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# Theta invariants and lattice-point counting in normed $\mathbb{Z}$-modules

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ABSTRACT. Euclidean lattices occupy a central position in number theory, the geometry of numbers, and modern cryptography. In the present article, the theory of Euclidean lattices is employed to investigate normed $\mathbb{Z}$-modules of finite rank. Specifically, let $\overline{E}$ be a normed $\mathbb{Z}$-module of finite rank. We establish several inequalities for the lattice-point counting function of $\overline{E}$, along with related results. Our arguments rely primarily on the analytic properties of the theta series associated with Euclidean lattices.

CONTENTS



## 1. Introduction

Euclidean lattices are fundamental objects in number theory and the geometry of numbers, as extensively studied in works such as [1, 2, 7, 9, 11, 15, 25]. In recent decades, they have also gained significant attention in cryptography, see [17, 18, 19]. Statements that relate the geometric invariants of a Euclidean lattice and its dual lattice are traditionally referred to as *transference theorems*. In his seminal work [1], Banaszczyk established remarkable transference inequalities involving the successive minima and the covering radius of Euclidean lattices. His approach relies on the analytical properties of the theta series $\theta_{\overline{E}}$ associated with Euclidean lattices $\overline{E}$, as well as on the Poisson summation formula (see Sections 2 and 3 for detailed definitions of these concepts). The functions $\theta_{\overline{E}}$ play a central role in the refined analysis of general Euclidean lattices, as demonstrated by Banaszczyk's method.

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Let $\overline{E} = (E, \|\cdot\|)$ be a normed $\mathbb{Z}$-module of finite rank $n$. We define $\hat{h}^0(\overline{E})$ as the real number given by

$$\hat{h}^0(\overline{E}) = \log \#\{s \in E \mid \|s\| \leq 1\}.$$

This invariant, $\hat{h}^0(\overline{E})$, plays a pivotal role in Arakelov geometry. It serves as an arithmetic analogue to the dimension of the space of sections of vector bundles over algebraic curves. We further define $\hat{h}^1(\overline{E}) := \hat{h}^0(\overline{E}^\vee)$, where $\overline{E}^\vee$ denotes the dual of the normed $\mathbb{Z}$-module $\overline{E}$. Additionally, we consider the arithmetic degree of $\overline{E}$, denoted by $\widehat{\deg}(\overline{E})$; for more details on these invariants, see Section 2.

Gillet and Soulé [10] established an arithmetic analogue of the geometric Riemann-Roch theorem for curves. This can be formulated as follows:

$$-\log(6)\cdot n \leq \hat{h}^0(\overline{E}) - \hat{h}^1(\overline{E}) - \widehat{\deg}(\overline{E}) \leq \log\left(\frac{3}{2}\right)\cdot n + 2\log n!. \tag{1.1}$$

Consequently, they demonstrated that the number of lattice points in a symmetric convex body is essentially determined by its successive minima, modulo a function that depends only on the rank of the convex body. This result has significant applications in Arakelov geometry and number theory. Henk [14] presented a remarkably simple proof of a result due to Gillet and Soulé, which relates the number of lattice points in a symmetric convex body to its successive minima.

In this paper, we present an alternative approach to studying lattice points. We establish several inequalities for the lattice-point counting function of $\overline{E}$, along with related results. Our bounds are somewhat coarser compared to those given by the Gillet-Soulé theorem. This is primarily because the proof of Gillet and Soulé employs a difficult result due to Bourgain and Milman [4] which gives a sharp lower bound for the product of the volumes of the unit balls associated with a normed $\mathbb{Z}$-module and its dual. Nevertheless, as noted by Boucksom in [3], these coarser bounds remain sufficient for various applications.

Our approach is rooted in the theory of Euclidean lattices, more particularly on theta series associated with Euclidean lattices. We define $h^0_\theta(\overline{E})$ as the real number given by:

$$h^0_\theta(\overline{E}) = \log \sum_{v\in E} e^{-\pi\|v\|^2}.$$

We call it *the theta invariant* of $\overline{E}$. We define $h^1_\theta(\overline{E}) := h^0_\theta(\overline{E}^\vee)$, where $\overline{E}^\vee$ denotes the dual of the Euclidean lattice $\overline{E}$ (see Section 3 for more details about these invariants).

A pivotal result in the theory of Euclidean lattices asserts that $h^0_\theta(\overline{E})$ and $\hat{h}^0(\overline{E})$ are essentially equal. More precisely, the difference $h^0_\theta(\overline{E}) - \hat{h}^0(\overline{E}) = O(n\log n)$ depends only on $n$, the rank of $E$ (see Proposition 3.1). This result plays a crucial role in this paper.

Next, we exhibit briefly a class of normed $\mathbb{Z}$-modules and Euclidean lattices that arise naturally in Arakelov geometry. For an introduction to Arakelov geometry, see [24]. Let $\mathcal{X}$ be a projective, integral, and flat scheme of dimension $n+1$ over $\mathbb{Z}$. Such schemes are referred to as arithmetic varieties over $\mathbb{Z}$. Assume $\mathcal{X}$ is an arithmetic variety over $\mathbb{Z}$ of dimension $n+1$. We assume that the generic fibre $\mathcal{X}_{\mathbb{Q}}$ is smooth. Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_{\overline{\mathcal{L}}})$ be a smooth Hermitian line bundle on $\mathcal{X}$.

For any $k \in \mathbb{N}$, we write $k\overline{\mathcal{L}} := \overline{\mathcal{L}}^{\otimes k}$, and denote by $n_k$ the rank of $H^0(\mathcal{X}, k\mathcal{L})$. We set $X := \mathcal{X}(\mathbb{C})$ and $L := \mathcal{L}(\mathbb{C})$. Let $\mu$ be a smooth volume form on $\mathcal{X}$. The space of global sections $H^0(X, L)$ is equipped with the $L^2$-norm:

$$\|s\|^2_{L^2,\overline{\mathcal{L}}} := \int_X \|s(x)\|^2_{\overline{\mathcal{L}}}\mu \quad \text{for any } s \in H^0(X, L).$$

Additionally, we consider the supremum norm, defined as:

$$\|s\|_{\sup,\overline{\mathcal{L}}} := \sup_{x\in X} \|s(x)\|_{\overline{\mathcal{L}}} \quad \text{for any } s \in H^0(X, L).$$

Thus, we obtain two normed $\mathbb{Z}$-modules: $\overline{H^0(\mathcal{X}, \mathcal{L})}_{L^2,\overline{\mathcal{L}}} = (H^0(\mathcal{X}, \mathcal{L}), \|\cdot\|_{L^2,\overline{\mathcal{L}}})$, which is Euclidean, and $\overline{H^0(\mathcal{X}, \mathcal{L})}_{\sup,\overline{\mathcal{L}}} = (H^0(\mathcal{X}, \mathcal{L}), \|\cdot\|_{\sup,\overline{\mathcal{L}}})$. Elements of $\overline{H^0(\mathcal{X}, \mathcal{L})}_{\sup,\overline{\mathcal{L}}}$ with norm less than or equal to 1 are called *small sections*.

The arithmetic volume $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})$ for a Hermitian line bundle $\overline{\mathcal{L}}$ on an arithmetic variety is a fundamental invariant in Arakelov geometry. It was introduced by Moriwaki in [20] as an analogue of the geometric volume function. Roughly speaking, this invariant measures the growth of the number of small sections of $k\overline{\mathcal{L}}$ as $k \to \infty$. Yuan [26] studied the bigness property of Hermitian line bundles on arithmetic varieties. The arithmetic volume function of $\overline{\mathcal{L}}$ is defined as follows:

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) := \limsup_{k\to\infty} \frac{\hat{h}^0(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{(\sup,k\overline{\mathcal{L}})})}{k^{n+1}/(n+1)!}.$$

The following theorem is our main result. We shall show that it is essentially a consequence of the theory of Euclidean lattices.

**Theorem 1.1.** *Let $\overline{E}$ be a normed $\mathbb{Z}$-module of rank $n$. Then the following inequality holds:*

$$-n\log n + \log\left(1 - \frac{1}{2\pi}\right) - \pi \le \widehat{h}^0(\overline{E}) - \widehat{h}^1(\overline{E}) - \widehat{\deg}(\overline{E}) \le n\log n + \pi - \log\left(1 - \frac{1}{2\pi}\right).$$

From this, we immediately obtain the asymptotic estimate

$$\widehat{h}^0(\overline{E}) - \widehat{h}^1(\overline{E}) - \widehat{\deg}(\overline{E}) = O(n\log n),$$

where the error term $O(n\log n)$ depends only on the rank $n$ of the lattice $E$. In view of (1.1), we may interpret Theorem 1.1 as an arithmetic Riemann–Roch theorem for normed $\mathbb{Z}$-modules.

A particularly important class of Euclidean lattices consists of those lattices $\overline{E}$ for which the underlying $\mathbb{Z}$-module $E$ admits a basis that is orthogonal with respect to the scalar product on $\overline{E}$. Such lattices arise naturally in the arithmetic geometry of toric varieties, as we now recall.

Let $\mathcal{X}$ be a smooth toric variety over $\operatorname{Spec}(\mathbb{Z})$, equipped with an action of the torus $\mathbb{T}$. Then $\mathcal{X}$ is determined by a complete nonsingular fan in the real vector space $N\otimes_{\mathbb{Z}}\mathbb{R}\simeq\mathbb{R}^n$, where $N$ is a free $\mathbb{Z}$-module of rank $n$. Denote by $M=N^\vee$ the dual $\mathbb{Z}$-module, and let $M_{\mathbb{R}}=M\otimes_{\mathbb{Z}}\mathbb{R}$.

Let $\mathcal{L}$ be a $\mathbb{T}$-equivariant line bundle on $\mathcal{X}$ that is generated by its global sections. Then there exists a $\mathbb{T}$-Cartier divisor $D$ on $\mathcal{X}$ such that $\mathcal{L}\simeq\mathcal{O}(D)$. The divisor $D$ determines a rational convex polytope $\Delta_D\subset M_{\mathbb{R}}$. The space of global sections of $\mathcal{O}(D)$ is then described combinatorially by the lattice points in $\Delta_D$, as follows:

$$H^0(\mathcal{X},\mathcal{O}(D))=\bigoplus_{m\in\Delta_D\cap M}\mathbb{Z}\cdot\chi^m,$$

where $\chi^m$ denotes the character associated with $m\in M$. For details, we refer the reader to [8, 23].

Following [5, Section 3], we define a Hermitian line bundle $\overline{\mathcal{O}(D)}:=(\mathcal{O}(D),\|\cdot\|)$ on $\mathcal{X}$ as toric if $D$ is a toric divisor and its associated Green function is invariant under the action of $\mathbb{S}$, the compact torus of $\mathcal{X}(\mathbb{C})$. Let $\mu$ denote a smooth volume form on $\mathcal{X}(\mathbb{C})$, which is invariant with respect to the action of $\mathbb{S}$. Furthermore, let $\overline{\mathcal{O}(D)}$ be a toric, continuous Hermitian line bundle on $\mathcal{X}$. One can see that $(\chi^m)_{m\in\Delta_D\cap M}$ forms a $\mathbb{Z}$-basis of $\overline{H^0(\mathcal{X},\mathcal{O}(D))}_{L^2,\overline{\mathcal{O}(D)}}$ that is orthogonal with respect to the Euclidean norm $\|\cdot\|_{L^2,\overline{\mathcal{O}(D)}}$. This observation plays a key role in the proof of the integral representation for the arithmetic volume of toric Hermitian line bundles (see [22, Lemma 2.2] and [13, 5]). For further background on the Arakelov geometry of toric varieties, we refer the reader to [5, 6].

The classical theory of Euclidean lattice reduction seeks to construct distinguished bases of Euclidean lattices, commonly referred to as *reduced bases*. Loosely speaking, this theory demonstrates that an Euclidean lattice of rank $n>0$ can be effectively approximated by a direct sum of rank one Euclidean lattices, expressed as $\overline{E}_1\oplus\overline{E}_2\oplus\ldots\oplus\overline{E}_n$, where $\overline{E}_1,\ldots,\overline{E}_n$ denote rank one Euclidean lattices. This approximation is achieved with a controlled error that depends on $n$. Consequently, the lattice can be approximately characterized by $n$ real parameters $\mu_i:=\widehat{\deg}\,\overline{E}_i$. For detailed presentations and relevant references, the reader is referred to [16, 17].

Motivated by the above discussion, we prove the following result.

**Theorem 1.2.** *Let $\overline{E} = (E, \|\cdot\|)$ be a Euclidean lattice of rank n, and suppose that $\overline{E}$ admits an orthogonal $\mathbb{Z}$-basis. Then the following inequality holds:*

$$-\frac{1}{2}n\log n + \log\left(1 - \tfrac{1}{2\pi}\right) \leq \hat{h}^0(\overline{E}) + \sum_{i=1}^{n} \log\min(\lambda_i(\overline{E}), 1) \leq \pi + n\log\frac{3}{2},$$

*where $\lambda_i(\overline{E})$ is the i-th successive minimum of $\overline{E}$, see Section* 4 *for the definition.*

We generalize this result in Corollary 4.3 by showing that, for any normed $\mathbb{Z}$-module $\overline{E}$ of rank $n$, the estimate

$$\hat{h}^0(\overline{E}) + \sum_{i=1}^{n} \log\min(\lambda_i(\overline{E}), 1) = O(n\log n)$$

holds, where the error term $O(n\log n)$ depends only on $n$.

In Paragraph 4.1, we study the notion of arithmetic bigness in Arakelov geometry. The results presented in this section are primarily applications of the theory developed in this paper. Let $\mathcal{X}$ be an arithmetic variety over $\mathbb{Z}$ of dimension $n+1$. We assume that the generic fibre $\mathcal{X}_{\mathbb{Q}}$ is smooth. Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_{\overline{\mathcal{L}}})$ be a smooth Hermitian line bundle on $\mathcal{X}$. Yuan [26] introduced the condition:

$$\liminf_{k\to\infty} \frac{\log \#\{s \in H^0(\mathcal{X}, k\mathcal{L}) \mid \|s\|_{\sup,k\overline{\mathcal{L}}} < 1\}}{k^{n+1}/(n+1)!} > 0,$$

as a criterion for defining an arithmetically big line bundle. Moriwaki [21] proposed an alternative definition for arithmetic big line bundles: $\overline{\mathcal{L}}$ is said to be arithmetically big if $\mathcal{L}_{\mathbb{Q}}$ is big and there exists a positive integer $k$ and a nonzero global section $s$ of $k\overline{\mathcal{L}}$ such that $\|s\|_{\sup,k\overline{\mathcal{L}}} < 1$. He demonstrated that Yuan's definition is equivalent to the existence of a nonzero section of a power of $\mathcal{L}$ with supremum norm less than 1, and that $\mathcal{L}_{\mathbb{Q}}$ is big [1].

We provide an alternative proof of Moriwaki's result; see Theorem 4.6 and the discussion following it.

## 2. Normed $\mathbb{Z}$-modules

A normed $\mathbb{Z}$-module $\overline{E} = (E, \|\cdot\|)$ is a $\mathbb{Z}$-module of finite type endowed with a norm $\|\cdot\|$ on the $\mathbb{C}$-vector space $E_{\mathbb{C}} = E \otimes_{\mathbb{Z}} \mathbb{C}$. Let $E_{\mathrm{tors}}$ denote the torsion-module of $E$, $E_{\mathrm{free}} = E/E_{\mathrm{tors}}$, and $E_{\mathbb{R}} = E \otimes_{\mathbb{Z}} \mathbb{R}$. We let $B(E, \|\cdot\|) = \{m \in E_{\mathbb{R}} \mid \|m\| \leq 1\}$. There exists a unique Haar measure on $E_{\mathbb{R}}$ such that the volume of $B(E, \|\cdot\|)$ is 1. We let

$$\widehat{\chi}(E, \|\cdot\|) = \log \#E_{\mathrm{tors}} - \log\mathrm{vol}(E_{\mathbb{R}}/(E/E_{\mathrm{tors}})).$$

[1] That is, $\mathrm{vol}(\mathcal{L}_{\mathbb{Q}}) > 0$, which by definition means

$$\limsup_{k\to\infty} \frac{h^0(\mathcal{X}_{\mathbb{Q}}, k\mathcal{L}_{\mathbb{Q}})}{k^n/n!} > 0,$$

where $n = \dim \mathcal{X}_{\mathbb{Q}}$.

Equivalently, we have

$$\widehat{\chi}(E, \|\cdot\|) = \log \# E_{\mathrm{tors}} - \log\left(\frac{\mathrm{vol}(E_{\mathbb{R}}/(E/E_{\mathrm{tor}}))}{\mathrm{vol}(B(E, \|\cdot\|))}\right),$$

for any choice of a Haar measure of $E_{\mathbb{R}}$.

The arithmetic degree of $(E, \|\cdot\|)$ is defined as follows

$$\widehat{\deg}(E, \|\cdot\|) = \widehat{\deg}\,\overline{E} = \widehat{\chi}(\overline{E}) - \widehat{\chi}(\overline{\mathbb{Z}}^n),$$

where $\widehat{\chi}(\overline{\mathbb{Z}}^n) = -\log\left(\Gamma(\frac{n}{2}+1)\pi^{-\frac{n}{2}}\right)$, with $n$ is the rank of $E_{\mathbb{R}}$.

When the norm $\|\cdot\|$ is induced by a Hermitian product $\langle\cdot,\cdot\rangle$, we have

$$\widehat{\deg}(\overline{E}) = \log \# E/(s_1, \dots, s_n) - \log\sqrt{\det(\langle s_i, s_j\rangle)_{1\le i,j\le n}},$$

where $s_1, \dots, s_n$ are elements of $E$ such that their images in $E \otimes_{\mathbb{Z}} \mathbb{Q}$ form a basis.

We define $\widehat{H}^0(\overline{E})$ and $\widehat{h}^0(\overline{E})$ to be

$$\widehat{H}^0(\overline{E}) = \{m \in E \mid \|m\| \le 1\} \quad \text{and} \quad \widehat{h}^0(\overline{E}) = \log \# \widehat{H}^0(\overline{E}).$$

We let

$$\widehat{H}^1(\overline{E}) := \widehat{H}^0(\overline{E}^\vee) \quad \text{and} \quad \widehat{h}^1(\overline{E}) := \widehat{h}^0(\overline{E}^\vee),$$

where $\overline{E}^\vee$ is the $\mathbb{Z}$-module $E^\vee = \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$ endowed with the dual norm $\|\cdot\|^\vee$ defined as follows

$$\|f\|^\vee = \sup_{m \in E_{\mathbb{R}} \setminus \{0\}} \frac{|f(m)|}{\|m\|}, \quad \forall f \in E^\vee.$$

Gillet and Soulé [10] proved an arithmetic analogue of geometric Riemann-Roch theorem for curves. It can be stated as follows:

$$-\log(6)\,\mathrm{rk}\,E \le \widehat{h}^0(\overline{E}) - \widehat{h}^1(\overline{E}) - \widehat{\deg}(\overline{E}) \le \log(\tfrac{3}{2})\mathrm{rk}\,E + 2\log((\mathrm{rk}\,E)!), \quad (2.1)$$

see [21, Proposition 2.1] and also [26].

In this paper, $\overline{E}_t$ will denote the normed $\mathbb{Z}$-module $E$ endowed with the norm $t\|\cdot\|$ where $t > 0$.

Let $v_n$ denote the volume of the unit ball in $\mathbb{R}^n$ endowed with its standard Euclidean structure. It is known that

$$v_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

Stirling's formula gives that

$$-\frac{n}{2}\log n + n\log 2 \le \log v_n \le -\frac{n}{2}\log n + \frac{n}{2}(\log(2\pi)+1) \quad \forall n \gg 1. \quad (2.2)$$

## 3. On theta invariants of Euclidean lattices

In this section, we review some elements of the theory of Euclidean lattices. Let $\overline{E} = (E, \|\cdot\|_{\overline{E}})$ be a Euclidean lattice over $\mathbb{Z}$ of rank $n$. We recall that this means that we are given the following data: A finite-dimensional $\mathbb{R}$-vector space $E_{\mathbb{R}}$, ($n := \dim_{\mathbb{R}} E_{\mathbb{R}}$), a lattice $E$ in $E_{\mathbb{R}}$, and $\{e_1, \dots, e_n\}$ a $\mathbb{R}$-basis of $E_{\mathbb{R}}$ such that $E = \bigoplus_{i=1}^n \mathbb{Z}e_i$, and a Euclidean norm $\|\cdot\|_{\overline{E}}$ associated to some Euclidean scalar product $\langle\cdot,\cdot\rangle$ on $E_{\mathbb{R}}$.

Let $\lambda_{\overline{E}}$ be the unique translation-invariant Radon measure on $E_{\mathbb{R}}$ which satisfies the following normalization condition: for every orthonormal basis $\{e_1, \dots, e_n\}$ of $(E_{\mathbb{R}}, \|\cdot\|_{\overline{E}})$,

$$\lambda_{\overline{E}}\Big(\sum_{i=1}^{n} [0,1[e_i\Big) = 1.$$

We set

$$\operatorname{covol}(\overline{E}) := \operatorname{vol}(E_{\mathbb{R}}/E),$$

which is, by definition, $\lambda_{\overline{E}}(\sum_{i=1}^{N} [0,1[v_i)$ for every $\mathbb{Z}$-basis $\{v_1, \dots, v_n\}$ of $E$. $\operatorname{covol}(\overline{E})$ is called the covolume of $\overline{E}$. Note that

$$\widehat{\deg}(\overline{E}) = -\log \operatorname{covol}(\overline{E}).$$

$\overline{E}$ induces a natural Euclidean structure on the dual $E^{\vee}$. We denote the resulting Euclidean lattice by $\overline{E}^{\vee}$.

We let

$$\theta_{\overline{E}}(t) = \sum_{v\in E} e^{-\pi t \|v\|_{\overline{E}}^2} \quad (t > 0).$$

$\theta_{\overline{E}}$ is called *the theta series* associated with $\overline{E}$.

By the Poisson summation formula, we obtain a relation between the theta series of $\overline{E}$ and $\overline{E}^{\vee}$. That is

$$\sum_{v\in E} e^{-\pi\|v\|_{\overline{E}}^2} = (\operatorname{covol}(\overline{E}))^{-1} \sum_{v^{\vee}\in E^{\vee}} e^{-\pi\|v^{\vee}\|_{\overline{E}^{\vee}}^2}. \tag{3.1}$$

One can attach to $\overline{E}$ another arithmetic invariant $h^0_{\theta}(\overline{E})$ called the *theta invariant* of $\overline{E}$. It is given as follows:

$$h^0_{\theta}(\overline{E}) := \log \theta_{\overline{E}}(1).$$

We let

$$h^1_{\theta}(\overline{E}) := h^0_{\theta}(\overline{E}^{\vee}).$$

The equation (3.1) may be written as follows:

$$h^0_{\theta}(\overline{E}) - h^1_{\theta}(\overline{E}) - \widehat{\deg}(\overline{E}) = 0. \tag{3.2}$$

**Proposition 3.1.** *Let $\overline{E}$ be an Euclidean lattice. We have*

$$\begin{aligned} h^0_\theta(\overline{E}) - \frac{1}{2}\mathrm{rk}\,E\log\mathrm{rk}\,E + \log\Big(1-\tfrac{1}{2\pi}\Big) &\leq \log\#\{v\in E|\,\|v\|_{\overline{E}}<1\} \\ &\leq \log\#\{v\in E|\,\|v\|_{\overline{E}}\leq 1\} \leq h^0_\theta(\overline{E})+\pi, \end{aligned} \tag{3.3}$$

*where* rk $E$ *denotes the rank of the lattice* $E$.

**Proof.** See [1] or [2]. For reader's convenience, we recall the proof of (3.3). By the Poisson summation formula, we have

$$\log\theta_{\overline{E}}(t) + \frac{1}{2}\mathrm{rk}\,E\log t + \log\mathrm{covol}(\overline{E}) = \log\theta_{\overline{E}^\vee}(\frac{1}{t}) \quad \forall t>0. \tag{3.4}$$

We differentiate this equation to get that

$$\sum_{v\in E}\|v\|^2_{\overline{E}}\frac{e^{-\pi t\|v\|^2_{\overline{E}}}}{\sum_{u\in E}e^{-\pi t\|u\|^2_{\overline{E}}}} + \frac{1}{t^2}\sum_{v^\vee\in E^\vee}\|v^\vee\|^2_{\overline{E}^\vee}\frac{e^{-\frac{\pi}{t}\|v^\vee\|^2_{\overline{E}^\vee}}}{\sum_{u^\vee\in E^\vee}e^{-\frac{\pi}{t}\|u^\vee\|^2_{\overline{E}^\vee}}} = \frac{\mathrm{rk}\,E}{2\pi t} \quad \forall t>0.$$

It follows that

$$\sum_{v\in E}\|v\|^2_{\overline{E}}e^{-\pi t\|v\|^2_{\overline{E}}} \leq \frac{\mathrm{rk}\,E}{2\pi t}\sum_{u\in E}e^{-\pi t\|u\|^2_{\overline{E}}} \quad \forall\, t>0.$$

From which we infer the following inequality

$$\Big(1-\frac{\mathrm{rk}\,E}{2\pi t}\Big)\sum_{u\in E}e^{-\pi t\|u\|^2_{\overline{E}}} \leq \sum_{\substack{u\in E\\ \|u\|_{\overline{E}}<1}}e^{-\pi t\|u\|^2_{\overline{E}}} \quad \forall t>0. \tag{3.5}$$

Let $t>\max(1,\frac{\mathrm{rk}\,E}{2\pi})$. We have

$$\begin{aligned} \log\#\{v\in E|\,\|v\|_{\overline{E}}<1\} &\geq \log\Big(\sum_{\substack{v\in E\\ \|v\|<1}}e^{-\pi t\|v\|^2}\Big) \\ &\geq \log\theta_{\overline{E}}(t) + \log\Big(1-\frac{\mathrm{rk}\,E}{2\pi t}\Big) \quad \text{(by (3.5))} \\ &\geq \log\theta_{\overline{E}}(1) - \frac{\mathrm{rk}\,E}{2}\log t + \log\Big(1-\frac{\mathrm{rk}\,E}{2\pi t}\Big) \quad \text{(by (3.4))}. \end{aligned}$$

By taking $t=\mathrm{rk}\,E$, we obtain

$$\log\#\big\{v\in E|\,\|v\|_{\overline{E}}<1\big\} \geq h^0_\theta(\overline{E}) - \frac{\mathrm{rk}\,E}{2}\log\mathrm{rk}\,E + \log\Big(1-\tfrac{1}{2\pi}\Big).$$

On the other hand, it is clear that

$$\log\#\big\{v\in E|\,\|v\|_{\overline{E}}\leq 1\big\} \leq h^0_\theta(\overline{E})+\pi.$$

This concludes the proof of the proposition. □

**Proposition 3.2.** *Let $\overline{E}$ be an Euclidean lattice. We have*

$$-\frac{1}{2}\mathrm{rk}(E)\log\mathrm{rk}(E)+\log\left(1-\tfrac{1}{2\pi}\right)-\pi\leq \widehat{h}^0(\overline{E})-\widehat{h}^1(\overline{E})-\widehat{\deg}(\overline{E})$$
$$\leq \frac{1}{2}\mathrm{rk}(E)\log\mathrm{rk}(E)+\pi-\log\left(1-\tfrac{1}{2\pi}\right).$$

**Proof.** We combine (3.3) with (3.2) to conclude the proof of the proposition. □

**Proof of Theorem 1.1.** Let $\overline{E}=(E,\|\cdot\|)$ be a normed $\mathbb{Z}$-module of rank $n$. There exists a Euclidean norm $\|\cdot\|_J$ on $E$ satisfying the following

$$\|\cdot\|\leq\|\cdot\|_J\leq n^{\frac{1}{2}}\|\cdot\|.$$

This norm is called John norm, see for instance [2, Appendix F, 355]. This gives us the following inequalities.

$$\widehat{\chi}(\overline{E}_J)\leq\widehat{\chi}(\overline{E})\leq\frac{n}{2}\log n+\widehat{\chi}(\overline{E}_J),$$

and

$$\widehat{h}^0(\overline{E}_J)\leq\widehat{h}^0(\overline{E})\leq\widehat{h}^0((\overline{E}_J)_{n^{-\frac{1}{2}}}),$$

and

$$\widehat{h}^0\left(\left((\overline{E}_J)_{n^{-\frac{1}{2}}}\right)^{\vee}\right)\leq\widehat{h}^0(\overline{E}^{\vee})\leq\widehat{h}^0\left(\left(\overline{E}_J\right)^{\vee}\right),$$

where $\overline{E}_J=(E,\|\cdot\|_J)$.

Let $\lambda_{\overline{E}_J}$ denote the unique Lebesgue measure on $E_{\mathbb{R}}$ that gives the volume 1 to the unit cube in $(E_{\mathbb{R}},\|\cdot\|_J)$. Then

$$\widehat{\chi}(E,\|\cdot\|_J)=\log\mathrm{vol}(B(E,\|\cdot\|_J))+\widehat{\deg}(\overline{E}_J).$$

Consequently, we get

$$\log v_n+\widehat{\deg}(\overline{E}_J)\leq\widehat{\chi}(\overline{E})\leq\log v_n+(n/2)\log n+\widehat{\deg}(\overline{E}_J).$$

So

$$\widehat{h}^0(\overline{E}_J)-\widehat{h}^1(\overline{E}_J)-\log v_n-\frac{n}{2}\log n-\widehat{\deg}(\overline{E}_J)\leq\widehat{h}^0(\overline{E})-\widehat{h}^1(\overline{E})-\widehat{\chi}(\overline{E})$$
$$\leq\widehat{h}^0\left((\overline{E}_J)_{n^{-\frac{1}{2}}}\right)-\widehat{h}^1\left((\overline{E}_J)_{n^{-\frac{1}{2}}}\right)-\log v_n-\widehat{\deg}(\overline{E}_J).$$

From Proposition 3.2, we obtain

$$-n\log n-\log v_n+\log\left(1-\tfrac{1}{2\pi}\right)-\pi\leq\widehat{h}^0(\overline{E})-\widehat{h}^1(\overline{E})-\widehat{\chi}(\overline{E})$$
$$\leq n\log n+\pi-\log\left(1-\tfrac{1}{2\pi}\right)-\log v_n.$$

We use (2.2) to end the proof of the Theorem. □

## 4. Successive minima and arithmetic bigness

Minkowski defined $n$ successive minima of a given convex body. In the context of normed $\mathbb{Z}$-modules, they are given as follows. Let $\overline{E}$ be a normed $\mathbb{Z}$-module of positive rank $n$. The successive minima $(\lambda_i(\overline{E}))_{i=1,\dots,n}$ of $\overline{E}$ are defined as follows:

$$\lambda_i(\overline{E}) = \inf\left\{r > 0 \mid \dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(E \cap \{m \in E_{\mathbb{R}} \mid \|m\| \leq r\})) \geq i\right\}$$

for $i = 1, \dots, n$. Clearly

$$\lambda_1(\overline{E}) \leq \dots \leq \lambda_n(\overline{E}).$$

**Lemma 4.1.** *Let $\overline{\mathbb{Z}}$ be the Euclidean lattice $\mathbb{Z}$ endowed with the standard norm on $\mathbb{R}$. We have*

$$1 \leq \min(1, \sqrt{t})\theta_{\overline{\mathbb{Z}}}(t) \leq \frac{3}{2} \quad \forall t > 0.$$

**Proof.** Let $t \geq 1$, then $\theta_{\overline{\mathbb{Z}}}(t) \leq \theta_{\overline{\mathbb{Z}}}(1)$. On the other hand, let $t \in (0,1)$. Since $\theta_{\overline{\mathbb{Z}}}(t) = \frac{1}{\sqrt{t}}\theta_{\overline{\mathbb{Z}}}(\frac{1}{t})$. Then $\theta_{\overline{\mathbb{Z}}}(t) \leq \frac{1}{\sqrt{t}}\theta_{\mathbb{Z}}(1)$. We infer that

$$\theta_{\overline{\mathbb{Z}}}(t) \leq \frac{\theta_{\overline{\mathbb{Z}}}(1)}{\min(1, \sqrt{t})} \quad \forall\, t > 0.$$

Since $\min(1, \sqrt{t})\theta_{\overline{\mathbb{Z}}}(t) = \min(1, \frac{1}{\sqrt{t}})\theta_{\overline{\mathbb{Z}}}(\frac{1}{t})$ for every $t > 0$ and $\theta_{\overline{\mathbb{Z}}}$ is a nondecreasing function and $\theta_{\mathbb{Z}}(t) \geq 1$, we obtain

$$1 \leq \min(1, \sqrt{t})\theta_{\overline{\mathbb{Z}}}(t) \quad \forall t > 0.$$

Using the geometric growth of the exponential terms, we estimate:

$$\theta_{\overline{\mathbb{Z}}}(1) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2} \leq \frac{1 + e^{\pi}}{1 - e^{-\pi}} \leq \frac{3}{2}.$$

Thus, we establish the inequality:

$$1 \leq \min(1, \sqrt{t})\theta_{\overline{\mathbb{Z}}}(t) \leq \frac{3}{2} \quad \forall t > 0.$$

This ends the proof of the lemma. □

**Proof of Theorem 1.2.** Let $\{e_1, \dots, e_n\}$ be an orthogonal $\mathbb{Z}$-basis of $\overline{E}$. Without loss of generality, assume that $\|e_1\| \leq \dots \leq \|e_n\|$. Then

$$\lambda_i(\overline{E}) = \|e_i\| \quad (i = 1, \dots, n).$$

We have

$$\begin{aligned}
\theta_{\overline{E}}(1)\prod_{i=1}^{n}\min(\lambda_i(\overline{E}),1) &= \prod_{i=1}^{n}\theta_{\overline{\mathbb{Z}}}(\|e_i\|^2)\prod_{i=1}^{n}\min(\|e_i\|,1)\\
&\leq \prod_{i=1}^{n}\frac{\frac{3}{2}}{\min(\|e_i\|,1)}\prod_{i=1}^{n}\min(\|e_i\|,1) \ \text{ (by Lemma 4.1)}\\
&\leq \left(\frac{3}{2}\right)^n .
\end{aligned}$$

On the other hand, by applying Lemma 4.1 once again, we obtain:

$$\begin{aligned}
1 &\leq \prod_{\substack{i=1,\dots,n\\ \lambda_i(\overline{E})\leq 1}}\theta_{\overline{\mathbb{Z}}}(\lambda_i(\overline{E})^2)\prod_{\substack{i=1,\dots,n\\ \lambda_i(\overline{E})\leq 1}}\lambda_i(\overline{E})\\
&= \prod_{\|e_i\|\leq 1}\theta_{\overline{\mathbb{Z}}}(\|e_i\|^2)\prod_{\|e_i\|\leq 1}\|e_i\|\\
&\leq \prod_{i=1}^{n}\theta_{\overline{\mathbb{Z}}}(\|e_i\|^2)\prod_{i=1}^{n}\min(\|e_i\|,1)\\
&= \theta_{\overline{E}}(1)\prod_{i=1}^{n}\min(\lambda_i(\overline{E}),1),
\end{aligned}$$

where the final inequality follows from the orthogonality property.

Combining these results, we obtain:

$$0 \leq \hat{h}^0_{\theta}(\overline{E}) + \sum_{i=1}^{n}\log\min(\lambda_i(\overline{E}),1) \leq n\log\frac{3}{2}.$$

Finally, by Proposition 3.1, we conclude:

$$-\frac{1}{2}n\log n + \log\left(1-\tfrac{1}{2\pi}\right) \leq \hat{h}^0(\overline{E}) + \sum_{i}\log\min(\lambda_i(\overline{E}),1) \leq \pi + n\log\tfrac{3}{2}.$$

This completes the proof. □

**Theorem 4.2.** *Let $\overline{E}$ be a Euclidean lattice. We have*

$$\begin{aligned}
-\pi - \log n! + \log\left(\tfrac{2}{e\pi}\right)\tfrac{n}{2} - \tfrac{n}{2}\log n + \log\left(1-\tfrac{1}{2\pi}\right) &\leq\\
\hat{h}^0(\overline{E}) + \sum_{i=1}^{n}\log\min(\lambda_i(\overline{E}),1) &\leq \pi - \log\left(1-\tfrac{1}{2\pi}\right) + n\log n,
\end{aligned}$$

*where $n$ is the rank of $E$.*

**Proof.** Let $E_0$ be the $\mathbb{Z}$-submodule of $E$ generated by the elements of the set $E\cap\{m\in E_{\mathbb{R}} \mid \|m\|<1\}$. Let $\overline{E}_0$ denote the Euclidean lattice $E_0$ equipped with the induced Euclidean norm from $\overline{E}$. It is clear that:

$$\widehat{h}^1(\overline{E}_0) = 0.$$

Now, consider the case where $\lambda_1(\overline{E}) > 1$. In this situation, we observe that:

$$\widehat{h}^0(\overline{E}) = 0.$$

Thus, the theorem holds true in this case.

Let us consider the case where $\lambda_1(\overline{E}) < 1$. In this case, the submodule $E_0$ has positive rank. It is straightforward to see that $\prod_i \lambda_i(\overline{E}_0) = \prod_i \min(\lambda_i(\overline{E}), 1)$. From equation (3.3), we obtain the bounds:

$$e^{-\pi} \leq \theta_{\overline{E}_0^\vee}(1) \leq \frac{2\pi}{2\pi - 1} n_0^{n_0/2},$$

where $n_0$ is the rank of $E_0$.

By Minkowski's theorem on successive minima ([12, Theorem 1, p. 59, Theorem 2, p. 62]),

$$\frac{2^{n_0}}{n_0!} \leq \lambda_1(\overline{E}_0) \cdots \lambda_{n_0}(\overline{E}_0) \operatorname{vol}(B(E_0, \|\cdot\|)) \leq 2^{n_0}.$$

where $\operatorname{vol}(\cdot)$ is the volume function with respect to the Lebesgue measure that gives volume 1 to $E_{\mathbb{R}}/E$. Note that $\theta_{\overline{E}_0}(1)\prod_i \lambda_i(\overline{E}_0) = \frac{\prod_i \lambda_i(\overline{E}_0)}{\operatorname{covol}(\overline{E}_0)} \theta_{\overline{E}_0^\vee}(1)$.

We conclude that

$$e^{-\pi}\frac{2^{n_0}}{n_0! v_{n_0}} \leq \theta_{\overline{E}_0}(1) \prod_i \lambda_i(\overline{E}_0) \leq \frac{2\pi}{2\pi - 1} n_0^{n_0/2} \frac{2^{n_0}}{v_{n_0}}.$$

We use (2.2) to obtain that

$$e^{-\pi}\frac{1}{n!}\left(\frac{2}{e\pi}\right)^{\frac{n}{2}} \leq \theta_{\overline{E}_0}(1) \prod_i \lambda_i(\overline{E}_0) \leq \frac{2\pi}{2\pi - 1} n^n.$$

Note that $2^n/n!$ is a decreasing function.

Since $\{m \in E_{\mathbb{R}} \mid \|m\| < 1\} \cap E = B_{\overline{E}_0}(0,1) \cap E_0$, we can use Proposition 3.1 to deduce that

$$\frac{e^{-\pi}}{n!}\left(\frac{2}{e\pi}\right)^{\frac{n}{2}} n^{-\frac{n}{2}}(1 - \frac{1}{2\pi}) \leq \#(B_{\overline{E}}(0,1) \cap E) \prod_i \min(\lambda_i(\overline{E}), 1) \leq e^{\pi} \frac{2\pi}{2\pi - 1} n^n. \quad (4.1)$$

It remains to consider the case when $\lambda_1(\overline{E}) = 1$. We see that (4.1) holds for $\overline{E}_t$ with $t \in (0,1)$. By letting $t \to 1$, we conclude that (4.1) holds for $\overline{E}$. This ends the proof of the theorem. □

**Corollary 4.3.** *Let $\overline{E}$ be a normed lattice of rank $n$. We have*

$$-\pi - \log n! + \frac{n}{2}\log\left(\frac{2}{e\pi}\right) - n\log n + \log\left(1 - \frac{1}{2\pi}\right)$$
$$\leq \hat{h}^0(\overline{E}) + \sum_{i=1}^{n} \log\min(\lambda_i(\overline{E}), 1) \leq 2\pi - 2\log\left(1 - \frac{1}{2\pi}\right) + \frac{3n}{2}\log n.$$

**Proof.** Let $\overline{E}$ be a normed lattice. Let $\|\cdot\|_J$ be the John norm on $E_{\mathbb{R}}$ that satisfies

$$\|\cdot\| \leq \|\cdot\|_J \leq n^{\frac{1}{2}}\|\cdot\|.$$

Let us denote by $\overline{E}_J$ the Euclidean lattice $E$ endowed with $\|\cdot\|_J$.

We have

$$\begin{aligned}
\#\left(B(E,\|\cdot\|)\cap E\right)&\prod_i \min(\lambda_i(\overline{E}),1)\\
&\leq \#\left(B(E,\tfrac{1}{\sqrt{n}}\|\cdot\|_J)\cap E\right)\prod_i \min(\lambda_i(\overline{E}_J),1)\\
&\leq e^{\pi}\theta_{(\overline{E}_J)_{\frac{1}{\sqrt{n}}}}(1)\prod_i \min(\lambda_i(\overline{E}_J),1)\\
&\leq e^{\pi} n^{\frac{n}{2}}\theta_{\overline{E}_J}(1)\prod_i \min(\lambda_i(\overline{E}_J),1)\\
&\leq n^{\frac{n}{2}} e^{\pi}\frac{2\pi}{2\pi-1}\#(B(E,\|\cdot\|_J)\cap E)\prod_i \min(\lambda_i(\overline{E}_J),1)\\
&\leq e^{2\pi}\left(\frac{2\pi}{2\pi-1}\right)^2 n^{\frac{3n}{2}},
\end{aligned}$$

where we have used Theorem 4.2 and that $t\mapsto \log\theta_{\overline{E}}(t)+\frac{n}{2}\log t$ is an increasing function.

Note that $\min(\lambda_i(\overline{E}),1)\geq \frac{1}{\sqrt{n}}\min(\lambda_i(\overline{E}_J),1)$ for every $i=1,\dots,n$. We deduce that

$$\hat{h}^0(\overline{E}_J)+\sum_i \log\min(\lambda_i(\overline{E}),1)-\frac{n}{2}\log n \leq \hat{h}^0(\overline{E})+\sum_i \log\min(\lambda_i(\overline{E}),1).$$

Using Theorem 4.2 once again, we derive the following inequality:

$$\begin{aligned}
-\pi-\log n! + \frac{n}{2}\log\left(\frac{2}{e\pi}\right) - n\log n + \log\left(1-\frac{1}{2\pi}\right)&\\
\leq \hat{h}^0(\overline{E})+\sum_i \log\min(\lambda_i(\overline{E}),1).&
\end{aligned}$$

This concludes the proof of the corollary. □

**4.1. Arithmetic bigness.** Let $\mathcal{X}$ be an arithmetic variety over $\mathbb{Z}$ of dimension $n+1$ and such that $\mathcal{X}_{\mathbb{Q}}$ is smooth. Let $\overline{\mathcal{L}}=(\mathcal{L},\|\cdot\|_{\overline{\mathcal{L}}})$ be a smooth Hermitian line bundle on $\mathcal{X}$. For any $k\in\mathbb{N}$, we write $k\overline{\mathcal{L}} := \overline{\mathcal{L}}^{\otimes k}$, we let $n_k$ denote the rank of $H^0(\mathcal{X},k\mathcal{L})$. We set $X := \mathcal{X}(\mathbb{C})$, and $L := \mathcal{L}(\mathbb{C})$. Let $\mu$ be a smooth volume form on $\mathcal{L}$. The space of global sections $H^0(X,L)$ is endowed with the $L^2$-norm

$$\|s\|^2_{L^2,\overline{\mathcal{L}}} := \int_X \|s(x)\|^2_{\overline{\mathcal{L}}}\mu \quad \text{for any } s\in H^0(X,L).$$

Also we consider the sup norm defined as follows

$$\|s\|_{\sup,\overline{\mathcal{L}}} := \sup_{x\in X}\|s(x)\|_{\overline{\mathcal{L}}} \quad \text{for any } s\in H^0(X,L).$$

For an introduction to Arakelov geometry, see [24].

There are several notions of arithmetic positivity for a Hermitian line bundle on an arithmetic variety. We refer the reader to [21, 27], or to [5, p. 227] for a detailed discussion of these concepts.

A Hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ is said to be *big* if:

- The generic fiber $\mathcal{L}_{\mathbb{Q}}$ is big,[2]
- There exists a positive integer $k$ and a nonzero section $s \in H^0(\mathcal{X}, k\mathcal{L})$ such that $\|s\|_{\sup, k\overline{\mathcal{L}}} < 1$.

A Hermitian line bundle $\overline{\mathcal{A}}$ is said to be ample if:

- $\mathcal{L}$ is ample on $\mathcal{X}$,
- The first Chern form $c_1(\overline{\mathcal{L}})$ is positive on $\mathcal{X}(\mathbb{C})$, and
- For a sufficiently large integer $k$, the space $H^0(\mathcal{X}, k\mathcal{L})$ is generated by the set
$$\{s \in H^0(\mathcal{X}, k\mathcal{L}) \mid \|s\|_{\sup,k\overline{\mathcal{L}}} < 1\},$$
as a $\mathbb{Z}$-module.

Following the convention in [21, Convention 9, p. 411], we write $\overline{\mathcal{L}} \leq \overline{\mathcal{M}}$ if there is an injective homomorphism $\phi : \mathcal{L} \to \mathcal{M}$ such that $\|\phi_{\mathbb{C}}(\cdot)\|_{\mathcal{M}} \leq \|\cdot\|_{\mathcal{L}}$ on $\mathcal{X}(\mathbb{C})$, where $\|\cdot\|_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{M}}$ are the Hermitian norms associated to $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$, respectively.

**Lemma 4.4.** *Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a big Hermitian line bundle on $\mathcal{X}$. Then there exists a positive integer $\ell_0$ such that*
$$\widehat{h}^0\left(\overline{H^0(\mathcal{X}, \ell\mathcal{L})}_{\sup, \ell\overline{\mathcal{L}}}\right) \neq 0 \quad \textit{for all } \ell \geq \ell_0.$$

**Proof.** Since $\overline{\mathcal{L}}$ is big, there exist a positive integer $k_0$ and a nonzero section $s \in H^0(\mathcal{X}, k_0\mathcal{L})$ such that $\|s\|_{\sup, k_0\overline{\mathcal{L}}} < \alpha$ for some real number $0 < \alpha < 1$.

It is known that the sequence
$$\left(\frac{1}{k}\log\lambda_1\left(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup, k\overline{\mathcal{L}}}\right)\right)_{k\in\mathbb{N}}$$
converges to a finite limit as $k \to \infty$. In particular,
$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k}\log\lambda_1\left(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup, k\overline{\mathcal{L}}}\right) &= \lim_{k\to\infty}\frac{1}{kk_0}\log\lambda_1\left(\overline{H^0(\mathcal{X}, kk_0\mathcal{L})}_{\sup, kk_0\overline{\mathcal{L}}}\right)\\ &\leq \frac{1}{k_0}\log\alpha\\ &< 0.\end{aligned}$$

[2]That is, $\mathrm{vol}(\mathcal{L}_{\mathbb{Q}}) > 0$, which by definition means
$$\limsup_{k\to\infty}\frac{h^0(\mathcal{X}_{\mathbb{Q}}, k\mathcal{L}_{\mathbb{Q}})}{k^n/n!} > 0,$$
where $n = \dim\mathcal{X}_{\mathbb{Q}}$.

Let $0 < \varepsilon < -\frac{1}{k_0}\log\alpha$. By the convergence above, there exists $\ell_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$,

$$\frac{1}{\ell}\log\lambda_1\left(\overline{H^0(\mathcal{X},\ell\mathcal{L})}_{\sup,\ell\overline{\mathcal{L}}}\right) \leq \lim_{k\to\infty}\frac{1}{k}\log\lambda_1\left(\overline{H^0(\mathcal{X},k\mathcal{L})}_{\sup,k\overline{\mathcal{L}}}\right) + \varepsilon$$
$$\leq \frac{\log\alpha}{k_0} + \varepsilon < 0.$$

Thus, for all $\ell \geq \ell_0$, we have $\log\lambda_1\left(\overline{H^0(\mathcal{X},\ell\mathcal{L})}_{\sup,\,\ell\overline{\mathcal{L}}}\right) < 0$, i.e.,

$$\lambda_1\left(\overline{H^0(\mathcal{X},\ell\mathcal{L})}_{\sup,\,\ell\overline{\mathcal{L}}}\right) < 1.$$

So $\widehat{h}^0(\overline{H^0(\mathcal{X},\ell\mathcal{L})}_{\sup,\ell\overline{\mathcal{L}}}) \neq 0$ for all $\ell \geq \ell_0$. □

**Proposition 4.5.** *Let $\overline{\mathcal{A}}$ be an ample Hermitian line bundle on $\mathcal{X}$. Then,*

$$\liminf_{k\to\infty}\frac{\widehat{h}^0(\overline{H^0(\mathcal{X},k\mathcal{A})}_{\sup,k\overline{\mathcal{A}}})}{k^{n+1}/(n+1)!} > 0.$$

**Proof.** Since $\overline{\mathcal{A}}$ is ample, there exists an integer $k_0 > 0$ such that the graded algebra $\bigoplus_{m\in\mathbb{N}} H^0(\mathcal{X}, mk_0\mathcal{A})$ is generated by the set

$$S := \left\{s \in H^0(\mathcal{X},k_0\mathcal{A}) \;\middle|\; \|s\|_{\sup,\,k_0\overline{\mathcal{A}}} < 1\right\}.$$

Define

$$\varepsilon := -\sup_{s\in S}\log\|s\|_{\sup,\,k_0\overline{\mathcal{A}}} > 0.$$

By construction, for each $k \geq 1$, we can find a basis of $H^0(\mathcal{X}, kk_0\mathcal{A})$ consisting of sections whose sup-norm is at most $e^{-\varepsilon k}$. Consequently, the $n_{kk_0}$-th successive minimum $\lambda_{n_{kk_0}}(kk_0\overline{\mathcal{A}})$ satisfies

$$\lambda_{n_{kk_0}}(kk_0\overline{\mathcal{A}}) \leq e^{-\varepsilon k},$$

where $n_k$ is the rank of $H^0(\mathcal{X}, k\mathcal{A})$.

By Corollary 4.3, we get the following inequality:

$$O(n_k\log n_k) + \varepsilon n_{kk_0}k \leq \hat{h}^0\left((\overline{H^0(\mathcal{X},kk_0\mathcal{A})}_{\sup,\,kk_0\overline{\mathcal{A}}}\right).$$

Consequently,

$$\varepsilon(n+1)\frac{\mathrm{vol}(\mathcal{A}_{\mathbb{Q}})}{k_0} \leq \liminf_{k\to\infty}\frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X},kk_0\mathcal{A})}_{\sup,\,kk_0\overline{\mathcal{A}}}\right)}{(kk_0)^{n+1}/(n+1)!},$$

noting that $\mathcal{A}_{\mathbb{Q}}$ is big.

Applying Lemma 4.4 to $\overline{\mathcal{A}}$, we obtain a positive integer $\ell_0$ such that, for every $\ell \geq \ell_0$,

$$\widehat{h}^0\left(\overline{H^0(\mathcal{X},\ell\mathcal{A})}_{\sup,\,\ell\overline{\mathcal{A}}}\right) \neq 0.$$

Given $m \geq k_1k_0 + k_0 + \ell_0$, write $m = kk_0 + r + \ell_0$, where $k \in \mathbb{N}$ and $r \in \{0, \dots, k_0 - 1\}$. By Lemma 4.4, we have

$$kk_0\overline{\mathcal{A}} \leq kk_0\overline{\mathcal{A}} + r\overline{\mathcal{A}} + \ell_0\overline{\mathcal{A}}.$$

Therefore,

$$\begin{aligned}\frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X}, m\mathcal{A})}_{\sup, m\overline{\mathcal{A}}}\right)}{m^{n+1}} &\geq \frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X}, kk_0\mathcal{A})}_{\sup, kk_0\overline{\mathcal{A}}}\right)}{(kk_0)^{n+1}} \frac{(kk_0)^{n+1}}{(kk_0 + r + \ell_0)^{n+1}} \\ &\geq \frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X}, kk_0\mathcal{A})}_{\sup, kk_0\overline{\mathcal{A}}}\right)}{(kk_0)^{n+1}} \frac{(kk_0)^{n+1}}{(kk_0 + k_0 + \ell_0)^{n+1}}.\end{aligned}$$

It follows that

$$\liminf_{m\to\infty} \frac{\widehat{h}^0(\mathcal{X}, m\overline{\mathcal{A}})}{m^{n+1}} > 0.$$

This completes the proof. □

Moriwaki in [20] introduced the *arithmetic volume* $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})$ for a Hermitian line bundle $\overline{\mathcal{L}}$ on arithmetic variety $\mathcal{X}$ which is an analogue of the geometric volume function. It is given as follows:

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) = \limsup_{k\to\infty} \frac{\hat{h}^0(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup, k\overline{\mathcal{L}}})}{k^{n+1}/(n+1)!}.$$

Yuan [26] employs the condition

$$\liminf_{k\to\infty} \frac{\log \#\{s \in H^0(\mathcal{X}, k\mathcal{L}) \mid \|s\|_{\sup, k\overline{\mathcal{L}}} < 1\}}{k^{n+1}/(n+1)!} > 0,$$

as a definition of an arithmetic big Hermitian line bundle. Moriwaki [21] proposed an alternative definition for arithmetic big line bundles: $\overline{\mathcal{L}}$ is said to be arithmetically big if $\mathcal{L}_{\mathbb{Q}}$ is big and there exists a positive integer $k$ and a nonzero global section $s$ of $k\overline{\mathcal{L}}$ such that $\|s\|_{\sup, k\overline{\mathcal{L}}} < 1$. He showed that Yuan's definition is equivalent to the existence of a nonzero section of a sufficiently high tensor power of $\mathcal{L}$ with sup-norm less than 1, and that $\mathcal{L}_{\mathbb{Q}}$ is big.

The following theorem is an application of the theory developed in this paper.

**Theorem 4.6.** *We keep the same notations as in the beginning of this section. We have*

(1)

$$\limsup_{k\to\infty} \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{\sup, k\overline{\mathcal{L}}} < 1\right\}}{k^{n+1}/(n+1)!} = \limsup_{k\to\infty} \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{\sup, k\overline{\mathcal{L}}} \leq 1\right\}}{k^{n+1}/(n+1)!}.$$

(2)

$$\liminf_{k\to\infty} \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{\sup,k\overline{\mathcal{L}}} < 1\right\}}{k^{n+1}/(n+1)!} = \liminf_{k\to\infty} \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{\sup,k\overline{\mathcal{L}}} \le 1\right\}}{k^{n+1}/(n+1)!}.$$

**Proof.** Let us prove (1). (2) can be proved in a similar way. Let $k$ be a positive integer. We denote by $\|\cdot\|_{J_k}$ the John norm on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ satisfying

$$n_k^{-\frac{1}{2}} \|\cdot\|_{J_k} \le \|\cdot\|_{\sup,k\overline{\mathcal{L}}} \le \|\cdot\|_{J_k},$$

where $n_k$ is the rank of $H^0(\mathcal{X}, k\mathcal{L})$. So

$$\begin{aligned}\limsup_{k\to\infty} & \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{J_k} \le 1\right\}}{k^{n+1}/(n+1)!} \\ &\le \limsup_{k\to\infty} \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \mid \|s\|_{\sup,k\overline{\mathcal{L}}} \le 1\right\}}{k^{n+1}/(n+1)!} \\ &\le \limsup_{k\to\infty} \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, n_k^{-\frac{1}{2}} \|s\|_{J_k} \le 1\right\}}{k^{n+1}/(n+1)!}.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\limsup_{k\to\infty} & \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{J_k} < 1\right\}}{k^{n+1}/(n+1)!} \\ &\le \limsup_{k\to\infty} \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{\sup,k\overline{\mathcal{L}}} < 1\right\}}{k^{n+1}/(n+1)!} \\ &\le \limsup_{k\to\infty} \frac{\log \#\left\{s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, n_k^{-\frac{1}{2}} \|s\|_{J_k} < 1\right\}}{k^{n+1}/(n+1)!}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}h^0_\theta\left(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{J_k}\right) &\le h^0_\theta\left(\overline{(H^0(\mathcal{X}, k\mathcal{L})_{J_k})}_{n_k^{-\frac{1}{2}}}\right) \\ &\le h^0_\theta\left(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{J_k}\right) - \frac{n_k \log n_k}{4}.\end{aligned}$$

where we have used the fact that $\log \theta_{\overline{E}}(t) + \frac{1}{2}\mathrm{rk}\, E \log t$ is a nondecreasing function, see (3.4). So

$$\limsup_{k\to\infty} \frac{h^0_\theta\left(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{J_k}\right)}{k^{n+1}/(n+1)!} = \limsup_{k\to\infty} \frac{h^0_\theta\left(\overline{(H^0(\mathcal{X}, k\mathcal{L})}_{J_k})_{n_k^{-\frac{1}{2}}}\right)}{k^{n+1}/(n+1)!}. \tag{4.2}$$

Combining the inequalities above with (4.2) and Proposition 3.1, we conclude that

$$\limsup_{k\to\infty} \frac{\log \# \left\{ s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{\sup, k\overline{\mathcal{L}}} < 1 \right\}}{k^{n+1}/(n+1)!}$$
$$= \limsup_{k\to\infty} \frac{\log \# \left\{ s \in H^0(\mathcal{X}, k\mathcal{L}) \,\middle|\, \|s\|_{\sup, k\overline{\mathcal{L}}} \leq 1 \right\}}{k^{n+1}/(n+1)!}.$$

This completes the proof of (1). □

From this theorem, we can deduce that $\overline{\mathcal{L}}$ is arithmetically big in the sense of Yuan if and only if it is arithmetically big in the sense of Moriwaki. Indeed, let us explain this in detail.

Let $\overline{\mathcal{L}}$ be a big Hermitian line bundle on $\mathcal{X}$ in the sense of Moriwaki. Let $\overline{\mathcal{A}}$ be an ample Hermitian line bundle on $\mathcal{X}$. By the argument in [21, p. 445], there exists a positive integer $p$ such that

$$p\overline{\mathcal{L}} \geq \overline{\mathcal{A}}.$$

This implies the following bound:

$$\liminf_{k\to\infty} \frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X}, pk\mathcal{L})}_{\sup, pk\overline{\mathcal{L}}}\right)}{(pk)^{n+1}} \geq \frac{1}{p^{n+1}} \liminf_{k\to\infty} \frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X}, k\mathcal{A})}_{\sup, k\overline{\mathcal{A}}}\right)}{k^{n+1}}.$$

By Proposition 4.5, we know that

$$\liminf_{k\to\infty} \frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X}, k\mathcal{A})}_{\sup, k\overline{\mathcal{A}}}\right)}{k^{n+1}} > 0,$$

since $\overline{\mathcal{A}}$ is ample.

Then

$$\liminf_{k\to\infty} \frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X}, pk\mathcal{L})}_{\sup, pk\overline{\mathcal{L}}}\right)}{(pk)^{n+1}} > 0.$$

Arguing as in the proof of Proposition 4.5, we deduce that

$$\liminf_{k\to\infty} \frac{\widehat{h}^0\left(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup, k\overline{\mathcal{L}}}\right)}{k^{n+1}} > 0.$$

Using Theorem 4.6, we conclude that $\overline{\mathcal{L}}$ is arithmetically big in the sense of Yuan.

Now, let us suppose that

$$\liminf_{k\to\infty} \frac{\log \#\{s \in H^0(\mathcal{X}, k\mathcal{L}) \mid \|s\|_{\sup,k\overline{\mathcal{L}}} < 1\}}{k^{n+1}/(n+1)!} > 0.$$

This assumption implies the existence of a positive constant $c$ and a positive integer $k_0$ such that

$$\log \#\{s \in H^0(\mathcal{X}, k\mathcal{L}) \mid \|s\|_{\sup,k\overline{\mathcal{L}}} < 1\} \geq ck^{n+1} \quad \text{for all } k \geq k_0. \tag{4.3}$$

Consequently, there exists a nonzero section $s \in H^0(\mathcal{X}, k_0\mathcal{L})$ satisfying

$$\|s\|_{\sup,k_0\overline{\mathcal{L}}} < 1.$$

Next, we aim to show that $\mathcal{L}_{\mathbb{Q}}$ is big. According to Corollary 4.3, we have

$$\begin{aligned}\log \#\{s \in H^0(\mathcal{X}, k\mathcal{L}) | \|s\|_{\sup,k\overline{\mathcal{L}}} < 1\} \\ \leq -n_k \log \lambda_1(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup,k\overline{\mathcal{L}}}) + O(n_k \log n_k).\end{aligned}$$

Combining this with our earlier inequality (4.3), we obtain

$$ck^{n+1} \leq -kn_k \log \lambda_1 \left(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup,k\overline{\mathcal{L}}}\right)^{\frac{1}{k}} + O(n_k \log n_k).$$

From this, we can infer that

$$\frac{c}{-\log \lambda_1(\overline{H^0(\mathcal{X}, k\mathcal{L})}_{\sup,k\overline{\mathcal{L}}})^{\frac{1}{k}}} \leq \liminf_{k\to\infty} \frac{n_k}{k^n} + O(\frac{\log k}{k}).$$

Then

$$0 < \liminf_{k\to\infty} \frac{n_k}{k^n}.$$

This shows that $\mathcal{L}_{\mathbb{Q}}$ is indeed big.

**Remark 4.7.** *Note that, as explained in* [21, p. 446]*, the proof that* $\mathcal{L}_{\mathbb{Q}}$ *is big under the assumption* $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) > 0$ *relies on* [21, Theorem 4.4]*, which in turn is based on the main technical result of the paper, namely* [21, Theorem 3.1].

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