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A CONTRIBUTION TO BEST APPROXIMATION IN THE L_2-NORM

S.R.H. Rizvi and *O.P. Juneja*

1. Let E denote a closed and bounded Jordan region in the complex plane with transfinite diameter $d > 0$. Let $w(z)$ be a positive continuous function on E and let $\mathcal{H}_2(E)$ denote the Hilbert space of functions analytic in D with inner product.

$$(1.1)\qquad (f,g) = \iint_E w(z)f(z)\overline{g(z)}dxdy, \quad f,g \in \mathcal{H}_2(E)$$

so that for any $f \in \mathcal{H}_2(E)$, we have

$$(1.2)\qquad \|f\| = \left[\iint_E w(z)|f(z)|^2 dxdy\right]^{1/2} < \infty$$

If $\mathcal{A}(E) \equiv \{p_{n-1}(z)\}_{n=1}^{\infty}$, $p_n(z)$ being a polynomial of degree n, is a complete orthonormal sequence in $\mathcal{H}_2(E)$[1], set, for any $f \in \mathcal{H}_2(E)$ and for $n = 0, 1, 2, \ldots$

$$(1.3)\qquad \Delta_n(f) \equiv \Delta_n(f;E) =$$
$$= \inf_{\{ci\}} \left[\iint_E w(z)|f(z) - c_0 - c_1p_1(z) - \cdots - c_np_n(z)|^2 dxdy\right]^{1/2}$$

$$(1.4)\qquad a_n \equiv a_n(f;E) = \iint_E w(z)f(z)\overline{p_n(z)}dxdy.$$

$\Delta_n(f;E)$ is called the minimum error of f in L^2-norm with respect to the system $\mathcal{A}(E)$ and a_n is called the nth Fourier coefficient of f with respect to the

[1] Such a sequence of polynomials always exists in $\mathcal{H}_2(E)$ as can be easily seen with the help of Faber polynomials ([**2**] and [**8**]).

system $\mathcal{A}(E)$. It is wel known (see e.g. [**3**] that if U denotes the unit disc and $|w(z)| \equiv 1$, then $\mathcal{A}(U) \equiv \{\sqrt{\frac{n}{\pi}}z^{n-1}\}_{n=1}^{\infty}$ forms a complete orthonormal sequence in $\mathcal{H}_2(U)$ and that if $f(z) = \sum_{n=0}^{\infty} b_n z^n (|z| < 1)$ is in $\mathcal{H}_2(t)$ then

$$b_n = \sqrt{\frac{n+1}{\pi}} a_n(f;U). \tag{1.5}$$

Hence it follows from (1.5) that if f can be extended to an entire function of order ρ, lower order λ and type T then in all the results that give ρ, λ, T in terms of the coefficients $b_n{}'s$ (see [**1**], [4], [**5**] etc.), one can replace b_n by $a_n(f;U)$. In fact, much more than this can be said. Authors have recently shown [**7**] that if $f \in \mathcal{H}_2(E)$, the Fourier series $\sum_{k=0}^{\infty} a_k p_k(z)$ converges uniformly to $f(z)$ on E and f can be extended to an entire function, if and only if,

$$\lim |a_n(f;E)|^{1/n} = 0. \tag{1.6}$$

Further, if f is of order $\rho (0 < \rho < \infty)$, type T and lower order λ then

$$\rho = \lim_{n\to\infty} \sup \frac{n \log n}{\log |a_n(f;E)|^{-1}}; \tag{1.7}$$

$$T d^{\rho} = \lim_{n\to\infty} \sup (n/e_{\rho}) |a_n(f;E)|^{\rho/n}; \tag{1.8}$$

$$\lambda = \sup_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log |a_{n_k}(f;E)|^{-1}}; \tag{1.9}$$

where supremum in (1.9) is taken over all increasing sequences $\{n_k\}$ of positive integers.

Attempts have also been made to connect the minimum error $\Delta(f;U)$ with the growth of the entire function f. Thus, for $w(z) \equiv 1$, Reedy [**6**] has obtained relations essentially involving $\Delta_n(f;U)$ and the order and the type of entire function f. The aim of the present paper is to generalize the results of Reddy to any bounded and closed Jordan region E and for any weight function $w(z)$, positive and continuous on E. We also obtain relations involving $\Delta_n(f;E)$ and the lower order of f the analogues of which for the unit disc U have not been obtained by Reddy. We shall assume troughout that E is a bounded and closed Jordan region with transfinite diameter $d > 0$ and that for $f \in \mathcal{H}_2(E)$, $\Delta_n(f)$ and a_n are given by (1.3) and (1.4) respectively.

2. We first prove few lemmas which will be required in the sequel.

LEMMA 1: *Let $f \in \mathcal{H}_2(E)$, then*

$$\Delta_n(f) = \left[\sum_{n=1}^{\infty} |a_k|_2\right]^{1/2}, \quad n = 1, 2, 3, \ldots \tag{2.1}$$

Proof. By (1.3), we have

$$\Delta_n(f) \le \left[\iint_E w(z) \left| f(z) - \sum_{k=0}^{n} a_k p_k(z) \right|^2 dxdy \right]^{1/2}.$$

Since $\sum_{k=0}^{\infty} a_k p_k(z)$ converges to $f(z)$ on E, we get,

$$\begin{aligned} \Delta_n(f) &\le \left[\iint_E w(z) \left| \sum_{n+1}^{\infty} a_k p_k(z) \right|^2 dxdy \right]^{1/2} \\ &= \left[\sum_{n+1}^{\infty} |a_k|_2 \right]^{1/2}. \end{aligned} \tag{2.2}$$

Again as for any $(n+1)$ complex numbers $\{a_o', a_1', \dots, a_n'\}$

$$\begin{aligned} &\iint_E w(z)|f(z) - a_0'p_0(z) - a_1'p_1(z) - \cdots - a_n'p_n(z)|^2 dxdy \\ &\quad = \iint_E \left| \sum_{k=0}^{n} (a_k - a_k')p(z) + \sum_{k=n+1}^{\infty} a_k p_k(z) \right|^2 dxdy \\ &\quad = \sum_{k=0}^{n} |a_k - a_k'|^2 + \sum_{k=n+1}^{\infty} |a_k|^2 \\ &\quad \ge \sum_{k=n+1}^{\infty} |a_k|^2. \end{aligned}$$

Since this is true for any $(n+1)$ complex numbers $a_0', a_1', \dots, a_n'$, we have

$$[\Delta_n(f)]^2 \ge \sum_{n+1}^{\infty} |a_k|^2. \tag{2.3}$$

Combining (2.2) and (2.3) we get (2.1).

Lemma 2 [**9**]: *Let $w = \varphi(z)$ be the function mapping the complement of E onto $|w| > 1$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$ and let $E_r = \{z: |\varphi(z)| = r\}$. Let f be an entire function of order $\rho (0 < \rho < \infty)$, lower order λ, type T and let*

$$\overline{M}(r) = \max_{z \in E_r} |f(z)|, \tag{2.4}$$

then

$$\begin{matrix} \rho \\ \lambda \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log \overline{M}(r)}{\log r} \tag{2.5}$$

$$d^{\rho}T = \limsup_{r\to\infty} \frac{\log \overline{M}(r)}{r^{\rho}} \tag{2.6}$$

LEMMA 3: *Let f be an entire function and $\overline{M}(r)$ be given by (2.4), then for any $\varepsilon > 0$*

$$\Delta_n(f) \leq A\frac{\overline{M}(r)}{r^n}e^{n\varepsilon} \tag{2.7}$$

for $r \geq r_0(\varepsilon)$, and $n \geq n_0(\varepsilon)$, where A is a constant independent of n and r.

PROOF. We known [**9**] that there exist polynomials $\{g_n(z)\}_{n=1}^{\infty}$ of respective degree less than or equal to n, such that for all $z \in E$

$$f(z) - g_n(z) \leq K\frac{\overline{M}(r)}{r^n}e^{n\varepsilon} \tag{2.8}$$

for $n \geq n_0(\varepsilon, E)$ and $r \geq r_0(\varepsilon)$ and where K is a constant independent of n and depends only on ε and E.

Now as each $g_n(z)$ can be written as a linear combination of $\{p_k(z)\}_{k=0}^{n}$, it follows that

$$[\Delta_n(f)]^2 \leq \iint\limits_E w(z)|f(z) - g_n(z)|^2 dxdy.$$

Using (2.8) we get

$$\Delta_n(f) \leq A\frac{\overline{M}(r)}{r^n}e^{n\varepsilon}$$

where A is a independent of n and r.

3. In this section we give necessary and sufficent conditions, in terms of $\Delta_n(f)$, for an entire function to be of order $\rho (0 < \rho < \infty)$ and Type T. First we have

THEOREM 1. *Let $f \in \mathcal{H}_2(E)$. Then f can be extended to an entire function if and only if,*

$$\lim_{n\to\infty} \Delta_n(f)^{1/n} = 0. \tag{3.1}$$

PROOF. Suppose f can be extended to an entire function. Then, by (1.6), we have

$$\lim_{n\to\infty} |a_n|^{1/n} = 0;$$

so that for any ε satisfying $0 < \varepsilon < 1$, there exists an $n_0 = n_0(\varepsilon)$ such that

$$|a_n| < \varepsilon^n \text{ for } n \geq n_0.$$

Using (2.1), we get

$$\Delta_n(f) \leq \frac{\varepsilon^n}{(1-\varepsilon^2)^{1/2}}.$$

So

$$\lim_{n\to\infty} \sup \Delta_n(f)^{1/n} \leq \varepsilon$$

i.e.

$$\lim_{k\to\infty} \sup \Delta_n(f)^{1/n} = 0.$$

Conversely, if $\lim_{n\to\infty} \Delta_n(f)^{1/n} = 0$, then as (2.1) gives

$$\Delta_n(f) > |a_{n+1}|$$

for every n, we get

$$\lim_{n\to\infty} |a_n|^{1/n} = 0.$$

Hence (1.6) yields that f is entire.

THEOREM 2. *Let $f \in \mathcal{H}_2(E)$; then f can be extended to an entire function of finite order ρ, if and only if,*

$$\mu \equiv \lim_{n\to\infty} \sup \frac{n \log n}{\log \Delta_n(f)^{-1}} < \infty. \tag{3.2}$$

Further $\mu = \rho$ also holds.

PROOF. Suppose first that (3.2) holds. Let ε be any positive number, then there exists $n_0 = n_0(\varepsilon)$ such that

$$\frac{n \log n}{\log \Delta_n(f)^{-1}} < \mu + \varepsilon \ \text{ for } \ n \geq n_0,$$

or

$$\Delta_n(f) < n^{-n/\mu+\varepsilon} \ \text{ for } \ n \geq n_0,$$

which gives $\lim_{n\to\infty} \Delta_n(f)^{1/n} = 0$.

So, in view of Theorem 1, f can be extended to an entire function. Let its order be ρ. Since, by (2.1)

$$\Delta_n(f) > |a_{n+1}| \tag{3.3}$$

for every n, (3.3) gives

$$\lim_{n\to\infty} \sup \frac{n \log n}{\log \Delta_n(f)^{-1}} \geq \lim_{n\to\infty} \sup \frac{n \log n}{\log |a_n|^{-1}}.$$

So, using (1.7) we get

$$\rho \leq \lim_{n\to\infty} \sup \frac{n \log n}{\log \Delta_n(f)^{-1}}$$

i.e.

$$\rho \le \mu \tag{3.4}$$

which shows that f is of finite order.

Conversely, suppose that f is an entire function of finite order ρ. Then (1.7) gives that, for any $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that

$$|a_n| \le n^{-n/\rho+\varepsilon} \text{ for } n \ge n_0.$$

Using (2.1) we get

$$[\Delta_n(f)]^2 \le \sum_{k=n+1}^{\infty} k^{-2k/\rho+\varepsilon} \text{ for } n \ge n_0$$

or

$$[\Delta_n(f)]^2 \le (n+1)^{-\frac{2(n+1)}{\rho+\varepsilon}}[1+o(1)] \text{ as } n \to \infty.$$

i.e.

$$\lim_{n\to\infty} \sup \frac{n \log n}{\log \Delta_n(f)^{-1}} \le (\rho + \varepsilon)$$

As this is true for every $\varepsilon > 0$, we get

$$\mu \le \rho \tag{3.5}$$

which shows that (3.2) holds.

Combining (3.4) and (3.5) we also get $\mu = \rho$.

THEOREM 3. *Let $f \in \mathcal{H}_2(E)$ and v defined by*

$$\lim_{n\to\infty} \sup n|\Delta_n(f)|^{\rho/n} = v \quad (0 < \rho < \infty) \tag{3.6}$$

satisfies $0 < v < \infty$). Then f can be extended to an entire function of order ρ and type T, if and only if

$$v = (e\rho T)d^\rho,$$

where d is the transfinite diameter of E.

PROOF. Suppose $v = (e\rho T)d^\rho$ and $0 < v < \infty$. Then (3.6) gives easily

$$\lim_{n\to\infty} \sup \frac{n \log n}{\log \Delta_n(f)^{-1}} = \rho$$

So, Theorem 2 gives that f can be extended to an entire function of order ρ. Now, we shall show that its type is $v/(e\rho d^\rho)$.

Let f be of type T. By (2.6), we have, for any $\varepsilon > 0$, there exists $r_0 = r_0(\varepsilon)$ such that

$$\log \overline{M}(r) < (T' + \varepsilon) r^\rho \ \text{ for } \ r \geq r_0, \tag{3.7}$$

where

$$T' = T d^\rho.$$

Using (2.7) and (3.7), we get

$$\log \Delta_n(f) \geq \log A + n\varepsilon - n \log r + (T' + \varepsilon) r^\rho$$

for $n \geq n_0$ and $r \geq r_0$.

Chose a sequence $r_n \to \infty$ as

$$r_n = \left[\frac{n}{\rho(T' + \varepsilon)} \right]^{1/\rho},$$

then for large n, we have

$$\log \Delta_n(f) \leq \log A + n\varepsilon - \frac{n}{\rho}[\log n - \log(T' + \varepsilon)\rho] + \frac{n}{\rho}$$

or,

$$n \Delta_n(f)^{\rho/n} \leq e\rho(T' + \varepsilon) e^{\rho\varepsilon} + o(1).$$

Hence

$$\lim_{n\to\infty} \sup n \Delta_n(f)^{\rho/n} \leq e\rho(T' + \varepsilon) e^{\rho\varepsilon} \ \text{ as } \ n \to \infty$$

as this true for every $\varepsilon > 0$ so we get

$$v \equiv \lim_{n\to\infty} \sup n \Delta_n(f)^{\rho/n} \leq e\rho T' = e\rho T d^\rho.$$

i.e.,

$$T \geq v/(e\rho d^\rho). \tag{3.8}$$

For reverse inequality, we obverse that, since

$$\Delta_n(f) \geq |a_{a+1}| \ \text{ for all } \ n,$$

so

$$\limsup_{n\to\infty} n|a_n|^{\rho/n} \leq \limsup_{n\to\infty} n \Delta_n(f)^{\rho/n}.$$

Using (1.8) we get

$$e\rho T d^\rho \leq \limsup_{n\to\infty} n \Delta_n(f)^{\rho/n} = v$$

i. e.,

$$T \le v/(e\rho d^\rho). \tag{3.9}$$

Combing (3.8) and (3.9) we get $e\rho d^\rho T = v$.

This proves the theorem.

4. We now obtain relations involving $\Delta_n(f)$ and the lower order of f. We thus have

THEOREM 4. *Let $f \in \mathcal{H}_2(E)$. Then f can be extended to an entire function of finite lower order λ, and only if,*

$$\lambda = \sup_{\{n_k\}} \liminf_{n\to\infty} \frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}} < \infty$$

where supremum is taken over all increasing sequences $\{n_k\}$ of positive integers.

PROOF. Suppose f can be extended to an entire function of finite lower order λ. We show that λ satisfies (4.1). Let $\{n_k\}$ be an increasing sequence of positive integers. Since (2.1) gives

$$\Delta_{n_k}(f) \ge |a_{n_k+1}|$$

for every k, we get

$$\liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}} \ge \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log |a_{n_k}|^{-1}}.$$

Hence

$$\sup_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}} \ge \sup_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log |a_{n_k}|^{-1}}.$$

Using (1.9) we get

$$\lambda \le \sup_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}}.$$

Now for reverse inequality we procead as follows. For any increasing sequence $\{n_k\}$ of positive integers define

$$\alpha \equiv \alpha(\{n_k\}) = \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}}. \tag{4.3}$$

Obviously $0 \le \alpha \le \infty$. Consider first the case when $0 < \alpha < \infty$. Let $\varepsilon > 0$ be such that $0 < \varepsilon < \alpha$, then, by (4.3), there exists $k_0 = k_0(\varepsilon)$ such that

$$\frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}} > \alpha - \varepsilon \;\text{ for }\; k \ge k_0.$$

Or

$$\Delta_{n_k}(f) \geq n_{k-1}^{-n_k/(\alpha-\varepsilon)} \text{ for } k \geq k_0. \tag{4.4}$$

Using (2.7) and (4.4), we get, for $r \geq r_0$ and $k \geq k_0$,

$$\log A + \log \overline{M}(r) > -\frac{n_k}{\alpha - \varepsilon} \log n_{k-1} + n_k \log r - n_k \varepsilon.$$

Define a sequence $r_k \to \infty$ as

$$r_k = e^{1+\varepsilon} n_{k-1}^{1/\alpha-\varepsilon}. \tag{4.5}$$

Then, if k is large and $r_k \leq r \leq r_{k+1}$,

$$\log A + \log \overline{M}(r) > -\frac{n_k}{\alpha - \varepsilon} \log n_{k-1} + n_k \log r_k - n_k \varepsilon.$$

Or,

$$\log A + \log \overline{M}(r) \geq \left(\frac{r_{n_k+1}}{e^{1+\varepsilon}}\right)^{\alpha-\varepsilon}$$

i.e.,

$$\log A + \log \overline{M}(r) \geq \left(\frac{r}{e^{1+\varepsilon}}\right)^{\alpha-\varepsilon}.$$

Hence

$$\lambda \equiv \liminf_{r\to\infty} \frac{\log\log \overline{M}(r)}{\log r} \geq \alpha - \varepsilon.$$

Since ε is arbitrary so we get $\lambda \geq \alpha$; which holds good in case $\alpha = 0$. Moreover, when $\alpha = \infty$, λ can not be finite. Hence

$$\lambda \geq \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}}.$$

Since, this is true for any increasing $\{n_k\}$ of positive integers, we get

$$\lambda \geq \sup_{\{n_k\}} \liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}}. \tag{4.6}$$

Combing (4.2) and (4.6) we get (4.1)

Conversely, suppose (4.1) is true while f can not be extended to an entire function (3.1) gives

$$\limsup_{n\to\infty} \Delta_n(f)^{1/n} = l = 0$$

Since for every $f \in \mathcal{H}_2(E)$, $\Delta_n(f) \to 0$, it is clear that $0 < l \leq 1$. Let $\varepsilon > 0$ be any positive number such that $0 < \varepsilon < l$, then there exists an increasing sequence $\{n_k\}$ of positive integers such that

$$\Delta_{n_k}(f) > (l - \varepsilon)^{n_k} \text{ for every } k.$$

Now on this sequence $\{n_k\}$

$$\liminf_{k\to\infty} \frac{n_k \log n_{k-1}}{\log \Delta_{n_k}(f)^{-1}} = \infty,$$

which contradicts the fact that (4.1) holds. Hence f can be extended to an entire function and then its lower order λ is given by (4.1).

The following theorem can also be proved in a similar manner.

THEOREM 5. *Let $f \in \mathcal{H}_2(E)$. Then f can be extended to an entire function of finite lower order λ, if and only if,*

$$\lambda = \sup_{\{n_k\}} \liminf_{k\to\infty} \frac{(n_k - n_{k-1}) \log n_{k-1}}{\log \left| \frac{\Delta_{n_{k-1}}(f)}{\Delta_{n_k}(f)} \right|} < \infty, \tag{4.7}$$

where supremum is taken over all increasing sequences $\{n_k\}$ of positive integers.

5. In this section we study about the sequence $\{n_k\}$ of those values of n for which $\Delta_{n-1}(f) > \Delta_n(f)$ i.e.,

$$\Delta_n(f) = \Delta_{n_k}(f) \text{ for } n_{n-1} \le n < n_k.$$

We shall see that for function of regular growth this sequence can not grow too rapidly. In fact we have

THEOREM 6. Let $f \in \mathcal{H}_2(E)$ be such that its extension is an entire function of order ρ and lower order $\lambda(0 \le \lambda \le \rho < \infty)$; and let $\{n_k\}$ be the sequence defined by (5.1); then

$$\lambda \le \lim_{k\to\infty} \inf \frac{\log n_k}{\log n_{k+1}}.$$

PROOF. Set $h(z) = \sum_{k=1}^{\infty} \pi_k(f) z^{nk}$, where $\pi_k(f) = \Delta_{n_{k-1}}(f) - \Delta_{n_k}(f)$. In wiew of (1.7), (1.9) and theorem 2 and 4 it can be easily seen that $h(z)$ is an entire function of order ρ and lower order λ.

So

$$\rho = \limsup_{k\to\infty} \frac{n_k \log n_k}{\log \pi_k^{-1}(f)}$$

and

$$\begin{aligned}
\lambda &= \sup_{\{m_h\}} \liminf_{h\to\infty} \frac{n_{m_h} \log n_{m_{h-1}}}{\log \pi_{m_h}^{-1}(f)} \\
&\le \sup_{\{m_h\}} \limsup_{h\to\infty} \frac{n_{m_h} \log n_{m_h}}{\log \pi_{m_h}^{-1}(f)} \times \sup_{\{m_h\}} \liminf_{h\to\infty} \frac{\log n_{m_{h-1}}}{\log n_{m_h}} \\
&= \rho \liminf_{k\to\infty} \frac{\log n_k}{\log n_{k+1}}
\end{aligned}$$

which completes the proof.

COROLLARY. *For entire functions or regular growth*

$$\log n_n \sim \log n_{k-1} \text{ as } k \to \infty$$

Finally we have

THEOREM 7. *Let f be an entire function. Then there exist integers*

$$0 \leq n_1 < n_2 < \cdots < n_k < \cdots$$

for which $a_{n_{k+1}} \neq 0$ for every k and

$$\Delta_{n_k} \sim \left|a_{n_{k+1}}\right| \text{ as } k \to \infty.$$

PROOF. Since f is entire, (1.6) gives

$$\lim_{k\to\infty} |a_k|^{1/k} = 0.$$

Let $\{\varepsilon_k\}$ be an arbitrary sequence of positive numbers each less than one and $\varepsilon \to \infty$ as $k \to \infty$. Now we shall show than there exists a sequence $\{n_k\}$ such that

$$|a_{n_k+j}| \leq |a_{n_k+1}|\varepsilon_k^{j-1} \text{ for } k = 1, 2, \ldots$$

Since for each $\varepsilon_k > 0$, there exists $N = N(k)$ such that

$$|a_n|^{1/n} \leq \varepsilon_k \text{ for } n \geq N(k); \tag{5.4}$$

let $\{n_k\}$ be the sequence such that

$$\max_{n\geq N=N(k)} |a_n|^{1/n} = |a_{n_k+1}|^{1/(n_k+1)},$$

then (5.4) gives

$$|a_{n_k+1}|^{1/(n_k+1)} < \varepsilon_k$$

for every k; so

$$|a_{n_k+j}|^{1/(n_k+j)} \leq |a_{n_k+1}|^{1/(n_k+1)} < \varepsilon_k$$

for everu j or

$$\begin{aligned} |a_{n_k+j}| &< |a_{n_k+1}|^{\frac{n_k+j}{n_k+1}} \varepsilon_k^{n_k+j} \\ &\leq |a_{n_k+1}||a_{n_k+1}|^{\frac{j-1}{n_k+1}} \end{aligned}$$

Or

$$|a_{n_k+j}| < |a_{n_k+1}|\varepsilon_k^{j-1}.$$

Now as

$$(\Delta_n(f))^2 = \sum_{n=1}^{\infty} |a_k|^2.$$

So we get

$$(\Delta_{n_k}(f))^2 < \sum_{j=1}^{\infty} |a_{n_k+1}|^2 \varepsilon_k^{(j-1)/2}$$

or

$$\Delta_{n_k}(f) < \frac{|a_{n_k+1}|}{(1-\varepsilon_k^2)^1/2}. \tag{5.5}$$

Combining (5.5) together with the fact that $\Delta_n(f) > |a_{n+1}|$ we get that

$$\lim_{k\to\infty} \frac{\Delta_{n_k}(f)}{|a_{n_k+1}|} = 1$$

i.e $\Delta_{n_k} \sim |a_{n_k+1}|$ as $k \to \infty$.

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Department of Mathematics
Indian Institute of Technology
Kanpur 208016,
INDIA