ALMOST TRIVIAL GROUPOIDS

A. Krapež

(Communicated January 16. 1980)

Berglund and Mislove defined semigroups with almost trivial multiplications and proved that such a semigroup belongs to one of five well known semigroup classes (see [1]). In a similar way, without associativity condition, we defined almost trivial groupoids and proved Th 1 analogous to Th 1 of [1].

Using this result, we solved generalized associativity equation on almost trivial groupoids. As an example we obtained 1344 solutions of generalized associativity equation on two-element groupoids.

$$*$$

1. If groupoid $(S, \cdot)$ (or simply $\cdot$) is given, then functions

$$\lambda_x, \rho_y \colon S \to S \quad (x, y \in S),$$

defined by:

$$\lambda_x y = \rho_y x = xy$$

are respectively left and right translations of $(S, \cdot)$.

In [1], semigroup with almost trivial multiplications is defined as a semigroup whose translations are either surjections or constant.

Almost trivial groupoids (ATG) we define as groupoids with translations which are either permutations or constant.

LEMMA 1.1. *Any isotope of* ATG *is also an* ATG.

PROOF: Let $(S, \cdot)$ be an ATG and $x * y = \varphi^{-1}(\alpha x \cdot \beta y)$ where $\alpha, \beta, \varphi \colon T \to S$ are bijections.

Let $\lambda'(\rho')$ be left (right) translations of $(T, *)$.

Then for all $a \in T$:

$$\lambda_a'x = a * x = \varphi^{-1}(\alpha a \cdot \beta x) = \varphi^{-1}\lambda_{\alpha_a}\beta x$$
$$\rho_a'x = x * a = \varphi^{-1}(\alpha x \cdot \beta a) = \varphi^{-1}\rho_{\beta_a}\alpha x$$

so λ_a' and ρ_a' are also either permutations or constant functions on T and consequently $(T, *)$ is also an ATG.

DEFINITION. $(S, \cdot)$ is a quasigroup with quasizero (p, q, r) iff:

– $px = r$

– $xq = r$

– for all $a, b \in S(a \neq p)$, equation $ax = b$ has the unique solution

– for all $a, b \in S(a \neq q)$, equation $xa = b$ has the unique solution.

$(S, \cdot)$ is a left (right) groupoid iff:

$$xy = \varphi x \qquad (xy = \varphi y)$$

where φ is a permutation of S.

LEMMA 1.2.   *Left (right) groupoid* $xy = \varphi x(xy = \varphi y)$ *is a semigroup iff* $\varphi = \varepsilon$.

PROOF: If $xy = \varphi x$ is associative, then

$$\varphi\varphi x = \varphi(xy) = xy \cdot z = x \cdot yz = \varphi x$$

and $\varphi = \varepsilon$. The converse is trivial.

It is clear that a quasigroup with quasizero is not a quasigroup. If $p = q = r$ (i.e. r is zero), S is a quasigroup with zero. Both these notions are generalizations of well known notion of group with zero (see [2]).

THEOREM 1.   *Any* ATG *is one of:*

(S_0) *zero-semigroup*

(S_L) *left groupoid*

(S_R) *right groupoid*

(Q) *quasigroup*

(Q_p) *quasigroup with quasizero.*

PROOF: It is easy to see that all groupoids we mention, are ATG.

To prove the converse, we define:

$$L_0 = \{a \in S \mid \lambda_a \text{ is constant}\}$$
$$L_1 = \{a \in S \mid \lambda_a \text{ is a permutation}\}$$
$$R_0 = \{a \in S \mid \rho_a \text{ is constant}\}$$
$$R_1 = \{a \in S \mid \rho_a \text{ is a permutation}\}$$

(a) Let $L_0 = \emptyset$.

If both $R_0 \neq \emptyset$ and $R_1 \neq \emptyset$ then there are $x \in R_0$ and $y \in R_1$ such that $Sx = b$ for some $b \in S$ and $Sy = S$, so there is an $a \in S$ such that $ay = b$. From $ax = b$ and $ay = b$ it follows that λ_a is constant contrary to hypothesis.

It must be either $R_0 = \emptyset$ or $R_1 = \emptyset$.

(a$'$) Let $R_0 = \emptyset$.

Then any translation is a permutation, so $\cdot$ is a quasigroup.

(a$''$) Let $R_1 = \emptyset$.

Then any left translation is a permutation while any right translation is constant and there is a permutation φ such that $xy = \varphi y$.

(b) Let $L_1 = \emptyset$.

If both $R_0 \neq \emptyset$ and $R_1 \neq \emptyset$ then there are $x \in R_0$ and $y \in R_1$ such that for some $a, b \in S (a \neq b)$:

$$ay = ax = bx = by$$

and ρ cannot be permutation, contrary to hypothesis.

It must be either $R_0 = \emptyset$ or $R_1 = \emptyset$.

(b$'$) Let $R_0 = \emptyset$.

Then any right translations is a permutation, while any left translation is constant and there is a permutation φ such that $xy = \varphi x$.

(b$''$) Let $R_1 = \emptyset$.

From $x, y, u, v \in S$ it follows that $x \in L_0$, $v \in R_0$ and $xy = xv = uv$ so $\cdot$ is a zero-semigroup.

(c) Let $L_0 \neq \emptyset$ and $L_1 \neq \emptyset$.

Then also $R_0 \neq \emptyset$ and $R_1 \neq \emptyset$. It is easy to see that L_0 and R_0 have only one element. Let $a \in L_0$, $b \in R_0$ and $c = ab$.

Except λ_a, all λ_x are permutations. Also, except ρ_b, all ρ_x are permutations and $\cdot$ is a quasigroup with quasizero (a, b, c).

DEFINITION. Principal ATG's are:

(S_0) zero-semigroup

(A_L) left zero semigroup

(A_R) right zero semigroup

(L) loop

(L_0) loop with zero.

Principal ATG's from a family $\{(S, *_i) \mid i \in I\}$ are compatible if:

– all ATG's (from the family) with zero, have a common zero d

– all ATG's (from the family) with unit, have a common unit e

– if d and c are as above, then $d \neq e$.

LEMMA 1.3. *Any* ATG *is (principal) isotope of some principal* ATG.

PROOF: In the first three cases the proof is trivial, while in the fourth it follows from the well known theorem of Albert ([3]).

Let $\cdot$ be a quasigroup with quasizero (p, q, r), α a transposition of p and r and β a transposition of q and r.

Let also $x * y = \alpha x \cdot \beta y$. Then:

$$r * x = \alpha r \cdot \beta x = p \cdot \beta x = r$$
$$x * r = \alpha x \cdot \beta r = \alpha x \cdot q = r.$$

Also, for $x \neq r$ $\lambda_x' y = x * y = \alpha x \cdot \beta y = \lambda_{\alpha x} \beta y$ so λ_x' is a permutation such that $\lambda_x' r = r$.

Analogously $\rho_x' \upharpoonright S \backslash \{r\}$ is a permutation and $* \upharpoonright (S \backslash \{r\})^2$ is a quasigroup. Since any quasigroup is (principally) isotopic to a loop, $(S, *)$ is (principally) isotopic to a loop with zero.

COROLLARY 1.4. *Any* ATG *is (principal) isotope of one of:*

(S_0) *zero-semigroup*

(A_L) *left zero semigroup*

(A_R) *right zero semigroup*

(L) *loop*

(L_0) *loop with zero*

COROLLARY 1.5. *Any* ATG *which is a semigroup is one of:*

(S_0) *zero-semigroup*

(A_L) *left zero semigroup*

(A_R) *right zero semigroup*

(G) *group*

(G_0) *group with zero*

PROOF:

(a) Zero-semigroup is a semigroup.

(b) According to L 1.1.

(c) According to L 1.1.

(d) Associative quasigroup is a group.

(e) Let $\cdot$ be a quasigroup with quasizero (p, q, r). For any $x, y, z \in S$ we have:

$$rz = py \cdot z = p \cdot yz = r = pz \quad \text{so} \quad p = r$$
$$xr = x \cdot yq = xy \cdot q = r = xq \quad \text{so} \quad q = r.$$

Consequently $\cdot$ is an associative quasigroup with zero i.e. group with zero.

Corollary 1.5 is smoewhat weaker than Th 1 from [1], where the same conclusion follows from weaker hypothesis about semigroups with almost trivial multiplications.

COROLLARY 1.6. *Any ATG with unit is one of:*

(L) *loop*

(L_0) *loop with zero.*

COLLORARY 1.7. *Any ATG with zero is one of:*

(S_0) *zero-semigroup*

(Q_0) *quasigroup with zero.*

$$*$$

2. In Th 2 the general solution is given of the generalized associativity equation on ATG:

(1) $$A(x, B(y, z)) = C(D(x, y), z).$$

We are using the following conventions:

– A, B, C, D are always given by the formulas (2) and their respective isotopes $\cdot$, $\circ$, $*$, $\triangle$.

– $S_\circ$, S_L, A_L, S_R, A_R, Q, L, G, Q_q, $Q_\circ$, $L_\circ$, $G_\circ$ are types i.e. various kinds of ATG's as follows:

$$S_\circ \ - \text{zero-semigroup}$$
$$S_L(S_R) \ - \text{left (right) groupoid}$$
$$A_L(A_R) \ - \text{left (right) zero semigroup}$$
$$Q(Q_q, Q_\circ) \ - \text{quasigroup (with quasizero, with zero)}$$
$$L(L_\circ) \ - \text{loop (with zero)}$$
$$G(G_\circ) \ - \text{group (with zero)}$$

– a, b, c, d denote types

– $\langle a, b, c, d \rangle$ means that $\cdot$ is of type a, $\circ$ of type b, $*$ of type c and $\triangle$ of type d. If for example $b = d$, then $\circ$ and $\triangle$ are the same operation.

$\langle a, b, c, b' \rangle$ means that operations $\circ$ and $\triangle$ are of the same type but not necessarily identical.

So $\langle S_\circ L S_\circ L' \rangle$ means that zero-semigroups $\cdot$ and $*$ are identical while $\circ$ and $\triangle$ are loops which are not necessarily identical.

THEOREM 2. *The general solution of generalized associativity equation* (1) *on ATG's, is given by:*

$$A(x, y) = A_1 x \cdot A_2 y$$

(2)
$$B(x, y) = A_2^{-1}(A_2 B_1 x \circ A_2 B_2 y)$$
$$C(x, y) = C_1 x * C_2 y$$
$$D(x, y) = C_1^{-1}(C_1 D_1 x \ \triangle \ C_1 D_2 y)$$

where A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 are arbitrary permutations on S such that:

$$(3) \qquad\qquad A_1 = C_1 D_1 \qquad A_2 B_1 = C_1 D_2 \qquad A_2 B_2 = C_2$$

and where $\cdot$, $\circ$, $*$, $\triangle$ are arbitrary principal ATG's such that either one of the following conditions are fulfilled:

$\langle S_\circ S_\circ S_\circ S_\circ\rangle$	$\langle S_\circ S_\circ S_\circ A_L\rangle$	$\langle S_\circ S_\circ S_\circ A_R\rangle$	$\langle S_\circ S_\circ S_\circ L\rangle$	$\langle S_\circ S_\circ S_\circ L_\circ\rangle$
$\langle S_\circ A_L S_\circ S_\circ\rangle$	$\langle S_\circ A_L S_\circ A_L\rangle$	$\langle S_\circ A_L S_\circ A_R\rangle$	$\langle S_\circ A_L S_\circ L\rangle$	$\langle S_\circ A_L S_\circ L_\circ\rangle$
$\langle S_\circ S_\circ A_L S_\circ\rangle$	$\langle S_\circ S_\circ L_\circ S_\circ\rangle$	$\langle S_\circ A_L A_L S_\circ\rangle$	$\langle S_\circ A_L L_\circ S_\circ\rangle$	$\langle S_\circ A_R S_\circ S_\circ\rangle$
$\langle S_\circ A_R S_\circ A_L\rangle$	$\langle S_\circ A_R S_\circ A_R\rangle$	$\langle S_\circ A_R S_\circ L\rangle$	$\langle S_\circ A_R S_\circ L_\circ\rangle$	$\langle S_\circ A_R A_L S_\circ\rangle$
$\langle S_\circ A_R L_\circ S_\circ\rangle$	$\langle S_\circ L S_\circ S_\circ\rangle$	$\langle S_\circ L S_\circ A_L\rangle$	$\langle S_\circ L S_\circ A_R\rangle$	$\langle S_\circ L S_\circ L'\rangle$
$\langle S_\circ L S_\circ L_\circ\rangle$	$\langle S_\circ L A_L S_\circ\rangle$	$\langle S_\circ L L_\circ S_\circ\rangle$	$\langle S_\circ L_\circ S_\circ S_\circ\rangle$	$\langle S_\circ L_\circ S_\circ A_L\rangle$
$\langle S_\circ L_\circ S_\circ A_R\rangle$	$\langle S_\circ L_\circ S_\circ L\rangle$	$\langle S_\circ L_\circ A_L S_\circ\rangle$	$\langle A_L S_\circ A_L A_L\rangle$	$\langle A_L A_L A_L A_L\rangle$
$\langle A_L A_R A_L A_L\rangle$	$\langle A_L L A_L A_L\rangle$	$\langle A_L L_\circ A_L A_L\rangle$	$\langle A_R S_\circ S_\circ S_\circ\rangle$	$\langle A_R S_\circ S_\circ A_L\rangle$
$\langle A_R S_\circ S_\circ A_R\rangle$	$\langle A_R S_\circ S_\circ L\rangle$	$\langle A_R S_\circ S_\circ L_\circ\rangle$	$\langle A_R S_\circ A_L S_\circ\rangle$	$\langle A_R S_\circ L_\circ S_\circ\rangle$
$\langle A_R A_L A_L A_R\rangle$	$\langle A_R A_R A_R S_\circ\rangle$	$\langle A_R A_R A_R A_L\rangle$	$\langle A_R A_R A_R A_R\rangle$	$\langle A_R A_R A_R L\rangle$
$\langle A_R A_R A_R L_\circ\rangle$	$\langle A_R L L A_R\rangle$	$\langle A_R L_\circ L_\circ A_R\rangle$	$\langle L A_L A_L L\rangle$	$\langle L A_R L A_L\rangle$
$\langle L_\circ S_\circ S_\circ S_\circ\rangle$	$\langle L_\circ S_\circ S_\circ A_L\rangle$	$\langle L_\circ S_\circ S_\circ A_R\rangle$	$\langle L_\circ S_\circ S_\circ L\rangle$	$\langle L_\circ S_\circ A_L S_\circ\rangle$
$\langle L_\circ S_\circ L_\circ' S_\circ\rangle$	$\langle L_\circ A_L A_L L_\circ\rangle$	$\langle L_\circ A_R L_\circ A_L\rangle$	$\langle G G G G\rangle$	$\langle G_\circ G_\circ G_\circ G_\circ\rangle$

where operations $\cdot$, $\circ$, $*$, $\triangle$ are compatible, or one of the following:

$\langle S_\circ L_\circ S_\circ L_\circ'\rangle$, $\cdot$ *and* $\circ$ *have a common zero,* $\circ$ *and* $\triangle$ *have a common unit*

$\langle S_\circ L_\circ L_\circ' S_\circ\rangle$, $\cdot$ *and* $*$ *have a common zero,* $\circ$ *and* $*$ *have a common unit*

$\langle A_R A_R L S_\circ\rangle$, *the unit of the loop* $*$ *is a zero of* $\triangle$

$\langle A_R A_R L_\circ S_\circ\rangle$, *the unit of the loop with zero* $*$ *is a zero of* $\triangle$

$\langle L S_\circ A_L A_L\rangle$, *the unit of loop* $\cdot$ *is a zero of* $\circ$

$\langle L_\circ S_\circ S_\circ L_\circ'\rangle$, $\cdot$ *and* $*$ *have a common zero,* $\cdot$ *and* $\triangle$ *have a common unit*

$\langle L_\circ S_\circ A_L A_L\rangle$, *the unit of the loop with zero* $\cdot$ *is a zero of* $\circ$.

Sketch of the proof: We can easily check that quadruple (A, B, C, D) given by (2), satisfying (3) and one of conditions $\langle abcd\rangle$, is a solution of (1).

Conversely, let T be a ternary operation defined by:

$$T(x, y, z) = A(x, B(y, z))$$

(a) T does not depnd on x, y, z.

For A, B we have the following possibilities:

$-$ $A(x, y) = 0$, B is arbitrary

– $A(x, y) = \alpha y$, $B(x, y) = b$

– A is a guasigroup with quasizero $(a, b, 0)$, $B(x, y) = b$ and analogous possibilities for C, D.

Only for three of 49 solutions, we prove that they are of the required form.

(a') $A(x, y) = 0$, B is a guasigroup

$C(x, y) = 0$, D is a guasigroup.

Let $p, q, r \in S$ and A_1 arbitrary permutation on S such that $A_1\, p \neq 0$. We define:

$$B_1 x = B(x, r), \quad B_2 x = B(q, x), \quad D_1 x = D(x, q), \quad D_2 x = D(p, x),$$
$$C_1 = A_1 D_1^{-1}, \quad A_2 = C_1 D_2 B_1^{-1}, \quad C_2 = A_2 B_2, \quad e = A_1 p$$

and

$$x \cdot y = A(A_1^{-1} x, A_2^{-1} y)$$
$$x \circ y = A_2 B((A_2 B_1)^{-1} x, \ (A_2 B_2)^{-1} y)$$
$$x \,\Delta\, y = C_1 D((C_1 D_1)^{-1} x, \ (C_1 D_2)^{-1} y).$$

Then it is easy to prove (3), $\langle S_\circ L S_\circ L' \rangle$ and compatibility of $\cdot$, $\circ$, Δ.

(a'') $A(x, y) = 0$, B is a quasigroup with quasizero (b_1, b_2, b)

$D(x, y) = d$, C is a quasigroup with quasizero $(d, c, 0)$.

Let $q, r, r' \in S$ such that $q \neq b_1$, $r \neq b_2$, $r \neq c$, $r' \neq d$ and D_1 arbitrary permutation on S. We define:

$$B_1 x = B(x, r), \quad B_2 x = B(q, x), \quad C_1 x = C(x, r), \quad C_2 x = C(r', x),$$
$$e = C(r', r), \quad A_1 = C_1 D_1, \quad A_2 = C_2 B_2^{-1}, \quad D_2 = C_1^{-1} A_2 B_1$$

and

$$x \cdot y = A(A_1^{-1} x, A_2^{-1} y)$$
$$x \circ y = A_2 B((A_2 B_1)^{-1} x, \ (A_2 B_2)^{-1} y)$$
$$x * y = C(C_1^{-1} x, \ C_2^{-1} y).$$

Then it is easy to prove (3) and $\langle S_\circ L_\circ L_\circ' S_\circ \rangle$ where $\cdot$ and $*$ have zero 0, $*$ and $\circ$ have unit e and $\circ$ has zero $A_2 b$. Operations $\cdot$, $\circ$, $*$ are compatible iff $A_2 b = 0$ i.e. $b_2 = c$.

(a''') A is a quasigroup with quasizero $(a, b, 0)$, $B(x, y) = b$

C is a quasigroup with quasizero $(d, c, 0)$, $D(x, y) = d$.

Let $p, p', r \in S$, $p \neq a$, $p' \neq b$, $r \neq c$ and let B_1 be an arbitrary permutation on S. Since C is a quasigroup with quasizero and $r \neq c$, there is unique solution of equation $C(x, r) = A(p, p')$. Let us denote it by r' and define:

$$A_1 x = A(x, p'), \quad A_2 x = A(p, x), \quad e = A(p, p'), \quad C_1 x = C(x, r),$$
$$C_2 x = C(r', x), \quad B_2 = A_2^{-1} C_2, \quad D_1 = C_1^{-1} A_1, \quad D_2 = C_1^{-1} A_2 B_1$$

and

$$x \cdot y = A(A_1^{-1} x, A_2^{-1} y)$$
$$x \circ y = A_2 B((A_2 B_1)^{-1} x, \ (A_2 B_2)^{-1} y)$$
$$x * y = C(C_1^{-1} x, \ C_2^{-1} y).$$

It is easy to prove (3), $\langle L_\circ S_\circ L_\circ' S_\circ \rangle$ and combatibility of $\cdot, \circ, *$.

(b) T depends on x only (and (a) does not hold).

We have the following possibilities for A and B:

– $A(x, y) = \alpha x$, B is arbitrary

– A is a quasigroup, $B(x, y) = b$

– A is a quasigroup with quasizero $(a_1, a_2, 0)$, $B(x, y) = b$, $b \neq a_2$.

For C and D there is only one posibility:

– $C(x, y) = \gamma x$, $D(x, y) = \delta x$.

Only for one of 7 solutions, we prove that it is of the requred form.

(b′) A is a quasigroup, $B(x, y) = b$,

$$C(x, y) = \gamma x, \quad D(x, y) = \delta x, \quad A(x, b) = \gamma \delta x.$$

Let $p \in S$ and B_1, B_2 be arbitrary permutations on S. We define:

$$C_1 = \gamma, \quad D_1 = \delta, \quad A_1 = C_1 D_1, \quad A_2 = A(p, x), \quad C_2 = A_2 B_2,$$
$$D_2 = C_1^{-1} A_2 B_1$$

and

$$x \cdot y = A(A_1^{-1} x, A_2^{-1} y)$$
$$x \circ y = A_2 B((A_2 B_1)^{-1} x, \ (A_2 B_2)^{-1} y)$$
$$x * y = C(C_1^{-1} x, C_2^{-1} y).$$

Consequenty (3), $\langle L S_\circ A_L A_L \rangle$ and $A_2 b$ is both unit of $\cdot$ and zero of zero semigroup $\circ$.

(c) T depends on y only (and (a) does not hold).

Only one case (of type $\langle A_R A_L A_L A_R \rangle$) is possible.

(d) T depends on z only (and (a) does not hold).

This case is "dual" to the case (b). There are 7 solutions.

(e) T depends on x, y only ((a), (b), (c) does not hold).

There are two possibilities for A and B:

– A is a quasigroup, $B(x, y) = \beta x$

– A is a quasigroup with quasizero $(a, b, 0)$, $B(x, y) = \beta x$.

There are two possibilities for C and D:

– $C(x, y) = \gamma x$, D is a quasigroup

– $C(x, y) = \gamma x$, D is a quasigroup with quasizero $(c, d, 0')$.

It follows from (1) that $A(x, \beta y) = \gamma D(x, y)$ i.e. A and D are isotopic and consequently both are either quasigroups or quasigroups with quasizero.

There are two solutions (of types $\langle L A_L A_L L \rangle$ and and $\langle L_\circ A_L A_L L_\circ \rangle$).

(f) T depends on x, z only ((a), (b), (d) does not hold).

As in the case (e), A and C are isotopic, so only two solutions (of types $\langle L A_R L A_L \rangle$ and $\langle L_\circ A_R L_\circ A_L \rangle$) remain.

(g) T depends on y, z only ((a), (c), (d) does not hold).

This case is "dual" to the case (e). The types of solutions are $\langle A_R L L A_R \rangle$ and $\langle A_R L_\circ L_\circ A_R \rangle$.

(h) T depends on x, y, z (and previous cases does not hold).

Any one of A, B, C, D can be either a quasigroup or a quasigroup with quasizero. Let $p \in S$ (and $p \neq a_1$ if (a_1, a_2, a) is a quasizero of A). Also, let $A_2 x = A(p, x)$ and $D_2 x = D(p, x)$. It follows from (1) that $A_2 B(y, z) = C(D_2 y, z)$ so D_2 is also a permutation and consequently B and C are isotopic.

Analogously, other operations are isotopic to B and C, so all are simultaneously either quasigroups or quasigroups with quasizero.

In the first case, using procedure from [4] (or [5]), we deduce (3) and $\langle GGGG \rangle$.

Analogously we obtain the solution of type $\langle G_\circ G_\circ G_\circ G_\circ \rangle$.

COROLLARY 2.1. *The general solution of generalized associativity equation* (1) *on* ATG*'s is given by* (2) *where* $A_1, A_2, \ldots, D_2$ *are arbitrary permutations on* S *satisfying* (3) *and* $\cdot$, $\circ$, $*$, $\triangle$ *are arbitrary principal* ATG*'s such that:*

(4)
$$x \cdot (y \circ z) = (x \triangle y) * z.$$

<div align="center">*</div>

EXAMPLE. Let $S = \{0, 1\}$. Then all groupoids on S are ATG. We shall solve the generalized associativity equation on S.

We denote the operations on S in a following way:

x	y	$x0y$	$x\wedge y$	$x\mapsto y$	$x\llcorner y$	$x\leftarrowtail y$	xRy	$x+y$	$x\vee y$	$x\downarrow y$	$x\leftrightarrow y$	$x\bar{R}y$	$x\leftarrow y$	$x\ulcorner y$	$x\rightarrow y$	$x\uparrow y$	$x1y$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

      Repeating the procedure from Th 2 we obtain all solutions of (1). Solutions like $(0,B,0,D)$ and $(0,B,L,0)$ stand for solutions where B and D are substituted by arbitrary operation on S.

(a)

$(0,B,0,D)$	$(1,B,1,D)$	$(0,B,\llcorner,0)$	$(0,B,\overline{\ulcorner},1)$
$(1,B,\llcorner,1)$	$(1,B,\overline{\ulcorner},0)$	$(0,B,\wedge,0)$	$(0,B,\mapsto,0)$
$(0,B,\leftarrowtail,1)$	$(0,B,\downarrow,1)$	$(1,B,\vee,1)$	$(1,B,\leftarrow,1)$
$(1,B,\rightarrow,0)$	$(1,B,\uparrow,0)$	$(R,0,0,D),$	$(R,1,1,D)$
$(\bar{R},0,1,D)$	$(\bar{R},1,0,D)$	$(R,0,\llcorner,0)$	$(R,0,\overline{\ulcorner},1)$
$(R,1,\llcorner,1)$	$(R,1,\overline{\ulcorner},0)$	$(\bar{R},0,\llcorner,1)$	$(\bar{R},0,\overline{\ulcorner},0)$
$(\bar{R},1,\llcorner,0)$	$(\bar{R},1,\overline{\ulcorner},1)$	$(R,0,\wedge,0)$	$(R,0,\mapsto,0)$
$(R,0,\leftarrowtail,1)$	$(R,0,\downarrow,1)$	$(R,1,\vee,1)$	$(R,1,\leftarrow,1)$
$(R,1,\rightarrow,0)$	$(R,1,\uparrow,0)$	$(\bar{R},1,\wedge,0)$	$(\bar{R},1,\mapsto,0)$
$(\bar{R},1,\leftarrowtail,1)$	$(\bar{R},1,\downarrow,1)$	$(\bar{R},0,\vee,1)$	$(\bar{R},0,\leftarrow,1)$
$(\bar{R},0,\rightarrow,0)$	$(\bar{R},0,\uparrow,0)$	$(\wedge,0,0,D)$	$(\mapsto,1,0,D)$
$(\leftarrowtail,0,0,D)$	$(\downarrow,1,0,D)$	$(\vee,1,1,D)$	$(\leftarrow,0,1,D)$
$(\rightarrow,1,1,D)$	$(\uparrow,0,1,D)$	$(\wedge,0,\llcorner,0)$	$(\wedge,0,\overline{\ulcorner},1)$
$(\mapsto,1,\llcorner,0)$	$(\mapsto,1,\overline{\ulcorner},1)$	$(\leftarrowtail,0,\llcorner,0)$	$(\leftarrowtail,0,\overline{\ulcorner},1)$
$(\downarrow,1,\llcorner,0)$	$(\downarrow,1,\overline{\ulcorner},1)$	$(\vee,1,\llcorner,1)$	$(\vee,1,\overline{\ulcorner},0)$
$(\leftarrow,0,\llcorner,1)$	$(\leftarrow,0,\overline{\ulcorner},0)$	$(\rightarrow,1,|,1)$	$(\rightarrow,1,\overline{\ulcorner},0)$
$(\uparrow,0,\llcorner,1)$	$(\uparrow,0\overline{\ulcorner},0)$	$(\wedge,0,\wedge,0)$	$(\wedge,0,\mapsto,0)$
$(\wedge,0,\leftarrowtail,1)$	$(\wedge,0,\downarrow,1)$	$(\mapsto,1,\wedge,0)$	$(\mapsto,1,\mapsto,0)$
$(\mapsto,1,\leftarrowtail,1)$	$(\mapsto,1,\downarrow,1)$	$(\leftarrowtail,0,\wedge,0)$	$(\leftarrowtail,0,\mapsto,0)$
$(\leftarrowtail,0,\leftarrowtail,1)$	$(\leftarrowtail,0,\downarrow,1)$	$(\downarrow,1,\wedge,0)$	$(\downarrow,1,\mapsto,0)$
$(\downarrow,1,\leftarrowtail,1)$	$(\downarrow,1,\downarrow,1)$	$(\vee,1,\vee,1)$	$(\vee,1,\leftarrow,1)$
$(\vee,1,\rightarrow,0)$	$(\vee,1,\uparrow,0)$	$(\leftarrow,0,\vee,1)$	$(\leftarrow,0,\leftarrow,1)$
$(\leftarrow,0,\rightarrow,0)$	$(\leftarrow,0,\uparrow,0)$	$(\rightarrow,1,\vee,1)$	$(\rightarrow,1,\leftarrow,1)$
$(\rightarrow,1,\rightarrow,0)$	$(\rightarrow,1,\uparrow,0)$	$(\uparrow,0,\vee,1)$	$(\uparrow,0,\leftarrow,1)$
$(\uparrow,0,\rightarrow,0)$	$(\uparrow,0,\uparrow,0)$		

(b) $(\llcorner, B, \llcorner, \llcorner)$ $\quad(\llcorner, B, \overline{\llcorner}, \overline{\llcorner})$ $\quad(\overline{\llcorner}, B, \llcorner, \overline{\llcorner})$ $\quad(\overline{\llcorner}, B, \overline{\llcorner}, \llcorner)$

$(+, 0, \llcorner, \llcorner)$ $\quad(+, 0, \overline{\llcorner}, \overline{\llcorner})$ $\quad(+, 1, \llcorner, \overline{\llcorner})$ $\quad(+, 1, \overline{\llcorner}, \llcorner)$

$(\leftrightarrow, 0, \llcorner, \llcorner)$ $\quad(\leftrightarrow, 0, \overline{\llcorner}, \llcorner)$ $\quad(\leftrightarrow, 1, \llcorner, \llcorner)$ $\quad(\leftrightarrow, 1, \overline{\llcorner}, \overline{\llcorner})$

$(\wedge, 1, \llcorner, \llcorner)$ $\quad(\wedge, 1, \overline{\llcorner}, \overline{\llcorner})$ $\quad(\mapsto, 0, \llcorner, \llcorner),$ $\quad(\mapsto, 0, \overline{\llcorner}, \overline{\llcorner})$

$(\leftmapsto, 1, \llcorner, \overline{\llcorner})$ $\quad(\leftmapsto, 1, \overline{\llcorner}, \llcorner)$ $\quad(\downarrow, 0, \llcorner, \overline{\llcorner})$ $\quad(\downarrow, 0, \overline{\llcorner}, \llcorner)$

$(\vee, 0, \llcorner, \llcorner)$ $\quad(\vee, 0, \overline{\llcorner}, \overline{\llcorner})$ $\quad(\leftmapsto, 1, \llcorner, \llcorner)$ $\quad(\leftmapsto, 1, \overline{\llcorner}, \overline{\llcorner})$

$(\rightarrow, 0, \llcorner, \overline{\llcorner})$ $\quad(\rightarrow, 0, \overline{\llcorner}, \llcorner)$ $\quad(\uparrow, 1, \llcorner, \overline{\llcorner})$ $\quad(\uparrow, 1, \overline{\llcorner}, \llcorner)$

(c) $(R, \llcorner, \llcorner, R)$ $\quad(R, \llcorner, \overline{\llcorner}, \bar{R})$ $\quad(R, \overline{\llcorner}, \llcorner, \bar{R})$ $\quad(R, \overline{\llcorner}, \overline{\llcorner}, R)$

$(\bar{R}, \llcorner, \llcorner, \bar{R})$ $\quad(\bar{R}, \llcorner, \overline{\llcorner}, R)$ $\quad(\bar{R}, \overline{\llcorner}, \llcorner, R)$ $\quad(\bar{R}, \overline{\llcorner}, \overline{\llcorner}, \bar{R})$

(d) (R, R, R, D) $\quad(R, \bar{R}, \bar{R}, D)$ $\quad(\bar{R}, R, \bar{R}, D)$ $\quad(\bar{R}, \bar{R}, R, D)$

$(R, R, +, 0)$ $\quad(R, R, \leftrightarrow, 1)$ $\quad(R, \bar{R}, +, 1)$ $\quad(R, \bar{R}, \leftrightarrow, 0)$

$(\bar{R}, R, +, 1)$ $\quad(\bar{R}, R, \leftrightarrow, 0)$ $\quad(\bar{R}, \bar{R}, +, 0)$ $\quad(\bar{R}, \bar{R}, \leftrightarrow, 1)$

$(R, R, \wedge, 1)$ $\quad(R, R, \leftmapsto, 0)$ $\quad(R, R, \vee, 0),$ $\quad(R, R, \rightarrow, 1)$

$(R, \bar{R}, \mapsto, 1)$ $\quad(R, \bar{R}, \downarrow, 0)$ $\quad(R, \bar{R}, \leftmapsto, 0)$ $\quad(R, \bar{R}, \uparrow, 1)$

$(\bar{R}, R, \mapsto, 1)$ $\quad(\bar{R}, R, \downarrow, 0)$ $\quad(\bar{R}, R, \leftmapsto, 0)$ $\quad(\bar{R}, R, \uparrow, 1)$

$(\bar{R}, \bar{R}, \wedge, 1)$ $\quad(\bar{R}, \bar{R}, \leftmapsto, 0)$ $\quad(\bar{R}, \bar{R}, \vee, 0)$ $\quad(\bar{R}, \bar{R}, \rightarrow, 1)$

(e) $(+, \llcorner, \llcorner, +)$ $\quad(+, \llcorner, \overline{\llcorner}, \leftrightarrow)$ $\quad(+, \overline{\llcorner}, \llcorner, \leftrightarrow)$ $\quad(+, \overline{\llcorner}, \overline{\llcorner}, +)$

$(\leftrightarrow, \llcorner, \llcorner, \leftrightarrow)$ $\quad(\leftrightarrow, \llcorner, \overline{\llcorner}, +)$ $\quad(\leftrightarrow, \overline{\llcorner}, \llcorner, +)$ $\quad(\leftrightarrow, \overline{\llcorner}, \overline{\llcorner}, \leftrightarrow)$

$(\wedge, \llcorner, \llcorner, \wedge)$ $\quad(\wedge, \llcorner, \overline{\llcorner}, \uparrow)$ $\quad(\wedge, \overline{\llcorner}, \llcorner, \mapsto)$ $\quad(\wedge, \overline{\llcorner}, \llcorner, \rightarrow)$

$(\mapsto, \llcorner, \llcorner, \mapsto)$ $\quad(\mapsto, \llcorner, \overline{\llcorner}, \rightarrow)$ $\quad(\mapsto, \overline{\llcorner}, \llcorner, \wedge),$ $\quad(\mapsto, \overline{\llcorner}, \overline{\llcorner}, \uparrow)$

$(\leftmapsto, \llcorner, \llcorner, \leftmapsto)$ $\quad(\leftmapsto, \llcorner, \overline{\llcorner}, \leftarrow)$ $\quad(\leftmapsto, \overline{\llcorner}, \llcorner, \downarrow)$ $\quad(\leftmapsto, \overline{\llcorner}, \overline{\llcorner}, \vee)$

$(\downarrow, \llcorner, \llcorner, \downarrow)$ $\quad(\downarrow, \llcorner, \overline{\llcorner}, \vee)$ $\quad(\downarrow, \overline{\llcorner}, \llcorner, \leftmapsto)$ $\quad(\downarrow, \overline{\llcorner}, \overline{\llcorner}, \leftarrow)$

$(\vee, \llcorner, \llcorner, \vee)$ $\quad(\vee, \llcorner, \overline{\llcorner}, \downarrow)$ $\quad(\vee, \overline{\llcorner}, \llcorner, \leftarrow)$ $\quad(\vee, \overline{\llcorner}, \overline{\llcorner}, \leftmapsto)$

$(\leftarrow, \llcorner, \llcorner, \leftarrow)$ $\quad(\leftarrow, \llcorner, \overline{\llcorner}, \leftmapsto)$ $\quad(\leftarrow, \overline{\llcorner}, \llcorner, \vee)$ $\quad(\leftarrow, \overline{\llcorner}, \overline{\llcorner}, \downarrow)$

$(\rightarrow, \llcorner, \llcorner, \rightarrow)$ $\quad(\rightarrow, \llcorner, \overline{\llcorner}, \mapsto)$ $\quad(\rightarrow, \overline{\llcorner}, \llcorner, \uparrow)$ $\quad(\rightarrow, \overline{\llcorner}, \overline{\llcorner}, \wedge)$

$(\uparrow, \llcorner, \llcorner, \uparrow)$ $\quad(\uparrow, \llcorner, \overline{\llcorner}, \wedge)$ $\quad(\uparrow, \overline{\llcorner}, \llcorner, \rightarrow)$ $\quad(\uparrow, \overline{\llcorner}, \overline{\llcorner}, \mapsto)$

(f) $(+, R, +, \llcorner)$ $\quad(+, R, \leftrightarrow, \overline{\llcorner})$ $\quad(+, \bar{R}, \leftrightarrow, \llcorner)$ $\quad(+, \bar{R}, +, \overline{\llcorner})$

$(\leftrightarrow, R, \leftrightarrow, \llcorner)$ $\quad(\leftrightarrow, R, +, \overline{\llcorner})$ $\quad(\leftrightarrow, \bar{R}, +, \llcorner)$ $\quad(\leftrightarrow, \bar{R}, \leftrightarrow, \overline{\llcorner})$

$(\wedge, R, \wedge, \llcorner)$ $\quad(\wedge, R, \leftmapsto, \overline{\llcorner})$ $\quad(\wedge, \bar{R}, \mapsto, \llcorner)$ $\quad(\wedge, \bar{R}, \downarrow, \overline{\llcorner})$

$(\mapsto, R, \mapsto, \llcorner)$ $\quad(\mapsto, R, \downarrow, \overline{\llcorner})$ $\quad(\mapsto, \bar{R}, \wedge, \llcorner),$ $\quad(\mapsto, \bar{R}, \leftmapsto, \overline{\llcorner})$

$(\leftmapsto, R, \leftmapsto, \llcorner)$ $\quad(\leftmapsto, R, \wedge, \overline{\llcorner})$ $\quad(\leftmapsto, \bar{R}, \downarrow, \llcorner)$ $\quad(\leftmapsto, \bar{R}, \mapsto, \overline{\llcorner})$

$(\downarrow, R, \downarrow, \llcorner)$ $\quad(\downarrow, R, \mapsto, \overline{\llcorner})$ $\quad(\downarrow, \bar{R}, \leftmapsto, \llcorner)$ $\quad(\downarrow, \bar{R}, \wedge, \overline{\llcorner})$

$(\vee, R, \vee, \llcorner)$ $\quad(\vee, R, \rightarrow, \overline{\llcorner})$ $\quad(\vee, \bar{R}, \leftarrow, \llcorner)$ $\quad(\vee, \bar{R}, \uparrow, \overline{\llcorner})$

$(\leftarrow, R, \leftarrow, \llcorner)$ $\quad(\leftarrow, R, \uparrow, \overline{\llcorner})$ $\quad(\leftarrow, \bar{R}, \vee, \llcorner)$ $\quad(\leftarrow, \bar{R}, \rightarrow, \overline{\llcorner})$

$(\rightarrow, R, \rightarrow, \mathsf{L})$ $(\rightarrow, R, \vee, \overline{\mathsf{L}})$ $(\rightarrow, \bar{R}, \uparrow, \mathsf{L})$ $(\rightarrow, \bar{R}, \leftarrow, \overline{\mathsf{L}})$

$(\uparrow, R, \uparrow, \mathsf{L})$ $(\uparrow, R, \leftarrow, \overline{\mathsf{L}})$ $(\uparrow, \bar{R}, \rightarrow, \mathsf{L})$ $(\uparrow, \bar{R}, \vee, \overline{\mathsf{L}})$

(g) $(R, +, +, R)$ $(R, +, \leftrightarrow, \bar{R})$ $(R, \leftrightarrow, +, \bar{R})$ $(R, \leftrightarrow, \leftrightarrow, R)$

$(\bar{R}, +, +, \bar{R})$ $(\bar{R}, +, \leftrightarrow, R)$ $(\bar{R}, \leftrightarrow, +, R)$ $(\bar{R}, \leftrightarrow, \leftrightarrow, \bar{R})$

$(R, \wedge, \wedge, R)$ $(R, \wedge, \leftmapsto, \bar{R})$ $(R, \mapsto, \mapsto, R)$ $(R, \mapsto, \downarrow, \bar{R})$

$(R, \leftmapsto, \leftmapsto, R)$ $(R, \leftmapsto, \wedge, \bar{R})$ $(R, \downarrow, \downarrow, R)$ $(R, \downarrow, \mapsto, \bar{R})$

$(R, \vee, \vee, R)$ $(R, \vee, \rightarrow, \bar{R})$ $(R, \leftarrow, \leftarrow, R)$ $(R, \leftarrow, \uparrow, \bar{R})$

$(R, \rightarrow, \rightarrow, R)$ $(R, \rightarrow, \vee, \bar{R})$ $(R, \uparrow, \uparrow, R)$ $(R, \uparrow, \leftarrow, \bar{R})$

$(\bar{R}, \wedge, \uparrow, R)$ $(\bar{R}, \wedge, \leftarrow, \bar{R})$ $(\bar{R}, \mapsto, \rightarrow, R)$ $(\bar{R}, \mapsto, \vee, \bar{R})$

$(\bar{R}, \leftmapsto, \leftarrow, R)$ $(\bar{R}, \mapsto, \uparrow, \bar{R})$ $(\bar{R}, \downarrow, \vee, R)$ $(\bar{R}, \downarrow, \rightarrow, \bar{R})$

$(\bar{R}, \vee, \downarrow, R)$ $(\bar{R}, \vee, \mapsto, \bar{R})$ $(\bar{R}, \leftarrow, \leftmapsto, R)$ $(\bar{R}, \leftarrow, \wedge, \bar{R})$

$(\bar{R}, \rightarrow, \mapsto, R)$ $(\bar{R}, \rightarrow, \downarrow, \bar{R})$ $(\bar{R}, \uparrow, \wedge, R)$ $(\bar{R}, \uparrow, \leftmapsto, \bar{R})$

(h) $(+, +, +, +)$ $(+, +, \leftrightarrow, \leftrightarrow)$ $(+, \leftrightarrow, +, \leftrightarrow)$ $(+, \leftrightarrow, \leftrightarrow, +)$

$(\leftrightarrow, +, +, \leftrightarrow)$ $(\leftrightarrow, +, \leftrightarrow, +)$ $(\leftrightarrow, \leftrightarrow, +, +)$ $(\leftrightarrow, \leftrightarrow, \leftrightarrow, \leftrightarrow)$

$(\wedge, \wedge, \wedge, \wedge)$ $(\wedge, \wedge, \leftmapsto, \uparrow)$ $(\wedge, \mapsto, \mapsto, \wedge)$ $(\wedge, \mapsto, \downarrow, \uparrow)$

$(\wedge, \leftmapsto, \wedge, \mapsto)$ $(\wedge, \leftmapsto, \leftmapsto, \rightarrow)$ $(\wedge, \downarrow, \mapsto, \mapsto)$ $(\wedge, \downarrow, \downarrow, \rightarrow)$

$(\mapsto, \vee, \mapsto, \mapsto)$ $(\mapsto, \vee, \downarrow, \rightarrow)$ $(\mapsto, \leftarrow, \wedge, \mapsto)$ $(\mapsto, \leftarrow, \leftmapsto, \rightarrow)$

$(\mapsto, \rightarrow, \mapsto, \wedge)$ $(\mapsto, \rightarrow, \downarrow, \uparrow)$ $(\mapsto, \uparrow, \wedge, \wedge)$ $(\mapsto, \uparrow, \leftmapsto, \uparrow)$

$(\leftmapsto, \wedge, \wedge, \leftmapsto)$ $(\leftmapsto, \wedge, \leftmapsto, \leftarrow)$ $(\leftmapsto, \mapsto, \mapsto, \leftmapsto)$ $(\leftmapsto, \mapsto, \downarrow, \leftarrow)$

$(\leftmapsto, \leftmapsto, \wedge, \downarrow)$ $(\leftmapsto, \leftmapsto, \leftmapsto, \vee)$ $(\leftmapsto, \downarrow, \mapsto, \downarrow)$ $(\leftmapsto, \downarrow, \downarrow, \vee)$

$(\downarrow, \vee, \mapsto, \downarrow)$ $(\downarrow, \vee, \downarrow, \vee)$ $(\downarrow, \leftarrow, \wedge, \downarrow)$ $(\downarrow, \leftarrow, \leftarrow, \vee)$

$(\downarrow, \rightarrow, \mapsto, \leftmapsto)$ $(\downarrow, \rightarrow, \downarrow, \leftarrow)$ $(\downarrow, \uparrow, \wedge, \leftmapsto)$ $(\downarrow, \uparrow, \leftmapsto, \leftarrow)$

$(\vee, \vee, \vee, \vee)$ $(\vee, \vee, \rightarrow, \downarrow)$ $(\vee, \leftarrow, \leftarrow, \vee)$ $(\vee, \leftarrow, \uparrow, \downarrow)$

$(\vee, \rightarrow, \vee, \leftarrow)$ $(\vee, \rightarrow, \rightarrow, \leftmapsto)$ $(\vee, \uparrow, \leftarrow, \leftarrow)$ $(\vee, \uparrow, \uparrow, \leftmapsto)$

$(\leftarrow, \wedge, \leftarrow, \leftarrow)$ $(\leftarrow, \wedge, \uparrow, \leftmapsto)$ $(\leftarrow, \mapsto, \vee, \leftarrow)$ $(\leftarrow, \mapsto, \rightarrow, \leftmapsto)$

$(\leftarrow, \leftmapsto, \leftmapsto, \vee)$ $(\leftarrow, \leftmapsto, \uparrow, \downarrow)$ $(\leftarrow, \downarrow, \downarrow, \vee)$ $(\leftarrow, \downarrow, \rightarrow, \downarrow)$

$(\rightarrow, \vee, \vee, \rightarrow)$ $(\rightarrow, \vee, \rightarrow, \mapsto)$ $(\rightarrow, \leftarrow, \leftarrow, \rightarrow)$ $(\rightarrow, \leftarrow, \uparrow, \mapsto)$

$(\rightarrow, \rightarrow, \leftmapsto, \uparrow)$ $(\rightarrow, \rightarrow, \rightarrow, \wedge)$ $(\rightarrow, \uparrow, \leftarrow, \uparrow)$ $(\rightarrow, \uparrow, \uparrow, \wedge)$

$(\uparrow, \wedge, \leftarrow, \uparrow)$ $(\uparrow, \wedge, \uparrow, \wedge)$ $(\uparrow, \mapsto, \vee, \uparrow)$ $(\uparrow, \mapsto, \rightarrow, \wedge)$

$(\uparrow, \leftmapsto, \leftarrow, \rightarrow)$ $(\uparrow, \leftmapsto, \uparrow, \mapsto)$ $(\uparrow, \downarrow, \vee, \rightarrow)$ $(\uparrow, \downarrow, \rightarrow, \mapsto)$

REFERENCES

[1] Berglund J. F., Mislove M. W., *A class of semigroups having almost trivial multiplications*, Semigroup Forum, vol. 4 (No. 2), (1972).

[2] Clifford A. H., Preston G. B., *The algebraic theory of semigroups I*, Amer. Math. Soc., Providence (1961).

[3] Albert A. A., *Quasigroups I*, Trans. Amer. Math. Soc., 54, (1943).

[4] Krapež A., *On solving a system of balanced functional equations on quasigroups I–III*, Publ. Inst. Math. tomes 23 (37) (1978), 25 (39) (1979), 26 (40) (1979).

[5] Aczel J., Belousov V. D., Hosszú M., *Generalized associativity and bisymmetry on quasigroups*, Acta. Math. Acad. Sci. Hung., 11 (1960).