ALGORITHMICAL DEFINITION OF FINITE MARKOV SEQUENCE¹

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0. Let $T = (t_1, t_2, \ldots, t_N)$ be a finite binary sequence. Following von Mises ideas, A.N. Kolmogorov [1] defined the randomness of T with respect to the algorithm R = (F, G, H) for selection of the subsequence of T. We give this definition in the following way:

The system of functions $F = (F_0, F_1, \dots F_{N-1})$, $F_0 = \text{const}$, defines a permutation (x_1, x_2, \dots, x_N) of $(1, 2, \dots, N)$ which depends on T, by

$$x_i = F_{i-1}(x_1, t_{x_1}; \dots, x_{i-1}, t_{x_{i-1}}), \quad i = 1, 2, \dots, N.$$

The systems of functions $H=(H_0,H_1,\ldots,H_N)$ and $G=(G_0,G_1,\ldots,G_{N-1})$ have the properties: $H_i,\,G_i\in\{0,1\},\,H_0={\rm const}\,,\,H_N=1,\,H_i(x_0,t_{x1};\ldots;x_1,t_{xi})\leq H_{i+1}(x_1,t_{x1};\ldots,x_{i+1},t_{x_{i+1}}),\,G_0={\rm const}\,.$

Let $s = s(T) = \min\{i: H_i = 1\}$ The system (F, G, H) defines the subset $A \subset \{1, 2, \dots, N\}$ in the following way: $x_k \in A$ if $1 \le k \le s$ and $G_{k-1}(x_1, t_{x_1}, \dots x_{k-1}, t_{xk-1}) = 1$. Let $A = \{x_{i_1}x_{i_2}, \dots, x_{i_v}\}, \ x_{i_1} < x_{i_2} < \dots x_{i_v}$. We select the subsequence $(t_{xi_1}, t_{xi_2}, \dots, t_{xi_v})$ of T by R = (F, G, H).

The sequence T is $(n,\varepsilon,p)\text{-random}\ (1\le n\le N,\, 0<\varepsilon,\, 0\le p\le 1)$ with respect to R if

$$v \ge n$$
, and $\left| \frac{1}{v} \sum_{k \in A} t_k - p \right| < \varepsilon$

or if $v < n \cdot T$ is (n, ε, p) -random with respect to the system $\mathcal{R} = \{R_1, R_2, \dots\}$ if it is (n, ε, p) -random with respect to each $R_i \in \mathcal{R}$.

Another approach to the algorithmical definition of randomness was given in [2] and later developed for infinite set of sequences ([3], [4]). However, for finite set of sequences which we consider here, this approach is too broad to be successful applied (see discution in [5]).

¹This research was supported by Odsek za mat. i mat., meh. i astr. nauke, PMF Belgrade

Kolmogorov difinition of random sequence in [1] corresponds in a way to Bernoulli secuence $\mathcal{B}(p)$ in Probability theory. In this paper we define the randomness of T corresponding to homogenous Markov sequence $\mathcal{M}(\alpha,\beta)$ with the states $\{0,1\}$ and the transition matrix $\begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$. In this definition we follow Kolmogorov's method.

1. It is reasonable that the definition of Markov sequence (MS) is based on the stability of the frequences of transition from 0 and 1. We select a subsequence $t_{i_1}, t_{i_2}, \ldots, t_{i_k}, \ 1 < i_1 < i_3 < \cdots i_k \leq N$. Let $v_1 = \sum_{j=1}^k t_{i_j-1}$ and $v_0 = k - v_1 = \sum_{j=1}^k (1-t_{i_j-1})$. Consider $\frac{1}{v_1} \sum_{j=1}^k t_{i_{j-1}} t_{i_j}$ -relative frequence of the transition from 1 to 1 and $\frac{1}{v_i} \sum_{j=u}^k (1-t_{i_j-1})(1-t_{i_j})$ -relative frequence of transition from 0 to 0. In accordance with the idea of MS, selection of a particular t_x in the subsequence should not depend on t_x, t_{x+1}, \ldots . It means that the selection of t_i occurs before the selection of t_j for i < j.

Let R = (G, H) be a system of functions $H = (H_0, H_1, \ldots, H_N)$ and $G = (G_0, G_1 \cdots G_{N-1})$ with the properties $H_i, G_i \in \{0, 1\}, H_0 = \text{const}, H_N = 1, H_i(t_1, \ldots, t_i) \leq H_{i+1}(t_1, \ldots, t_{i+1}), G_0 = 0.$

Definition 1. The system R = (G, H) is an algorithm for selection of the subsequence S of T, given by:

Let $s = s(T) = \min\{i: H_i = 1\}$. Let $A \subset \{1, 2, ..., N\}$ be defined by $j \in A$ iff $G_{i-1}(t_1, ..., t_{i-1}) = 1$. Let $A = \{i_1, i_2, ..., i-v\}$. Then $S = (t_{i_1}, t_{i_2}, ..., t_{i_v})$.

By definition $2 \le i_1 < i_2 < \cdots i_v \le N$. From Definition 1 it follows that the algorithm R = (G, H) is a particular Kolmogorov algorithm R' = (F, G, H) where $F_{i-1} = i, i = 1, 2, \ldots, N$.

DEFINITION 2. The secuence T is $(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha, \beta)$ -Markov (denoted by $\mathcal{M}(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha, \beta)$, $(1 \leq n_i \leq N, 0 < \varepsilon_i, i = 0, 1, 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1)$ with respect to R if

(a)
$$v_0 \ge n_0$$
 and $\Delta_0 = \left| \frac{1}{v_0} \sum_{j \in A} (1 - t_{j-1})(1 - t_j) - \alpha \right| < \varepsilon_0$ or $v_0 < n_0$
(b) $v_1 \ge n_1$ and $\Delta_1 = \left| \frac{1}{v_1} \sum_{j \in A} t_{j-1} t_j - \beta \right| < \varepsilon_1$ or $v_1 < n_1$

The sequence T is $\mathcal{M}(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha, \beta)$ with respect to the system $\mathcal{R} = \{R_1, R_3, \dots\}$ if it is $\mathcal{M}(n_0, n_1, \varepsilon_v, \varepsilon_1, \alpha, \beta)$ with respect to each $R_i \in \mathcal{R}$.

DEFINITION 3. The sequence T is $(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha)$ -Bernoulli (denoted by $\mathcal{B}(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha)$ with respect to $R(\mathcal{R})$ if it is $\mathcal{M}(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha, 1 - \alpha)$ with respect to $R(\mathcal{R})$.

Definition 3-follows from the idea that Bernoulli sequence is a particular Markov sequence for $\beta=1-\alpha$.

PROPOSITION 1. Let T be $\mathcal{B}(n_0,n_1,\varepsilon_0,\varepsilon_1,\alpha)$ with respect to R=(G,H). Then T is $(\max\{n_0,n_1\},\max\{\varepsilon_0,\varepsilon_1\},1-\alpha)$ -random in the sence of Kolmogorov, with respect to the system $\{R_0,R_1\},\,R_i=(F^j,G^j,H^j),\,j=0,1$, where $H^j=H,\,j=0,1,\,F^j_{i-1}=i,\,i=1,2,\ldots,N,\,j=0,1$ and

$$G_{i-1}^{j} = \begin{cases} G_{i-1} & t_{i-1} = j, \\ 0 & t_{i-1} = 1 - j \end{cases}, \quad j = 0, 1.$$

PROOF. It is clear that R_0 is Kolmogorov algorithm. Let R select the subsequence $S = \{t_i\}$, $i \in A$ of T. Then R_0 selects the subsequence $S_0 = \{t_i\}$, $i \in B$ of S, which consits of elements proceeding zeros in T. Let S_0 have v elements. Evidently, $V = V_0$ and

$$\left|\frac{1}{v}\sum_{i\in B}t_i-(1-\alpha)\right|=\left|\frac{1}{v}\sum_{i\in B}(1-t_i)-\alpha\right|=\left|\frac{1}{v}\sum_{i\in A}(1-t_{i-1})(1-t_i)-\alpha\right|=\Delta_0.$$

Since T is $\mathcal{B}(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha)$ it follows that $v_0 < n_0$ or $v_0 \ge n_0$ and $\Delta_0 < \varepsilon_0$, e.i. $v < n_0$ or $v \ge n_0$ and $\left|\frac{1}{v}\sum_{i \in B} t_i - (1-\alpha)\right| < \varepsilon_0$. It means that T is $(n_0, \varepsilon_0, 1-\alpha)$ -random with respect to R_0 . Similarly, T is $(n_1, \varepsilon_1, 1-\alpha)$ -random with respect to R_1 .

Generally, let T be (n, ε, p) -random. Then T is (n, δ, p) - random for $m \ge n$, $\delta \ge \varepsilon$. Now since T is $(n_j, \varepsilon_j, 1 - \alpha)$ -random with respect to R_j , j = 0, 1, it means that T is $(\max\{n_0, n_1\}, \max\{\varepsilon_0, \varepsilon_1\}, 1 - \alpha)$ -random with respect to the system $\mathcal{R} = \{R_0, R_1\}$.

2. In this section we consider the existence of at least one MS for a given system \mathcal{R} with ρ algorithms.

Let $p(n, \varepsilon, \alpha) = P\left(\sup_{k \geq n} \left| \frac{S_k}{k} - \alpha \right| \geq \varepsilon\right)$ where random variable S_k have binomial distribution $b(k, \alpha)$.

Proposition 2. Let system \mathcal{R} have ρ algorithms. If

$$\rho < \frac{1}{p(n_0, \varepsilon_0, \alpha) + p(n_1, \varepsilon_1, \beta)}$$

then there exists at least one $\mathcal{M}(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha, \beta)$ sequence with respect to R.

PROOF. Consider Markov probability distribution on the set $\{T\}$, with given initial distribution and transition matrix $\begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$. Let $P(R)(P(\mathcal{R}))$ be the probability that T is non-Markov with respect to $R(\mathcal{R})$. Then (using the same notation as in Definition 2)

$$P(R) = P((v_0 \ge n_0, \Delta_0 \ge \varepsilon_0) \cup (v_1 \ge n_1, \Delta_1 \ge \varepsilon_1)) \ge P(v_0 \ge n_0, \Delta_0 \ge \varepsilon_0) + P(v_1 \ge n_1, \Delta_1 \ge \varepsilon_1).$$

Let $\xi_1,\xi_2,\ldots,\xi_i\in\{0,1\}$, be homogenous Markov chain with transition matrix $\begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$. We select the sequence of indices $i_1,i_3\ldots,1\leq i_1< i_2\ldots$, such that $\xi_{i_1}=\xi_{i_2}=\cdots=0$ and that the selection of i_j is independant of $\xi_k,\,k>i_j,\,j=1,2,\ldots$ Then the sequence $\xi_{i_1+1},\xi_{i_2+1},\ldots$ is Bernoulli sequence where the probability of occuring 0 is α . Consider the sequence $\xi_1,\xi_2,\ldots\xi_N$, as a part of infinite sequence ξ_1,ξ_2,\ldots , and a subsequence $\xi_{i_1+1},\xi_{i_2+1},\ldots\xi_{i_{v_0}+1}$ selected by R from $\xi_1,\xi_2,\ldots\xi_N$. Let R^* be the algorithm defined for infinite sequence as the extension of R in the following way. R^* selects the same subsequence $\xi_{i_1+1},\xi_{i_2+1},\ldots\xi_{i_k+1}$ as R until $k\leq v_0$. For $k>v_0$ the selection is arbitrary (but in accordance with described rules of selection). Let η_1,η_2,\ldots be the selected subsequence. We define the stopping rule for R^* as

$$v_{+}^{0} = n_{0} \text{ if } \left| \frac{1}{n_{0}} \sum_{i=1}^{n_{0}} \eta_{i} - (1 - \alpha) \right| \ge \varepsilon_{0} \text{ and}$$
 $v_{0}^{*} = k, \ k > n_{0}, \text{ if } \left| \frac{1}{j} \sum_{i=0}^{j} \eta_{i} - (1 - \alpha) \right| < \varepsilon_{0}, \ j = n_{0}, \ n_{0} + 1, \dots k - 1 \text{ and}$
 $\left| \frac{1}{k} \sum_{i=1}^{k} \eta_{i} - (1 - \alpha) \right| \ge \varepsilon_{0}.$

Then

$$P(v_0^* \ge n_0, \Delta_0^* \ge \varepsilon_0) = P\left(\sup_{k \ge n^0} \left| \frac{S_k}{k} - (1 - \alpha) \right| \ge \varepsilon_0\right) =$$

$$p(n_0, \varepsilon_0, 1 - \alpha) = p(n_0, \varepsilon_0, \alpha), \quad \left(\Delta_0^* = \left| \frac{1}{V_0^*} \sum_{i=1}^{v_0} \eta_i - (1 - \alpha) \right| \right).$$

If T is non-Markov with respect to R, than each infinite sequence begining with T is non-Markov with respect to R^* . So $P(v_0 \geq n_0, \Delta_0 \geq \varepsilon_0) \leq p(n_0, \varepsilon_0, \alpha)$. In the same way $P(v_1 \geq n_1, \Delta_1 \geq \varepsilon_1) \leq p(n_1, \varepsilon_1, \beta)$ i.e. $P(R) \leq p(n_0, \varepsilon_0, \alpha) + p(n_1, \varepsilon_1, \beta)$ and $P(R) \leq \sum_{R \in \mathcal{R}} P(R) \leq P[p(n_0, \varepsilon_0, \alpha) + p(n_1, \varepsilon_1, \beta)]$. If $\rho[p(n_0, \varepsilon_0, \alpha) + p(n_1, \varepsilon_1, \beta)] < 1$, then P(R) < 1 and the probability measure of the set of Markov secuence is 1 - P(R) < 0, i.e. there exists at least one Markov sequence with respect to R.

Kolmogorov [1] gave the estimation $p(n, \varepsilon, \alpha) \leq 2e^{-n\varepsilon^{2(1-\varepsilon)}}$. If

$$\rho < \frac{1}{2} \left[e^{-n_0 \varepsilon_0^2 (1 - \varepsilon_0)} + e^{-n_1 \varepsilon_1^2 (1 - \varepsilon_1)} \right]^{-1}$$

than for each system with ρ algorithms and each α and β there exists $\mathcal{M}(n_0, n_1, \varepsilon_0, \varepsilon_1, \alpha, \beta)$ sequence.

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