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A NEW SPECTRAL METHOD FOR DETERMINING THE NUMBER OF SPANNING TREES

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Abstract. As is known, the number of spanning tress of a regular graph can be determined by the graph spectrum. In this paper we describe a new variant of the spectral method for determining the number of spanning trees, which enables to solve the problem for a class of non-regular graphs.

Introductory comments

It was noted by H. Hutschenreuter [5] that the number of spanning of a regular graph can be expressed in terms of its spectrum. Let G be a regular graph of degree r on n verticies, let $P_G(\lambda)$ be its characteristic polynomial and $\lambda_1 = r, \lambda_2, \dots, \lambda_n$ its eigenvalues. Then the following formulas hold for $t(G)$, the number of spanning trees of G,

$$t(G) = \frac{1}{n}\prod_{i=2}^{n}(r-\lambda_i) = \frac{1}{n}P_G{}'(r). \tag{1}$$

The first author of this paper used formula (1) to determine the number of spanning trees for several classes of regular graphs [2] (see also [8]). Moreover, it is shown in [2] that this method, together with other spectral methods [12], is sufficient for deriving practically all known results concerning the number of spanning trees (see [3], pp. 217–220). Some additional results along these lines can be found in the references[1] [1], [6], [9], [10] and [11].

Formula (1) can be extended to non-regular graphs if one considers the characteristic polynomial and spectrum of the so called admittance matrix, instead of the characteristic polynomial and spectrum of the adjacency matrix (see [3] p. 39, [6] and [11]). Since the latter objects play a dominant role in the spectral graph theory, it would be of interest to extend (1) to non-regular graphs so that $t(G)$ is

[1] It should be pointed out that especially [1] and [6] simply ignore the paper [2], although this is a relevant reference for [1] and [6].

still expressed in terms of the characteristic polynomial or the spectrum of the adjacency matrix. This has been done in the present paper for class of graphs which are close to regular graphs: these are graphs in which all vertices but one have a fixed degree r. We shall call such graphs nearly regular of degree r while the vertex being not of this degree will be called an exceptional vertex.

Spanning trees in nearly regular graphs

PROPOSITION 1. Let G be a nearly regular graph of degree r and let H be its subgraph obtained by removing the exceptional vertex. Then

$$t(G) = P_H(r). \tag{2}$$

PROOF. By the matrix-tree theorem (see, for example, [3], p. 38, $t(G)$ is equal to any cofactor in the admittance matrix $D - A$. (A is the adjacency matrix and D is the matrix of vertex degrees). Formula (2) is obtained if we take the cofactor of the diagonal element corresponding to the exceptional vertex of G.

EXAMPLE. The characteristic polynomial of a circuit C_n is known to be equal to $2T_n\left(\frac{\lambda}{2}\right) - 2$, where $T_n(\lambda)$ is the Chebyshev polynomial of the first kind. Introducing a new vertex connected by an edge to all vertices of C_n we get a wheel W_n, which is a nearly regular graph. By Proposition 1 we have $t(W_n) = 2T_n(3/2) - 2$ (cf. [8]).

An application

The inner dual G^{**} of a plane graph G is the subgraph of the usual dual G^*, obtained by deleting the vertex corresponding to the infinite region of the original plane graph [4].

Let G be a plane graph in which any finite region is bounded by a circuit of a fixed length r. (The so called animals – see for example, [4] – belong to this class of graphs). Then G^* is a nearly regular graph. On the other hand, it is known that the graphs G and G^* have the same number of spanning trees [1], p. 38. Combining these observations with Proposition 1 we immediately reach the following corollary.

PROPOSITION 2. If G is a plane graph in which any finite region is bounded by a circuit of length r, then

$$t(G) = P_{G^{**}}(r). \tag{3}$$

EXAMPLE. For the graph G on Fig. 1 the inner dual G^{**} is the path P_n on n vertices, Since $P_{P_n}(\lambda) = U_n(\lambda/2)$, where $U_n(\lambda)$ is the Cheybyshev polynomial of the second kind, we have $t(G) = U_n(3) = \sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}6^{n-2k}$.

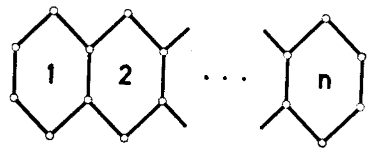
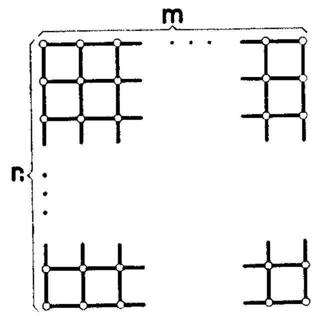


Fig. 1

If we replace in Fig. 1 hexagons with k-gons, then we get $t(G) = U_n(k/2)$.

EXAMPLE. For the graph $G_{m,n}$ on Fig. 2 we have $G_{m,n}^{**} = G_{m-1,n-1}$.



$G_{m,n}$

Fig. 2

The spectrum of $G_{m-1,n-1}$ consists of eigenvalues (c.f [2], p. 74)

$$2\cos\frac{i\pi}{m} + 2\cos\frac{j\pi}{n} \quad (i = 1, 2, \ldots, m-1, j = 1, 2, \ldots, n-1).$$

Therefore,

$$t(G_{m,n}) = 4^{(m-1)(n-1)} \prod_{i=1}^{m-1}\prod_{j=1}^{n-1}\left(\sin^2\frac{i\pi}{2m} + \sin^2\frac{j\pi}{2n}\right).$$

This result was obtained by other spectral methods in [6].

Non-isomorphic plane graphs from Proposition 2 can have isomorphic inner duals. Consequently, such graphs have the same number of spanning trees.

For example, the five graphs (animals) $G_1, \ldots, G_5$ on Fig. 3 haye equal number of spanning trees, namely $4^6 - 5 \cdot 4^4 + 5 \cdot 4^2 = 2896$, since $r = 4$ and $P_{G_i}{}^{**}(\lambda) = \lambda^6 - 5\lambda^4 + 5\lambda^2$

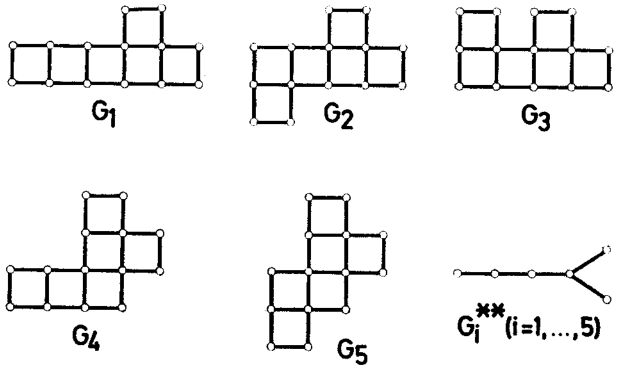


Fig. 3

Consider the set $\mathcal{T}$ of graphs from Proposition 2 with fixed r and with a fixed number of regions, for which the inner dual is a tree.

Let T be a tree which is an inner dual of a graph $G \in \mathcal{T}$. Then $\lambda_1(T) \leq r$. Since among trees with fixed number n of vertices the inequalty $P_{P_n}(\lambda) < P_T(\lambda)$ holds for all $\lambda > \lambda_1(T)$, $T \neq P_n$ [7], it follows that those graphs G from $\mathcal{T}$ for which $G^{**} = P_n$ have a maximal number of spanning trees.

A similar argument shows that minimal $t(G)$ have those graphs $G \in \mathcal{T}$ for which G^{**} is a star (provided that such graphs exist).

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