PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 29 (43), 1981, pp. 117–124

LOCALLY NEARLY PARACOMPACT SPACES

Ilija Kovačević

In the present paper, a new class of topological spaces, namely the class of locally nearly paracompact spaces has been introduced. This class contains the class of nearly paracompact spaces and is contained in the class of locally almost paracompact spaces.

Notation is standard except that $\alpha(A)$ will be used to denote interior of the closure of A. The topology τ^* is the semi regularization of τ and has as a base the regularly open sets from τ.

Definitions: A subset of a space is said to be regularly open iff it is the interior of some closed set or equivalently iff it is the interior of its own closure. A set is said to be regularly closed iff it is the closure of some open set or equivalently iff it is the closure of its own interior. A space X is said to be almost regular iff for any regularly closed set F and any point $x \notin F$, there exist disjoint open sets containing F and x respectively. A space X is nearly paracompact iff every regularly open cover of X has a locally finite open refinement, [5]. Let X be a topological space, and A a subset of X. The set A is α-nearly paracompact iff every X-regular open cover of A has an X-open refinement which covers A and is locally finite for every point in X (denot X-locally finite), [2].

Theorem 1. *If A is an α-nearly paracompact subset of a Hausdorff space X and x is a point of $X \backslash A$, then there are disjoint regular open neighbourhoods of x and A.*

Consequently, each α-nearly paracompact subset of Husdorff space (X, τ) is τ^*-closed.

Proof. Let A be any α-nearly paracompact subset of a Hausdorff space X, and x be any point of $X \backslash A$. Then for each point $a \in A$ there exist disjoint open sets U_a and V_a such that $a \in U_a$ and $x \in V_a$. Now, $U_a \cap V_a = \emptyset \Rightarrow \bar{U}_a \cap V_a = \emptyset$. Therefore $x \notin \bar{U}_a$. Then, the family

$$\mathcal{W} = \{\alpha(U_a) \colon a \in A\}$$

is a regular open (in X) covering of A, hence there exists an X-locally finite, X-open family $\mathcal{H} = \{H_i : i \in I\}$ which refines $\mathcal{W}$ and covers A. Since $\mathcal{H}$ is X-locally finite for $x \in X \backslash A$ there exists an open set M such that $x \in M$ and M intersects finitely many members of $\mathcal{H}$. Let $I_0 \subset I$ is a finite subset such that $M \cap H_i \neq \emptyset$ for each $i \in I_0$ and $M \cap H_i = \emptyset$ for each $i \in I \backslash I_0$. For every $i \in I_0$ there is $a_i \in A$ such that $H_i \subset \alpha(U_{a_i})$ Let

$$U = \cup\{H_i : i \in I\}, \text{ and } V_x = M \cap \{\cap V_{a_i} : i \in I_0\}$$

Then, $\alpha(U)$ and $\alpha(V_x)$ are regular open disjoint neighbourhoods of A and x respectively.

THEOREM 2. *If X is an almost regular topological space, A an α- nearly paracompact subset, U a regular open neighbourhood of A, then exists a regular closed neighbourhood V of A such that $A \subset V \subset U$.*

PROOF. Since X is almost regular, for each x in A there is a regular open neighbourhood W_x of x such that $x \in W_x \subset \bar{W}_x \subset U$. Now, the family

$$\mathcal{W} = \{W_x : x \in A\}$$

is an X-regular open covering of A, hence there exists an X-locally finite X-open family $\mathcal{U}$, which refines $\mathcal{W}$ and covers A. Let

$$W = \cup\{\bar{V}_i : V_i \in \mathcal{U}\}$$

Then,

$$V = \bar{W} = \overline{\cup\{V_i : V_i \in \mathcal{U}\}} = \cup\{\bar{V}_i : V_i \in \mathcal{U}\}$$

is a regular closed neighbourhood of A such that $A \subset V \subset U$.

COROLLARY 1. *If X is an almost regular toological space, A an α-nearly strongly paracompact subset, U a regular open neighbourhood of A, then there exists a regular closed neighbourhood V of A such that $A \subset V \subset U$.*

PROOF. Every α-nearly strongly paracompact subset is α-nearly paracompact.

LEMMA 1. *If B has an open neighbourhood U, such that $\bar{U}$ is α-nearly paracompact then B also has a regularly open neighbourhood V such that $\bar{V}$ is α-nearly paracompact and $U \subset \bar{V} \subset \bar{U}$.*

PROOF. If $B \subset U$, and $\bar{U}$ is α-nearly paracompact then $B \subset \alpha(U) \subset \overline{\alpha(U)} \subset \bar{U}$. Therefore $\alpha(U)$ is the desired neighbourhood.

DEFINITION 1. *A* space X is locally nearly paracompact iff each point of X has an open neighbourhood U such that $\bar{U}$ is α-nearly paracompact.

Obviously, every nearly paracompact space is locally nearly paracompact. But a locally nearly paracompact space may fail to be nearly paracompact as is shown by the following example.

EXAMPLE 1. Let Ω_0 be the set of all ordinal numbers less than the first uncountable ordinal Ω and let the topology be the order topology (the order topology has a subbase consisting of all sets of the form $\{x: x < a\}$ or $\{x: a < x\}$ for some a in Ω_0). Then Ω_0 is a Hausdorff locally compact space. Therefore Ω_0 is regular space. Since every locally compact space is locally nearly paracompact, Ω_0 is locally nearly paracompact space. Ω_0 is well known not to be paracompact, therefore Ω_0 is not nearly paracompact (Every regular nearly paracompact space is paracompact).

Clearly, every Hausdorff locally paracompact space is locally nearly paracompact. But a locally nearly paracompact space may fail to be locally paracompact as can be seen from the following example.

EXAMPLE 2. Let $X = \{a_{ij}, a_i, a: i, j = 1, 2, 3, \dots\}$. Let each point a_{ij} be isolated. Let $\{U^k(a_i): k = 1, 2, \dots\}$be the fundamental system of neighbourhoods of a_i where $U^k(a_i) = \{a_i, a_{ij}: j \geq k\}$ and let the fundamental system of neighbourhoods of a be $\{V^k(a): k = 1, 2, \dots\}$ where $V^k(a) = \{a, a_{ij}: i \geq k, j \geq k\}$. Then X is a Hausdorff space which is not regular at a and hence X is not locally paracompact (Every Hausdorff locally paracompact space is completely regular i.e. regular, [6]). But X is nearly paracompact ([5]), i.e. X is locally nearly paracompact.

THEOREM 3. *A Hausdorff space (X, τ) is locally nearly paracompact iff (X, τ^*) is locally paracompact.*

PROOF. Let (X, τ) be a Hausdorff locally nearly paracompact space, and x be any point of X. Then by lemma 1 there exists a regularly open set V containing x such that $\bar{V}$ is α-nearly paracompact. Then $\bar{V}$ is an α-paracompact set of (X, τ^*) (Lemma 2.1. in [2]). Since V is regularly open, then $\bar{V}$ is τ^*-closed set. Thus V is τ^*-open neighbourhood of x such that $\bar{V}_{\tau^*} = \bar{V}_\tau$ is α-paracompact set of (X, τ^*) and hence (X, τ^*) is locally paracompact.

Now, let (X, τ^*) be a locally paracompact Hausdorff space. Since (X, τ^*) is a Hausdorff space, therefore by Theorem 1 in [6], (X, τ^*) is completely regular i.e. (X, τ^*) is regular. Then, by Theorem 2 in [6] every point $x \in X$ has a τ^*-open neighbourhood V such that $\bar{V}_{\tau^*}$ is α-paracompact set of (X, τ^*). Then by Lemma 2.1. in [2] $\bar{V}_{\tau^*}$ is α -nearly paracompact subset of (X, τ). Since $\bar{V}_\tau \subset \bar{V}_{\tau^*}$, therefore $\bar{V}_\tau$ is α-nearly paracompact subset of (X, τ). Hence, (X, τ) is locally nearly paracompact.

THEOREM 4. *Every locally nearly paracompact Hausdorff space is almost regular.*

PROOF. Let (X, τ) be a locally nearly paracompact Haysdorff space. By Theorem $3 (X, \tau^*)$ is locally paracompact Hausdorff space. (X, τ^*) is regular. Hence, (X, τ) is almost regular ([2]).

Every locally nearly strongly paracompact space (A space X is locally nearly strongly paracompact iff each point has an open neighbourhood whose closure is α-nearly strongle paracompact subset of X, [3]) is locally nearly paracompact. The converse statement is not necesarily true. For our purpose , let X be any regular locally paracompact space which is not locally strongly paracompact. Because every regular locally paracompact space is locally nearly paracompact, X is locally nearly paracompact. But X is not locally strongly paracompact (Every regular locally nearly strongly paracompact space is locally strongly paracompact).

LEMMA 2. *If X is a locally nearly paracompact almost regular space then for each x in X and each open neighbourhood U of x there is an open set V such that $x \in V \subset \bar{V} \subset \alpha(U)$ and $\bar{V}$ is α-nearly paracompact.*

PROOF. Since X is locally nearly paracompact, there is an open set W with $x \in W \subset \bar{W}$ and $\bar{W}$ is α-nearly paracompact. The set $\alpha(W \cap U)$ is regularly open and is contained in $\bar{W}$. Since X is almost regular, there exists an open set V such that $x \in V \subset \bar{V} \subset \alpha(U \cap W) \subset \alpha(U)$. The set $\bar{V}$ is regularly closed and contained in $\bar{W}$, hence $\bar{V}$ is α-nearly paracompact.

LEMMA 3. *Let X be any topological space. A subset A of X is α-neraly paracompact iff for every X-open covering of A there exists an X-locally finite family of X-open sets which refines it and the interios of the X-closures of whose members cover the set A.*

PROOF. To prove the "if" part, let $\mathcal{U} = \{U_\lambda : \lambda \ \in I\}$ be any regular open covering off A. Then, there exists an X-locally finite family $\mathcal{V} = \{V_\beta : \beta \in J\}$ of X-open subsets of X such that each V_β is contained in some U_λ and $A \subset U\{\alpha(V_\beta) : \beta \in J\}$. Consider the family

$$\mathcal{U}^* = \{\alpha(V_\beta) : \beta \in J\}.$$

Clearly, $\mathcal{V}^*$ is X locally finite, X-open refinement of $\mathcal{U}$ and hence A is α-nearly paracompact.

To prove the "only if" part, let $\mathcal{U} = \{U_\lambda : \lambda \in I\}$ be any X-open covering of A. Then $\{\alpha(U_\lambda : \in I\}$ is regular open covering of A. Since A is α-nearly paracompact subset of X, there exists an X-locally finite X-open refinement $\{H_\lambda : \lambda \in I\}$ of $\{\alpha(U_\lambda) : \lambda \in I\}$ such that $H_\lambda \subset \alpha(U_\lambda)$ for each $\lambda \in I$ and $A \subset U\{H_\lambda : \lambda \in I\}$. For each $\lambda \in I$ let $M_\lambda = H_\lambda \backslash (\bar{U}_\lambda \backslash U_\lambda)$. Since $H_\lambda \subset \alpha(U_\lambda) \subset \bar{U}_\lambda$ therefore $M_\lambda = H_\lambda \cap U_\lambda$. Thus $\{M_\lambda : \lambda \in I\}$ is an X-locally finite family of X-open sets which refines $\mathcal{U}$. We shall prove that $A \subset U\{\alpha(M_\lambda) : \lambda \in I\}$. Let $x \in A$. Then $x \in H_\lambda$ for some $\lambda \in I$. Now

$$\alpha(M_\lambda) = \alpha(H_\lambda \cap U_\lambda) = \alpha(H_\lambda \cap \bar{U}_\lambda) = \alpha(H_\lambda). \text{ Thus } x \in \alpha(H_\lambda) = \alpha(M_\lambda).$$

Hence $\{M_\lambda : \lambda \in I\}$ is an X-locally finite family of X-open sets which refines $\mathcal{U}$ and the interiors of the closures of whose members cover A.

THEOREM 5. *Every locally nearly paracompact space is locally almost paracompact (A space X is locally almost paracompact iff each point of X has an open neighbourhood U, such that $\bar{U}$ is α-almost paracompact,* [1]*).*

PROOF. It follows easily from lemma 3.

We can show, however, that the converse of theorem 5 is not necesarily true. Following example will serve the purpose.

EXAMPLE 3. Let $X = \{a_{ij}, b_{ij}, c_i, a: i, j = 1, 2, \dots\}$. Let each point a_{ij} and b_{ij} be isolated. Let $\{U^k(c_i): k = 1, 2, \dots\}$. be the fundamental system of neighbourhoods of c_i where $U^k(c_i) = \{c_i, a_{ij}, b_{ij}: j \geq k\}$ and let $\{V^k(a): k = 1, 2, \dots\}$ be that of a where $V^k(a) = \{a, a_{ij}: i, j \geq k\}$. It can be shown that X is a Hausdorff semi regular space which is almost paracompact i.e. locally almost paracompact, but is not locally nearly paracompact. Note that X fails to be almost regular.

THEOREM 6. *Let A be an α-nearly paracompact subset of a locally nearly paracompact almost regular space (X, τ), and let G be any regular open set containing A. Then there is a regular closed α-nearly paracompact neighbourhood of A contained in G.*

PROOF. Let (X, τ) be a locally nearly paracompact almost regular space and A be any α-nearly paracompact subset, and G regular open containing A. Then, (X, τ^*) is a locally paracompact regular space, A is an α-paracompact subset of (X, τ^*) and G is a τ^*-open set containing A. Therefore, by Theorem 5 in [6], there is τ^*-closed α-paracompact in (X, τ^*) neighbourhood V of A contained in G. Now, $\bar{V}$ is α-nearly paracompact of the space (X, τ). Then $\overline{\alpha(V)} \subset \bar{V}$, therefore $V_1 = \overline{\alpha(V)}$ is regularly closed α-nearly paracompact neighbourhood of A contained in G.

COROLLARY 2. *Let A be any α-nearly paracompact subset of a locally nearly Hausdorff space, and let G be any regular open set containing it. Then there is a regular closed α-nearly paracompact neighbourhood of A contained in G.*

PROOF. Every locally nearly paracompact Hausdorff space is almost regular.

THEOREM 7. *The product off a locally nearly paracompact and locally nearly compact space is locally nearly paracompact.*

PROOF. Let (x, y) be any point of $X \times Y$ where X is locally nearly paracompact space and Y locally nearly compact space. Then there exists a regular open neighbourhood A of x in X such that $\bar{A}$ is an α-nearly paracompact subset of X. Also, there exists a regular open neighbourhood B of y in Y such that $\bar{B}$ is an α-nearly compact subset of Y. Then $A \times B$ is a regularly open neighbourhood of (x, y) in $X \times Y$ such that $\overline{A \times B} = \bar{A} \times \bar{B}$ is α-nearly paracompact, therefore $X \times Y$ is locally nearly paracompact space.

LEMMA 4. *If f is closed, continuous mapping of a space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each point $y \in Y$, then the inverse image $f^{-1}(K)$ any α-paracompact set in Y is α-nearly paracompact set in X.*

PROOF. Let $\mathcal{U} = \{U_\alpha: \alpha \in I\}$ be any regular open covering of $f^{-1}(K)$. For each $y \in Y$ select a finite union H_y of elements of $\{U_\alpha: \alpha \in I\}$ that contains the α-nearly compact set $f^{-1}(y)$. Since f is closed $G_y = Y \backslash f(X \backslash H_y)$ is an open set containing y and $\{G_y : y \in K\}$ is an Y-open covering of K. Since K is α-paracompact there exists an Y-locally finite open cover $\{K_\beta : \beta \in J\}$ of K which refines $\{G_y : y \in K\}$. We note that for each $y \in Y f^{-1}(G_y)$ is contained in H_y. Thus for each $\beta \in J$ we may and do choose a finite subset I_β of I such that $f^{-1}(K_\beta) \subset \cup\{U_\alpha : \alpha \in I\}$. We not assert that $\{f^{-1}(K_\beta) \cap U_\alpha : \alpha \in J$ and $\alpha \in I_\beta)$ is an X-locally finite X-open covering of $f^{-1}(K)$. In order to see this let $x \in X$. Since $\{K_\beta : \beta \in J\}$ is Y-locally finite, there exists an open set P of Y containing $f(x)$ such that P meets only finite many K_β. Then $f^{-1}(P)$ is an open subset of X-containing x that meets only finite many of the sets $f^{-1}(K_\beta)$ and thus meets only finitely many of the sets $f^{-1}(K_\beta) \cap U_\alpha : \alpha \in I_\beta$. Thus every regularly open cover of $f^{-1}(K)$ has an X-locally finite open refinement and $f^{-1}(K)$ is α-nearly paracompact.

THEOREM 8. *If f is a closed, continuous mapping of a Hausdorff space X onto a locally paracompact space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$, then X is locally nearly paracompact.*

PROOF. Let f be a closed, continuous mapping of Hausdorff space X onto a locally paracompact space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$. It follows that Y is Hausdorff space. Thus Y is Hausdorff locally paracompact. Hence, for each $x \in X$, there exists an open neighbourhood V of $f(x)$ in Y such that $\bar{V}$ is α-paracompact subset of Y. Then, $f^{-1}(\bar{V})$ is α-nearly paracompact subset of X. Since $\overline{f^{-1}(V)} \subset f^{-1}(\bar{V})$, then $\overline{f^{-1}(V)}$ is α-nearly paracompact subset of X ([2]). Thus, $f^{-1}(V)$ is open neighbourhood of a point x such that $\overline{f^{-1}(V)}$ is α-nearly paracompact subset of X, hence X is locally neraly paracompact.

DEFINITION 2. A mapping $f : X \to Y$ is said to be almost continuous iff the inverse image of every regularly open subset of Y is an open subset of X. f is called almost open (almost closed) iff the image of every regularly open (regularly closed) subset of X is an open (a closed) subset of Y [5].

LEMMA 5. *If X is an almost regular and $f : X \to Y$ is an almost continuous and almost closed surjection such that $f^{-1}(y)$ is α-nearly compact for each point $y \in Y$, then Y is almost regular.*

PROOF. Suppose $y \in Y$ and V is regularly open set of Y containing y. Then, since f is almost continuous and almost closed, $f^{-1}(V)$ is a regularly open set in X containing $f^{-1}(y)$. Since X is almost regular and $f^{-1}(y)$ is α-nearly compact, there exists a regularly open set U in X such that $f^{-1}(y) \subset U \subset \bar{U} \subset f^{-1}(V)$. By Lemma 3 in [4] there exists an open set W in Y such that $y \in W$ and $f^{-1}(W) \subset U$. Therefore, we have $y \in W \subset f(U) \subset f(\bar{U}) \subset V$. Since f is almost closed $f(\bar{U})$ is closed. Hence we have $y \in W \subset \alpha(W) \subset \bar{W} \subset V$, where $\alpha(W)$ is regularly open set of Y and $\overline{\alpha(W)} = \bar{W}$. Hence Y is almost regular.

LEMMA 6. *If $f : X \to Y$ is almost closed, alsmot continuous and alsmot open mapping of locally nearly paracompact space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$, then Y is locally nearly paracompact.*

PROOF. For any point $y \in Y$ there exists a point $x \in X$ such that $f(x) = y$. There exists a regular open neighbourhood M of x such that $\bar{M}$ is α-nearly paracompact subset of X. Since f is almost open, then $f(M)$ is open set in Y containing y such that $\overline{f(M)} \subset f(\bar{M})$($f$ is almost closed mapping, therefore $f(\bar{M})$ is closed subset of Y) and $f(\bar{M})$ is α-nearly paracompact subset of Y ([2]). Therefore, since $\overline{f(M)}$ is regularly closed subset of $f(\bar{M})$, $\overline{f(M)}$ is α-nearly paracompact subset of Y. Hence the result.

THEOREM 9. *If $f: X \to Y$ is an almost closed, almost conitinuous and almost open mapping of an almost regular (Hausdorff) locally nearly paracompact space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$, then Y is almost regular locally nearly paracompact.*

PROOF. If X is an almost regular locally nearly paracompact space, then Y is almost regular locally nearly paracompact (by lemmas 5 and 6). If X is a Hausdorff locally nearly paracompact space, then Y is Hausdorff locally nearly paracompact (The Hausdorff property is invariant under almost closed almost conitinuous mappings with α-nearly compact point inverses, Theorem 3, [4]). Since every Hausdorff locally nearly paracompact space is almost regular (Theorem 4) then Y is almost regular locally nearly paracompact.

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Kovačević Ilija
University of Novi Sad
Faculty of Techical Science
Department of mathematics
Veljka Vlahovića 3
21000 Novi Sad
Yugoslavia