\lesssim \sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)}2^{jn}\int_{Q_{j\ell}}|A_{Q_{j\ell}}(\varphi_j*\vec{f})(z)|^A\mathrm{d}z, \tag{8.4}$$

where $r:=\beta/p$ and $a\in[r,\infty)$. Let $A\in(0,1]$ satisfy $q/A>1$, from this, (8.4), and the disjointness of the dyadic cubes Q_{jk} for any $k\in\mathbb{Z}^n$, we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}_j}\left\{2^{j\alpha}\sup_{y\in\mathbb{R}^n}\frac{\left|A_Q(\varphi_j*\vec{f})(y)\right|}{(1+2^j|\cdot-y|)^a}\mathbf{1}_Q(\cdot)\right\}^q\\
&=\sum_{k\in\mathbb{Z}^n}2^{j\alpha q}\left\{\sup_{y\in\mathbb{R}^n}\frac{\left|A_{Q_{jk}}(\varphi_j*\vec{f})(y)\right|^A}{(1+2^j|\cdot-y|)^{aA}}\mathbf{1}_{Q_{jk}}(\cdot)\right\}^{q/A}\\
&\lesssim\left\{\sum_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)}2^{jn}\right.\\
&\qquad\left.\times\int_{Q_{j\ell}}\left|2^{j\alpha}A_{Q_{j\ell}}(\varphi_j*\vec{f})(z)\right|^A\mathrm{d}z\mathbf{1}_{Q_{jk}}(\cdot)\right\}^{q/A}.
\end{aligned}$$

Since $a\in(n/\min\{1,p,q\}+r,\infty)$, we have $\min\{1,p,q\}(a-r)>n$, therefore, we can choose $A\in(0,1]$ such that $A(a-r)>n$, $p/A>1$, and $q/A>1$. Thus, by Lemma 4.7 and the Fefferman-Stein vector-valued maximal inequality in weak Lebesgue spaces, for any $\vec{f}\in(\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we obtain

$$\begin{aligned}
&\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}\\
&=\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j}\left\{2^{j\alpha}\sup_{y\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|}{(1+2^j|\cdot-y|)^a}\mathbf{1}_Q\right\}^q\right\}^{1/q}\right\|_{L^{p,\infty}}\\
&\lesssim\left\|\left\{\sum_{j\in\mathbb{Z}}\left(\mathcal{M}\Big(\sum_{Q\in\mathcal{D}_j}\left(2^{j\alpha}|A_Q(\varphi_j*\vec{f})|\mathbf{1}_Q\right)^A\Big)\right)^{q/A}\right\}^{A/q}\right\|^{1/A}_{L^{p/A,\infty}}\\
&\lesssim\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}.
\end{aligned}$$

Next, for any $\vec{f}\in(\mathcal{S}'_\infty(\mathbb{R}^n))^m$, define

$$\left\|\vec{f}\right\|^{\star\star}_{\dot{F}^{a,q}_{p,\infty}(A_Q)}:=\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j}2^{j\alpha q}\sup_{z\in Q}\sup_{y\in\mathbb{R}^n}\frac{\left|A_Q(\varphi_j*\vec{f})(y)\right|^q}{(1+2^j|z-y|)^{aq}}\mathbf{1}_Q\right\}^{1/q}\right\|_{L^{p,\infty}}. \tag{8.5}$$

By (8.3) and Theorem 7.2, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we have

$$\|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \tag{8.6}$$

Therefore, to prove (8.1), it suffices to show that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|^{\star\star}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}. \tag{8.7}$$

We first prove the left-hand side of the above inequality. For any $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let

$$h_j(x) := 2^{j\alpha} \sup_{y\in\mathbb{R}^n} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|}{(1+2^j|x-y|)^a},$$

$$k_j(x) := \sum_{Q\in\mathcal{D}_j} |Q|^{-\alpha/n} \sup_{z\in Q} \sup_{y\in\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|}{(1+2^j|z-y|)^a} \mathbf{1}_Q(x),$$

and

$$\gamma_j(x) := \sum_{Q\in\mathcal{D}_j} \|W^{1/p}(x)A_Q^{-1}\| \mathbf{1}_Q(x).$$

Obviously, for any $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} h_j(x) &= \sum_{Q\in\mathcal{D}_j} 2^{j\alpha} \sup_{y\in\mathbb{R}^n} \frac{|W^{1/p}(x)A_Q^{-1}A_Q(\varphi_j * \vec{f})(y)|}{(1+2^j|x-y|)^a} \mathbf{1}_Q(x) \\ &\le \sum_{Q\in\mathcal{D}_j} 2^{j\alpha} \|W^{1/p}(x)A_Q^{-1}\| \sup_{y\in\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|}{(1+2^j|x-y|)^a} \mathbf{1}_Q(x) \\ &\le \gamma_j(x)k_j(x). \end{aligned} \tag{8.8}$$

Note that on any given dyadic cube $Q \in \mathcal{D}_j$, k_j is a constant, i.e.,

$$E_j(k_j) = k_j, \tag{8.9}$$

then by (8.8) and (8.9) and Lemma 4.9, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we have

$$\begin{aligned} \|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} &= \left\|\{h_j\}_{j\in\mathbb{Z}}\right\|_{L^{p,\infty}(\ell^q)} \le \left\|\{\gamma_j E_j(k_j)\}_{j\in\mathbb{Z}}\right\|_{L^{p,\infty}(\ell^q)} \\ &\lesssim \left\|\{E_j(k_j)\}_{j\in\mathbb{Z}}\right\|_{L^{p,\infty}(\ell^q)} \sim \left\|\{k_j\}_{j\in\mathbb{Z}}\right\|_{L^{p,\infty}(\ell^q)} \sim \|\vec{f}\|^{\star\star}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}. \end{aligned} \tag{8.10}$$

Thus, the first inequality in (8.7) holds. Next, we will prove that the second inequality in (8.7) also holds. For any $x, z \in Q_{jk}$ and $y \in Q_{js}$, there is a geometric relation $1+2^j|x-y| \sim 1+|s-k| \sim 1+2^j|z-y|$. From this, for any $a \in (0,\infty)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $x \in Q_{jk}$, we have

$$\sup_{z\in Q_{jk}} \sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|}{(1+2^j|z-y|)^a} \sim \sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|}{(1+2^j|x-y|)^a}.$$

Therefore, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we have

$$\|\vec{f}\|^{\star\star}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}. \tag{8.11}$$

Combining (8.10) and (8.11), we can obtain that (8.7) holds, i.e.,

$$\|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}.$$

Moreover, by (8.6), we have

$$\|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.$$

That is, (8.1) is proved. Then for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.$$

Therefore, Theorem 8.1 is proved. □

Theorem 8.2. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$. Suppose $\{\varphi_j\}_{j\in\mathbb{Z}_+} \in \mathbf{\Phi}$. Then $\vec{f} \in \dot{F}^{\alpha,q}_{p,\infty}(W)$ if and only if $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$ and $\|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} < \infty$, where*

$$\|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} := \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \fint_{B(\cdot,2^{-j})} |W^{1/p}(\cdot)(\varphi_j * \vec{f})(y)|^q \mathrm{d}y \right\}^{1/q} \right\|_{L^{p,\infty}}.$$

When $q = \infty$, the usual modification is made. Moreover, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)},$$

where the positive equivalence constants are independent of $\vec{f}$.

Proof. By Theorem 8.1, to prove

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)},$$

it is equivalent to proving

$$\|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.$$

We first prove that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|^{\star}_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \tag{8.12}$$

From the fact that for any $y \in B(0, 2^{-j})$, $1 + 2^j|y| \sim 1$, and the definition of $(\varphi_j^* \vec{f})_a^{(W,p)}$, we infer that for any given $q \in (0,\infty)$, $a \in (0,\infty)$, and any $j \in \mathbb{Z}$, $x \in \mathbb{R}^n$,

$$\fint_{B(x,2^{-j})} |W^{1/p}(x)(\varphi_j * \vec{f})(y)|^q \mathrm{d}y = \fint_{B(0,2^{-j})} |W^{1/p}(x)(\varphi_j * \vec{f})(x+y)|^q \mathrm{d}y$$

$$\begin{aligned} &\lesssim \sup_{y\in B(0,2^{-j})} |W^{1/p}(x)(\varphi_j * \vec{f})(x+y)|^q \\ &\sim \sup_{y\in B(0,2^{-j})} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(x+y)|^q}{(1+2^j|y|)^{aq}} \\ &\lesssim \left((\varphi_j^* \vec{f})_a^{(W,p)}(x)\right)^q, \end{aligned}$$

which implies that (8.12) holds. Next, we prove that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|^\diamond_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \tag{8.13}$$

By (8.7), to prove (8.13), it suffices to show that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{f}\|^\diamond_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \tag{8.14}$$

For any given $A \in (0,1]$ satisfying $q/A > 1$ and $p/A > 1$, we choose a sufficiently large $a \in (0,\infty)$ such that $A(a-r) > n$, where $r := \beta/p$, and β is the doubling index of W. Then by (3.21) in [35], for any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $x \in Q_{jk}$, we have

$$\begin{aligned} \sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^A}{(1+2^j|x-y|)^{aA}} &\lesssim \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(a-r)} 2^{jn} \\ &\quad \times \sum_{\{t\in\mathbb{Z}^n : =|t|_\infty \le 1\}} \int_{Q_{j(\ell+t)}} \fint_{B(0,2^{-j})} |A_{Q_{j(\ell+t)}}(\varphi_j * \vec{f})(s+z)|^A \mathrm{d}s\mathrm{d}z. \end{aligned} \tag{8.15}$$

We now prove (8.14) by considering two cases for p.

When $p \in (0,1]$. By $\mathbf{1}_{Q_{j(\ell+t)}} = \sum_{Q\in\mathcal{D}_j} (\mathbf{1}_Q \mathbf{1}_{Q_{j(\ell+t)}})$, we have

$$\begin{aligned} &\int_{Q_{j(\ell+t)}} 2^{j\alpha} \fint_{B(0,2^{-j})} |A_{Q_{j(\ell+t)}}(\varphi_j * \vec{f})(s+z)|^A \mathrm{d}s\mathrm{d}z \\ &= \int_{Q_{j(\ell+t)}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha} \fint_{B(0,2^{-j})} |A_Q(\varphi_j * \vec{f})(s+z)|^A \mathrm{d}s \mathbf{1}_Q(z)\mathrm{d}z \\ &= \int_{Q_{j(\ell+t)}} g_j(z)\mathrm{d}z, \end{aligned} \tag{8.16}$$

where for any $z \in \mathbb{R}^n$,

$$g_j(z) := \sum_{Q\in\mathcal{D}_j} 2^{j\alpha} \fint_{B(0,2^{-j})} |A_Q(\varphi_j * \vec{f})(s+z)|^A \mathrm{d}s \mathbf{1}_Q(z).$$

For any given $x \in Q_{jk}$, let $B_x := B(x_{k,\ell,t}, r_{k,\ell,t})$ be the smallest ball containing x and $Q_{j(\ell+t)}$. Then $r_{k,\ell,t} \sim 2^{-j}(1+|k-\ell-t|)$. Also, since $|t|_\infty \le 1$, we have

$$r_{k,\ell,t} \sim 2^{-j}(1+|k-\ell|).$$

From this and (8.16), for any $x \in Q_{jk}$, we obtain

$$\int_{Q_{j(\ell+t)}} 2^{j\alpha} \fint_{B(0,2^{-j})} |A_{Q_{j(\ell+t)}}(\varphi_j * \vec{f})(s+z)|^A \mathrm{d}s\mathrm{d}z \\ \leq \int_{B_x} g_j(z)\mathrm{d}z \lesssim 2^{-jn}(1+|k-\ell|)^n \mathcal{M}(g_j)(x). \tag{8.17}$$

By (8.15) and (8.17), assuming $A(a-r) > n$, for any $x \in \mathbb{R}^n$,

$$2^{j\alpha} \sum_{Q\in\mathcal{D}_j} \sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^A}{(1+2^j|x-y|)^{aA}} \mathbf{1}_Q(x) \\ \lesssim \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(a-r)+n} \mathcal{M}(g_j)(x) \lesssim \mathcal{M}(g_j)(x). \tag{8.18}$$

From (8.18), for any $x \in \mathbb{R}^n$,

$$\sum_{Q\in\mathcal{D}_j} \left\{ 2^{j\alpha} \sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|}{(1+2^j|x-y|)^a} \mathbf{1}_Q(x) \right\}^q \lesssim 2^{j\alpha q} 2^{-\frac{j\alpha q}{A}} (\mathcal{M}(g_j))^{\frac{q}{A}}. \tag{8.19}$$

By (8.19), the Fefferman-Stein vector-valued maximal inequality in weak Lebesgue spaces with $p/A > 1$ and $q/A > 1$, Hölder's inequality, and part (i) of Lemma 3.12, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\begin{aligned}
&\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} 2^{-\frac{j\alpha q}{A}} \left(\mathcal{M}(g_j)\right)^{\frac{q}{A}} \right\}^{\frac{A}{q}} \right\|^{1/A}_{L^{p/A,\infty}} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} 2^{-\frac{j\alpha q}{A}} (g_j)^{\frac{q}{A}} \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&= \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} \left(\fint_{B(0,2^{-j})} \left| A_Q(\varphi_j * \vec{f})(x+z) \right|^A \mathrm{d}z \right)^{\frac{q}{A}} \mathbf{1}_Q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} \fint_{B(0,2^{-j})} \left| A_Q(\varphi_j * \vec{f})(x+z) \right|^q \mathrm{d}z \mathbf{1}_Q \right)^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} \fint_{B(0,2^{-j})} \left\| A_Q W^{-\frac{1}{p}} \right\|^q \left| W^{\frac{1}{p}}(\varphi_j * \vec{f})(x+z) \right|^q \mathrm{d}z \mathbf{1}_Q \right)^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&\lesssim \|\vec{f}\|^\diamond_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.
\end{aligned}$$

Thus, when $p \in (0,1]$, (8.14) holds.

When $p \in (1,\infty)$. For any $x \in Q_{jk}$, by (3.27) in [35], we have

$$\sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^A}{(1+2^j|x-y|)^{aA}} \lesssim \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(a-r)} 2^{jn(1-\frac{A}{p'})} \sum_{\{t\in\mathbb{Z}^n:=|t|_\infty\le 1\}} \left\{ \int_{Q_{j(\ell+t)}} \left(\fint_{B(0,2^{-j})} \left| W^{1/p}(z)(\varphi_j * \vec{f})(s+z) \right|^A \mathrm{d}s \right)^{\frac{p'}{p'-A}} \mathrm{d}z \right\}^{\frac{p'-A}{p'}}. \tag{8.20}$$

Note that for any $M > n$,

$$\sup_{k\in\mathbb{Z}^n} \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-M} = \sup_{k\in\mathbb{Z}^n} \sum_{k-\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-M} = \sum_{\ell\in\mathbb{Z}^n} (1+|\ell|)^{-M} \lesssim 1. \tag{8.21}$$

For any given $x \in \mathbb{R}^n$, let $B_x := B(x_{k,\ell,t}, r_{k,\ell,t})$ be the smallest ball containing x and $Q_{j(\ell+t)}$, with the same assumptions as in the case $p \in (0,1]$. Then $r_{k,\ell,t} \sim 2^{-j}(1+|k-\ell|)$. By (3.28) in [35], we have

$$\sum_{k\in\mathbb{Z}^n} \sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^q}{(1+2^j|x-y|)^{aq}} \mathbf{1}_{Q_{jk}}(x) \lesssim \left\{ \mathcal{M}\left(\left(\fint_{B(0,2^{-j})} \left| W^{1/p}(\cdot)(\varphi_j * \vec{f})(\cdot+z) \right|^A \mathrm{d}z \right)^{\frac{p'}{p'-A}} \right)(x) \right\}^{\frac{(p'-A)q}{Ap'}}. \tag{8.22}$$

Note that $p(p'-A)/(Ap') = (A+(1-A)p)/A > 1$, choosing a sufficiently small $A \in (0,1)$, then we have $(p'-A)q/(Ap') > 1,\ q/A > 1$. For brevity, hereafter we set $T_{j,z} := W^{1/p}(x)(\varphi_j * \vec{f})(x+z)$, so that, by (8.22); the Fefferman-Stein vector-valued maximal inequality in weak Lebesgue spaces and Hölder's inequality, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we deduce that

$$\begin{aligned}
&\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \left(\mathcal{M}\left(\left(\fint_{B(0,2^{-j})} \left| T_{j,z} \right|^A \mathrm{d}z \right)^{\frac{p'}{p'-A}} \right) \right)^{\frac{(p'-A)q}{Ap'}} \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}} \\
&\sim \left\| \left\{ \sum_{j\in\mathbb{Z}} \left(\mathcal{M}\left(\left(\fint_{B(0,2^{-j})} \left| 2^{j\alpha} T_{j,z} \right|^A \mathrm{d}z \right)^{\frac{p'}{p'-A}} \right) \right)^{\frac{(p'-A)q}{Ap'}} \right\}^{\frac{Ap'}{(p'-A)q}} \right\|^{\frac{p'-A}{Ap'}}_{L^{\frac{p(p'-A)}{p'A},\infty}}
\end{aligned}$$

$$\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \left(\fint_{B(0,2^{-j})} \left| T_{j,z} \right|^{A} \mathrm{d}z \right)^{q/A} \right\}^{1/q} \right\|_{L^{p,\infty}}$$

$$\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \left(\fint_{B(0,2^{-j})} \left| T_{j,z} \right|^{q} \mathrm{d}z \right) \right\}^{1/q} \right\|_{L^{p,\infty}}$$

$$\sim \|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)}.$$

Thus, (8.14) holds for $p \in (0, \infty)$.

Therefore, (8.13) holds. Combining with (8.12) and Theorem 8.1. Theorem 8.2 is proved. □

Theorem 8.3. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and $\lambda \in (1/\min\{1, p, q\} + \beta/(np), \infty)$, where β is the doubling exponent of W. Suppose $\{\varphi_j\}_{j\in\mathbb{Z}_+} \in \mathbf{\Phi}$. Then $\vec{f} \in \dot{F}^{\alpha,q}_{p,\infty}(W)$ if and only if $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$ and $\|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} < \infty$, where*

$$\|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} := \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} 2^{jn} \int_{\mathbb{R}^n} \frac{|W^{1/p}(\cdot)(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|\cdot - y|)^{\lambda n q}} \mathrm{d}y \right\}^{1/q} \right\|_{L^{p,\infty}}.$$

When $q = \infty$ the usual modification is made. Moreover, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)},$$

where the positive equivalence constants are independent of $\vec{f}$.

Proof. We first prove that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \tag{8.23}$$

In fact, for any $x \in \mathbb{R}^n$, $y \in B(x, 2^{-j})$, by geometric observation, we have $1 + 2^j|x - y| \sim 1$, this implies that for any $x \in \mathbb{R}^n$, and $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\begin{aligned}\fint_{B(x,2^{-j})} |W^{1/p}(x)(\varphi_j * \vec{f})(y)|^q \mathrm{d}y &\sim 2^{jn} \int_{B(x,2^{-j})} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|x - y|)^{\lambda n q}} \mathrm{d}y \\ &\lesssim 2^{jn} \int_{\mathbb{R}^n} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|x - y|)^{\lambda n q}} \mathrm{d}y,\end{aligned}$$

which means $\|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)}$. By Theorem 8.2, we have $\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|^{\diamond}_{\dot{F}^{\alpha,q}_{p,\infty}(W)}$. Therefore, (8.23) holds. Thus, to prove Theorem 8.3, it remains to prove that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \tag{8.24}$$

Let $\{A_Q\}_{Q\in\mathcal{D}}$ be the p-th order reducing operators for W. For any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, define

$$\|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} := \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} 2^{jn} \sup_{z\in Q} \int_{\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1+2^j|z-y|)^{\lambda n q}} \mathrm{d}y \mathbf{1}_Q \right\}^{\frac{1}{q}} \right\|_{L^{p,\infty}}. \tag{8.25}$$

To prove (8.24), we first prove that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \lesssim \|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}. \tag{8.26}$$

For any given $p \in (0, \infty)$, $q \in (0, \infty]$, and any $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, let

$$\gamma_j(x) := \sum_{Q\in\mathcal{D}_j} \|W^{1/p}(x) A_Q^{-1}\| \mathbf{1}_Q(x),$$

$$h_j(x) := 2^{j\alpha} 2^{jn/q} \left[\int_{\mathbb{R}^n} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|^q}{(1+2^j|x-y|)^{\lambda n q}} \mathrm{d}y \right]^{1/q},$$

$$f_j(x) := \sum_{Q\in\mathcal{D}_j} |Q|^{-\alpha/n} 2^{jn/q} \left[\sup_{z\in Q} \int_{\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1+2^j|z-y|)^{\lambda n q}} \mathrm{d}y \right]^{1/q} \mathbf{1}_Q(x).$$

Then, by (3.37) in [35], for any $j \in \mathbb{Z}$, and $x \in \mathbb{R}^n$,

$$h_j(x) \le \gamma_j(x) f_j(x). \tag{8.27}$$

Note that f_j is a constant on any given $Q \in \mathcal{D}_j$, i.e., $E_j(f_j) = f_j$. According to this, the definitions of $\|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)}$ and $h_j(x)$, (8.27), Lemma 4.9, and (8.25), for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\begin{aligned} \|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(W)} &= \|\{h_j\}_{j\in\mathbb{Z}}\|_{L^{p,\infty}(\ell^q)} \le \|\{\gamma_j E_j(f_j)\}_{j\in\mathbb{Z}}\|_{L^{p,\infty}(\ell^q)} \\ &\lesssim \|\{E_j(f_j)\}_{j\in\mathbb{Z}}\|_{L^{p,\infty}(\ell^q)} \sim \|\{f_j\}_{j\in\mathbb{Z}}\|_{L^{p,\infty}(\ell^q)} \sim \|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})}. \end{aligned}$$

Therefore, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, (8.26) holds.

Next, we prove that for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^{\circ}_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)}. \tag{8.28}$$

By the proof of Theorem 3.14 in [35], for any $x \in \mathbb{R}^n$,

$$\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j} 2^{j\alpha q}2^{jn}\sup_{z\in Q}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1+2^j|z-y|)^{\lambda nq}}\mathrm{d}y\mathbf{1}_Q(x)$$
$$\lesssim \sum_{j\in\mathbb{Z}}\Big(\mathcal{M}\Big(\sum_{Q\in\mathcal{D}_j}\big(|2^{j\alpha}A_Q(\varphi_j * \vec{f})|\mathbf{1}_Q\big)^A\Big)(x)\Big)^{q/A}.$$

Thus, for $A \in (0, \min\{p, q\})$, using the Fefferman-Stein vector-valued maximal inequality in weak Lebesgue spaces and Theorem 7.2, for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|^\circ_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \lesssim \left\|\left\{\sum_{j\in\mathbb{Z}}\Big(\mathcal{M}\Big(\sum_{Q\in\mathcal{D}_j}\big(|2^{j\alpha}A_Q\varphi_j * \vec{f}|\mathbf{1}_Q\big)^A\Big)\Big)^{q/A}\right\}^{1/q}\right\|_{L^{p,\infty}}$$
$$\sim \left\|\left\{\mathcal{M}\Big(\sum_{Q\in\mathcal{D}_j}\big(|2^{j\alpha}A_Q\varphi_j * \vec{f}|\mathbf{1}_Q\big)^A\Big)\right\}_{j\in\mathbb{Z}}\right\|^{1/A}_{L^{p/A,\infty}(\ell^{q/A})}$$
$$\lesssim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(\{A_Q\})} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)},$$

which further shows that (8.28) holds. By (8.26) and (8.28), we obtain (8.24). Combining (8.24) and (8.23), for any $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$, we have $\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \sim \|\vec{f}\|^\circ_{\dot{F}^{\alpha,q}_{p,\infty}(W)}$, thus completing the proof of Theorem 8.3. □

9. Calderón-Zygmund operators

In this section, we prove that classical convolution Calderón-Zygmund operators (CZOs) are bounded on $\dot{F}^{\alpha,q}_{p,\infty}(W)$. We need to recall the definition of smooth atoms (see Definition 6.1) and the definition of classical convolution Calderón-Zygmund operators (see Definition 9.5). Then we use the general criterion for the boundedness of operators: if an operator T maps smooth atoms into smooth molecules, then T is bounded on $\dot{F}^{\alpha,q}_{p,\infty}(W)$.

Before starting, we first need to discuss the extension problem of a given operator $T : \mathcal{S}_\infty(\mathbb{R}^n) \to \mathcal{S}'_\infty(\mathbb{R}^n)$ to $\widetilde{T} : \dot{F}^{\alpha,q}_{p,\infty}(W) \to \dot{F}^{\alpha,q}_{p,\infty}(W)$. Let the matrix $B := \{b_{QP}\}_{Q,P\in\mathcal{D}}$. For any sequence $s := \{s_Q\}_{Q\in\mathcal{D}}$, define $Bs := \{(Bs)_Q\}_{Q\in\mathcal{D}}$, where $(Bs)_Q := \sum_{P\in\mathcal{D}} b_{QP}s_P$.

For Proposition 3.18 in Reference [3], taking $\tau = 0$, and combining with the fact that the classical matrix-weighted Triebel-Lizorkin space $\dot{F}^{\alpha,q}_p(W) \subset \dot{F}^{\alpha,q}_{p,\infty}(W)$, the following corollary is easily obtained.

Corollary 9.1. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$. Then $(\mathcal{S}_\infty(\mathbb{R}^n))^m \subset \dot{F}^{\alpha,q}_{p,\infty}(W)$. Moreover, there exist an $M \in \mathbb{Z}_+$ and a positive constant*

C such that for any $\vec{f} \in (\mathcal{S}_\infty(\mathbb{R}^n))^m$,

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \le C\|\vec{f}\|_{\vec{S}_M} := C \sup_{\gamma\in\mathbb{Z}^n_+, |\gamma|\le M} \sup_{x\in\mathbb{R}^n} \left|\left(\partial^\gamma \vec{f}\right)(x)\right|(1+|x|)^{n+M+|\gamma|}.$$

Lemma 9.2. *Let* $\varphi \in \mathcal{A}$, *and* $T \in \mathcal{L}(\mathcal{S}_\infty(\mathbb{R}^n), \mathcal{S}'_\infty(\mathbb{R}^n))$, *and define a finite matrix* $\widehat{T} := \{\langle T\psi_P, \varphi_Q\rangle\}_{Q,P\in\mathcal{D}}$. *Then* $\widehat{T}$ *maps* $S_{\varphi,\infty} := \{\{f, \varphi_Q\}_{Q\in\mathcal{D}} : f \in \mathcal{S}_\infty(\mathbb{R}^n)\}$ *to* $S_{\varphi,\infty'} := \{\{f, \varphi_Q\}_{Q\in\mathcal{D}} : f \in \mathcal{S}'_\infty(\mathbb{R}^n)\}$, *and satisfies* $\widehat{T}\circ S_\varphi = S_\varphi \circ T$ *on* $\mathcal{S}_\infty(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}_\infty(\mathbb{R}^n)$. For $\varphi \in \mathcal{A}$, let $\hat{\psi} = \hat{\varphi}/(\sum_{j\in\mathbb{Z}} |\widehat{\varphi}_j|^2)$, then $\psi \in \mathcal{A}$. By Lemma 6.3, we have $f = \sum_{P\in\mathcal{D}}\langle f, \varphi_P\rangle\psi_P$ converges in $\mathcal{S}_\infty(\mathbb{R}^n)$. Since $T \in \mathcal{L}(\mathcal{S}_\infty(\mathbb{R}^n), \mathcal{S}'_\infty(\mathbb{R}^n))$, it follows that $Tf = \sum_{P\in\mathcal{D}}\langle f, \varphi_P\rangle T\psi_P$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Through $\varphi_Q \in \mathcal{S}_\infty(\mathbb{R}^n)$, we further obtain the conclusion that the convergence in $\mathcal{S}'_\infty(\mathbb{R}^n)$ implies

$$\langle Tf, \varphi_Q\rangle = \sum_{P\in\mathcal{D}} \langle T\psi_P, \varphi_Q\rangle\langle f, \varphi_P\rangle.$$

Since $\{\langle f, \varphi_P\rangle\}_{P\in\mathcal{D}}$ is an arbitrary element in $S_{\varphi,\infty}$, it is evident that

$$\widehat{T} : S_{\varphi,\infty} \to \mathbb{C}^{\mathcal{D}},$$

where

$$\mathbb{C}^{\mathcal{D}} := \{s : s = \{s_Q\}_{Q\in\mathcal{D}} \subset \mathbb{C}\},$$

is well-defined. The fact that $Tf \in \mathcal{S}'_\infty(\mathbb{R}^n)$ also indicates that $\widehat{T} : S_{\varphi,\infty} \to S_{\varphi,\infty'}$ and $\widehat{T}\circ S_\varphi = S_\varphi\circ T$. □

The φ-transform is defined as the mapping of each $\vec{f} \in (\mathcal{S}'_\infty(\mathbb{R}^n))^m$ to the sequence $S_\varphi\vec{f} := \{(S_\varphi\vec{f})_Q\}_{Q\in\mathcal{D}}$, where $(S_\varphi\vec{f})_Q := \langle f, \varphi_Q\rangle$. The inverse φ-transform is defined as the mapping of the sequence $\vec{s} := \{\vec{s}_Q\}_{Q\in\mathcal{D}} \subset \mathbb{C}^m$ to $T_\psi\vec{s} := \sum_{Q\in\mathcal{D}} \vec{s}_Q\psi_Q$ in $(\mathcal{S}'_\infty(\mathbb{R}^n))^m$.

Lemma 9.3. *Let* $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, $\varphi \in \mathcal{A}$, *and* $\hat{\psi} = \hat{\varphi}/(\sum_{j\in\mathbb{Z}} |\widehat{\varphi}_j|^2) \in \mathcal{A}$. *Assume* $T \in \mathcal{L}(\mathcal{S}_\infty(\mathbb{R}^n), \mathcal{S}'_\infty(\mathbb{R}^n))$, *and suppose* $\widehat{T} := \{\langle T\psi_P, \varphi_Q\rangle\}_{Q,P\in\mathcal{D}}$ *has an extension* $\widetilde{\widehat{T}} \in \mathcal{L}(\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\}))$. *Then* $\widetilde{T} := T_\psi\circ\widetilde{\widehat{T}}\circ S_\varphi$ *is an extension of* T *and* $\widetilde{T} \in \mathcal{L}(\dot{F}^{\alpha,q}_{p,\infty}(W))$.

Proof. For the operator S_φ, by replacing φ and $\widetilde{\varphi}$ in (7.1), we have

$$\|S_\varphi\vec{f}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})} \le C\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W,\widetilde{\varphi})},$$

which means that $S_\varphi : \dot{F}^{\alpha,q}_{p,\infty}(W, \widetilde{\varphi}) \to \dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})$ is bounded. For the operator T_ψ, by the conclusion (**b**) of Theorem 6.4 and Theorem 7.2, we have

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \le C\|\{\langle\vec{f}, \varphi_Q\rangle\}_{Q\in\mathcal{D}}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})},$$

moreover, for $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$, where $\vec{s}_Q := \langle \vec{f}, \varphi_Q\rangle$, and for $\varphi \in \mathcal{A}$, let $\hat{\psi} = \hat{\varphi}/(\sum_{j\in\mathbb{Z}} |\widehat{\varphi}_j|^2)$ by Lemma 6.3 we have $\vec{f} = \sum_{Q\in\mathcal{D}}\langle \vec{f}, \varphi_Q\rangle\psi_Q$, thus

$$\|T_\psi \vec{s}\|_{\dot{F}^{\alpha,q}_{p,\infty}(W)} \le C\|\vec{s}\|_{\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\})},$$

which means that $T_\psi : \dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\}) \to \dot{F}^{\alpha,q}_{p,\infty}(W)$ is bounded. Therefore, combining $\widetilde{\widehat{T}} \in \mathcal{L}(\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\}))$, it further shows that $\widetilde{T} \in \mathcal{L}(\dot{F}^{\alpha,q}_{p,\infty}(W))$. Since $\widetilde{\widehat{T}}$ is an extension of $\widehat{T}$, by Lemma 9.2, we infer that, for any $\vec{f} \in (\mathcal{S}_\infty(\mathbb{R}^n))^m$,

$$\widetilde{T}\vec{f} = T_\psi \circ \widetilde{\widehat{T}} \circ S_\varphi \vec{f} = T_\psi \circ S_\varphi \circ T\vec{f} = T\vec{f}.$$

From this and the Corollary 9.1, $\widetilde{T}$ is an extension of T. □

Corollary 9.4. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, $\varphi \in \mathcal{A}$, $\hat{\psi} = \hat{\varphi}/\left(\sum_{j\in\mathbb{Z}} |\widehat{\varphi}_j|^2\right) \in \mathcal{A}$, and $T \in \mathcal{L}(\mathcal{S}_\infty(\mathbb{R}^n), \mathcal{S}'_\infty(\mathbb{R}^n))$. If the matrix $\widehat{T} := \{\langle T\psi_P, \varphi_Q\rangle\}_{Q,P\in\mathcal{D}}$ is almost diagonal, then T admits an extension $\widetilde{T} \in \mathcal{L}(\dot{F}^{\alpha,q}_{p,\infty}(W))$.*

Proof. If the matrix $\widehat{T} := \{\langle T\psi_P, \varphi_Q\rangle\}_{Q,P\in\mathcal{D}}$ is almost diagonal, then by Theorem 5.5, $\widehat{T}$ admits an extension $\widetilde{\widehat{T}} \in \mathcal{L}(\dot{f}^{\alpha,q}_{p,\infty}(\{A_Q\}))$. Therefore, by Lemma 9.3, it follows that T admits an extension $\widetilde{T} \in \mathcal{L}(\dot{F}^{\alpha,q}_{p,\infty}(W))$. □

Next, we review the definition of the classical Calderón-Zygmund operators.

Definition 9.5. *Let $L \in \mathbb{N}$. We call T an L-smooth classical Calderón-Zygmund operator on $\mathbb{R}^n$. If $Tf(x) = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B(0,\epsilon)} K(y)f(x-y)\mathrm{d}y$, where the kernel K satisfies:*

- (M1) $|K(x)| \le \frac{c}{|x|^n}$, $x \in \mathbb{R}^n \setminus \{0\}$,
- (M2) $|D^\gamma K(x)| \le \frac{c}{|x|^{n+|\gamma|}}$, $x \in \mathbb{R}^n \setminus \{0\}$ *and* $|\gamma| \le L$,
- (M3) $\int_{R_1<|x|<R_2} K(x)\mathrm{d}x = 0$, $0 < R_1 < R_2 < \infty$.

Proposition 9.6. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, $T \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$, $0 < \delta \le 1$, $N > -\alpha + (\beta - n)/p + n/\min(1,p,q) - n - 1$, $K + \delta > \alpha + n/p$, $M > n/\min(1,p,q) + \beta/p$. Suppose there exists $N_0 \in \mathbb{Z}_+$ such that for every smooth N_0-atom a_Q associated with Q, the function $m_Q = Ta_Q$ satisfies conditions (i), (ii), (iii), (iv) in Definition 6.2, forming a family of smooth (N, K, M, δ)-molecules, where each constant C is independent of Q. Then the operator T is bounded on $\dot{F}^{\alpha,q}_{p,\infty}(W)$.*

Proof. By Lemma 5.3 in reference [15], we have $|\langle T\psi_P, \varphi_Q\rangle| \le c\omega_{QP}$, which implies that the matrix $\widehat{T} := \{\langle T\psi_P, \varphi_Q\rangle\}_{Q,P\in\mathcal{D}}$ is almost diagonal, i.e., $\widehat{T} \in \mathbf{ad}_{\mathbf{p}}^{\alpha,\mathbf{q}}(\beta)$. According to Corollary 9.4, the operator T is bounded on $\dot{F}^{\alpha,q}_{p,\infty}(W)$. □

By the above lemmas, suppose that a_Q is a smooth N_0-atom with $N_0 \in \mathbb{Z}_+$. For the classical Calderón-Zygmund operator T, let $Ta_Q = m_Q$. To prove that the Calderón-Zygmund operator T is bounded on $\dot{F}^{\alpha,q}_{p,\infty}(W)$, it suffices to show that m_Q forms a family of smooth molecules. For this purpose, the following lemma is further introduced to assist the proof, see Lemma 5.7 in Reference [15].

Lemma 9.7. *Let $N_0 \in \mathbb{Z}_+$, $L \in \mathbb{N}$, and a_Q be a smooth N_0-atom associated with Q. Let T be an L-smooth classical Calderón-Zygmund operator on $\mathbb{R}^n$, and set $Ta_Q = m_Q$. Then m_Q satisfies*

$$\int x^\gamma m_Q(x)\mathrm{dx} = 0,\ |\gamma| \le \mathrm{N}_0, \tag{9.1}$$

and

$$|D^\gamma m_Q(x)| \le c|Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}}\left(1+\frac{|x-x_Q|}{\ell(Q)}\right)^{-n-L},\ |\gamma| \le L. \tag{9.2}$$

Theorem 9.8. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$ with doubling exponent β. Let $L \in \mathbb{N}$ satisfy:* $(\mathrm{I}) : \mathrm{L} > \alpha + \mathrm{n/p}$, $(\mathrm{II}) : \mathrm{L} > -\alpha + (\beta - \mathrm{n})/\mathrm{p} + \mathrm{n}/\min(1,\mathrm{p},\mathrm{q}) - \mathrm{n}$, $(\mathrm{III}) : \mathrm{L} > \mathrm{n}/\min(1,\mathrm{p},\mathrm{q}) - \mathrm{n} + \beta/\mathrm{p}$. *If T is an L-smooth classical Calderón-Zygmund operator on $\mathbb{R}^n$, then the operator T is bounded on $\dot{F}^{\alpha,q}_{p,\infty}(W)$.*

Proof. Let N, K, and M satisfy the conditions required by Proposition 9.6, and take $\delta = 1$, $K = L - 1$. Let a_Q be a smooth N_0-atom for Q, and choose $N_0 > -\alpha + (\beta - n)/p + n/\min(1,p,q) - n - 1$. Set $Ta_Q = m_Q$. To prove that the Calderón-Zygmund operator T is bounded on $\dot{F}^{\alpha,q}_{p,\infty}(W)$, by Proposition 9.6, it suffices to show that m_Q satisfies conditions (i), (ii), (iii), and (iv) of Definition 6.2.

By Lemma 9.7, m_Q satisfies (9.1), and thus for $N = N_0$, m_Q satisfies condition (i) of Definition 6.2. Also, by (III) and (9.2), m_Q satisfies condition (iii) of Definition 6.2, where $M = L + n > n/\min(1,p,q) + \beta/p$ and $|\gamma| \le L$. For $\delta = 1$, $K = L - 1$, and $|\gamma| = L$, noting that by (I) we have $K + 1 = L > \alpha + n/p$, by the mean value theorem, m_Q satisfies condition (iv) of Definition 6.2. The decay of order $-M$ in condition (ii) of Definition 6.2 is obtained from (9.2) with $|\gamma| = 0$. According to this, along with (II) and (9.2), m_Q satisfies condition (ii) of Definition 6.2, where $N > -\alpha + (\beta - n)/p + n/\min(1,p,q)$.

In summary, m_Q satisfies conditions (i), (ii), (iii), and (iv) of Definition 6.2, and thus is a family of smooth (N,K,M,δ)-molecules. □

Remark 9.9. *In particular, the Hilbert transform $\mathbb{H}$ (for $n = 1$) and Riesz transforms $\mathcal{R}_j$ with $j = 1,\dots,n$ (for $n \ge 2$) are classical Calderón-Zygmund operators that are L-smooth for each value of L. Therefore, for all $\alpha \in \mathbb{R}$ and $0 < p < \infty$, $0 < q \le \infty$, they can be extended to be bounded on $\dot{F}^{\alpha,q}_{p,\infty}(W)$.*

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